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Some difficult problems

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5. Some Difficult Problems

As we discuss in this lecture, many of the basic problems formulated years ago for algebraic K-theory remain unsolved. This remains a subject in which much exciting work remains to be done.

5.1. $K_*(\mathbb{Z})$. Unfortunately, there are few examples (rings or varieties) for which a complete computation of the K-groups is known. As we have seen earlier, one such complete computation is the K-theory of an arbitrary finite field, $K_*(\mathbb{F}_q)$. Indeed, general theorems of Quillen give us the complete computations

$$K_*(\mathbb{F}_q[t]) \; = \; K_*(\mathbb{F}_q), \quad K_*(\mathbb{F}_q([t,t^{-1}]) \; = \; K_*(\mathbb{F}_q) \oplus K_{*-1}(\mathbb{F}_q).$$

Perhaps the first natural question which comes to mind is the folowing: "what is the K-theory of the integers."

In recent years, great advances have been made in computing $K_*(\mathcal{O}_K)$ of a ring of integers in a number field K (e.g., $\mathbb{Z}$ inside $\mathbb{Q}$).

- $K_0(\mathcal{O}_K) \otimes \mathbb{Q}$ is 1 dimensional by the finiteness of the class number of K (Minkowski).
- $K_1(\mathcal{O}_K) \otimes \mathbb{Q}$ has dimension $r_1 + r_2 - 1$, where $r_1, \; r_2$ are the numbers of real and complex embeddings of K. (Dirichlet).
- Quillen proved that $K_i(\mathcal{O}_K)$ is a finitely generated abelian group for any i.
- For $i > 1$, Borel determined

$$K_i(\mathcal{O}_K) \otimes \mathbb{Q} \; = \; \begin{cases} 0, & i \equiv 0 \; (mod \; 4) \\ r_1 + r_2, & i \equiv 1 \; (mod \; 4) \\ 0, & i \equiv 2 \; (mod \; 4) \\ r_2, & i \equiv 3 \; (mod \; 4) \end{cases} \tag{1}$$

  in terms of the numbers $r_1, \; r_2$.
- $K_*(O_K, \mathbb{Z}/2)$ has been computed by Rognes-Weibel as a corollary of Voevodsky's proof of the Milnor Conjecture.
- $K_*(\mathbb{Z}, \mathbb{Z}/p)$ follows in all degrees not divisible by 4 from the Bloch-Kato Conjecture, now seemingly proved by Rost and Voevodsky.

Here is a table of the values of $K_*(\mathbb{Z})$ whose likely inaccuracy is due to my confusion of indexing of Bernoulli numbers.

Theorem 5.1. *The K-theory of $\mathbb{Z}$ is given by (according to Weibel's survey paper):*

$$\begin{cases} K_{8k} \ =?0?, \quad 0<k \\ K_{8k+1} = \mathbb{Z} \oplus \mathbb{Z}/2, \quad 0<k \\ K_{8k+2} \ = \ \mathbb{Z}/2c_{2k+1} \oplus \mathbb{Z}/2 \\ K_{8k+3} = \mathbb{Z}/2d_{4k+2}, \quad i \equiv 3 \\ K_{8k+4} \ = \ ?0? \\ K_{8k+5} \ = \ \mathbb{Z} \\ K_{8k+6} \ = \ \mathbb{Z}/c_{2k+2} \\ K_{8k+7} \ = \ Z/d_{4k+4} \end{cases} \tag{2}$$

Here, c_k/d_k is defined to be the reduced expression for $B_k/4k$, where B_k is the k-th Bernoulli number (defined by

$$\frac{t}{e^t-1} = 1 + \sum_{k=1}^{\infty} \frac{B_k}{(2k)!} t^{2k} \ .$$

Challenge 5.2. *Prove the vanishing of $K_{4i}(\mathbb{Z})$, $i>0$.*

5.2. **Bass Finiteness Conjecture.** This is one of the most fundamental and oldest conjectures in algebraic K-theory. Very little progress has been made on this in the past 35 years.

Conjecture 5.3. *(Bass finiteness) Let A be a commutative ring which is finitely generated as an algebra over $\mathbb{Z}$. Is $K'_n(A)$ (i.e., the Quillen K-theory of mod(A)) finitely generated for all n?*

In particular, if A is regular as well as commutative and finitely generated over $\mathbb{Z}$, is each $K_n(A)$ a finitely generated abelian group?

This conjecture seems to be very difficult, even for $n=0$. There are similar finiteness conjectures for the K-theory of projective varieties over finite fields.

Example 5.4. Here is an example of Bass showing that we must assume A is regular or consider $G_*(A)$. Let $A = \mathbb{Z}[x,y]/x^2$. Then the ideal (x) is infinitely additively generated by $x, xy, xy^2, \ldots$. On the other hand, if $t \in (x)$, then $1+t \in A^*$, so that we see that $K_1(A)$ is not finitely generated.

Example 5.5. As pointed out by Bass, it is elementary to show (using general theorems of Quillen and Quillen's computation of the K-theory of finite fields) that if A is finite, then $G_n(A) \simeq G_n(A/radA)$ is finite for every $n \geq 0$. Subsequently, Kuku proved that $K_n(A)$ is also finite whenever A is finite.

There are many other finiteness conjectures involving smooth schemes of finite type over a finite field, $\mathbb{Z}$ or $\mathbb{Q}$. Even partial solutions to these conjectures would represent great progress.

5.3. **Milnor K-theory.** We recall Milnor K-theory, a major concept in Professor Vishik's lectures. This theory is motivated by Matsumoto's presentation of $K_2(F)$ (mentioned in Lecture 1),

Definition 5.6. (Milnor) Let F be a field with multiplicative group of units $F^\times$. The Milnor K-group $K_n^{Milnor}(F)$ is defined to be the n-th graded piece of the graded ring defined as the tensor algebra $\bigoplus_{n\geq 0}(F^\times)^{\otimes n}$ modulo the ideal generated by elements $\{a, 1-a\} \in F^* \otimes F^*, a \neq 1 \neq 1-a$.

In particular, $K_1(F) = K_1^{Milnor}(F), K_2(F) = K_2^{Milnor}(F)$ for any field F, and $K_n^{Milnor}(F)$ is a quotient of $\Lambda^n(F^\times)$. For F an infinite field, Suslin proved that there are natural maps

$$K_n^{Milnor}(F) \to K_n(F) \to K_n^{Milnor}(F)$$

whose composition is $(-1)^{n-1}(n-1)!$. This immediately implies, for example, that the higher K-groups of a field of high transcendence degree are very large.

Remark 5.7. It is difficult to even briefly mention K_2 of fields without also mentioning the deep and import theorem of Mekurjev and Suslin: for any field F and positive integer n,

$$K_2(F)/nK_2(F) \ \simeq \ H^2(F, \mu_n^{\otimes 2}).$$

In particular, $H^2(F, \mu_n^{\otimes 2})$ is generated by products of elements in $H^1(F, \mu_n) = \mu_n(F)$.

Moreover,if F contains the n^{th} roots of unity, then

$$K_2(F)/nK_2(F) \ \simeq \ {}_nBr(F),$$

where ${}_nBr(F)$ denotes the subgroup of the Brauer group of F consisting of elements which are n-torsion. In particular, ${}_nBr(F)$ is generated by "cyclic central simple algebras."

The most famous success of K-theory in recent years is the following theorem of Voevodsky, establishing a result conjectured by Milnor.

Theorem 5.8. *Let F be a field of characteristic $\neq 2$. Let $W(F)$ denote the Witt ring of F, the quotient of the Grothendieck group of symmetric inner product spaces modulo the ideal generated by the hyperbolic space $\langle 1\rangle \oplus \langle -1\rangle$ and let $I = ker\{W(F) \to \mathbb{Z}/2\}$ be given by sending a symmetric inner product space to its rank (modulo 2). Then the map*

$$K_n^{Milnor}(F)/2 \cdot K_n^{Milnor}(F) \to I^n/I^{n+1}, \quad \{a_1, \ldots, a_n\} \mapsto \prod_{i=1}^{n}(\langle a_i\rangle - 1)$$

is an isomorphism for all $n \geq 0$. Here, $\langle a\rangle$ is the 1 dimensional symmetric inner product space with inner product $(-,-)_a$ defined by $(c,d)_a = acd$.

Suslin proved that the natural map $K^M_*(F) \to K_*(F)$ (whose existence is given by Masumoto's Theorem and the ring structure on $K_*(F)$) is complemented by a natural map $K_*(F) \to K^M_*(F)$ whose composition in degree n is multiplication by $(n-1)!$. In particular, this tells us that the cardinality of $K_n(F),\ n > 0$ is large if the cardinality of F is large.

On the other hand, Suslin has proved the following theorem.

Theorem 5.9. *Let F be an algebraic closed field. If F has characteristic 0 and $i > 0$, then $K_{2i}(F)$ is a $\mathbb{Q}$ vector space and $K_{2i-1}(F)$ is a direct sum of $\mathbb{Q}/\mathbb{Z}$ and a rational vector space. If F has characteristic $p > 0$ and $i > 0$, then $K_{2i}(F)$ is a $\mathbb{Q}$ vector space and $K_{2i-1}(F)$ is a direct sum of $\oplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a rational vector space.*

Question 5.10. *What information is reflected in the uncountable vector spaces $K_n(\mathbb{C}) \otimes \mathbb{Q}$? Are there interesting structures to be obtained from these vector spaces?*

5.4. **Negative K-groups.** Bass introduced *negative* algebraic K-groups, groups which vanish for regular rings or, more generally, smooth varieties. These negative K-groups measure the failure of K-theory in positive degree to obey "homotopy invariance" and "localization" (i.e.,

$$K_*(X) \stackrel{?}{=} K_*(X \times \mathbb{A}^1), \quad K_*(X) \oplus K_{*-1}(X) \stackrel{?}{=} K_*(X \times \mathbb{A}^1 \backslash \{0\}).$$

Very recently, there has been important progress in computing these negative K-groups by Cortinas, Haesemeyer, Schlicting, and Weibel.

Question 5.11. *Can negative K-groups give useful invariants for the geometric study of singularities?*

5.5. **Algebraic versus topological vector bundles.** Let X be a complex projective variety, and let X^{an} denote the topological space of complex points of X equipped with the analytic topology. Then any algebraic vector bundle $E \to X$ naturally determines a topological vector bundle $E^{an} \to X^{an}$. This determines a natural map

$$K_0(X) \ \to \ K^0_{top}(X^{an}).$$

Challenge 5.12. *Understand the kernel and image of the above map, especially after tensoring with $\mathbb{Q}$:*

$$CH^*(X) \otimes \mathbb{Q} \ \simeq \ K_0(X) \otimes \mathbb{Q} \ \to \ K^0_{top}(X^{an}) \otimes \ \simeq \ H^{ev}(X^{an}, \mathbb{Q}). \tag{3}$$

The kernel of (3) can be identified with the subspace of $CH^*(X) \otimes \mathbb{Q}$ consisting of rational equivalence classes of algebraic cycles on X which are homologically equivalent to 0.

The image of (3) can be identified with those classes in $H^*(X^{an}, \mathbb{Q})$ represented by algebraic cycles – the subject of the Hodge Conjecture!

In positive degree, the analogue of our map is uninteresting.

Proposition 5.13. *If X is a complex projective variety, then the natural map*

$$K_i(X) \otimes \mathbb{Q} \;\to\; K_{top}^{-i}(X^{an}), \quad i > 0$$

is the 0-map.

5.6. **K-theory with finite coefficients.** Although the map in positive degrees

$$K_i(X) \;\to\; K_{top}^{-i}(X^{an})$$

is typically of little interest, the situation changes drastically if we consider K-theory mod-n.

As an example, we give the following special case of a theorem of Suslin. Recall that $(\operatorname{Spec}\mathbb{C})^{an}$ is a point, which we denote by $\star$.

Theorem 5.14. *The map*

$$K_i(\operatorname{Spec}\mathbb{C}) \;\to\; K_{top}^{-i}(\star)$$

is the 0-map for $i > 0$. On the other hand, for any positive integer n and any interger $i \geq 0$, the map

$$K_i(\operatorname{Spec}\mathbb{C}, \mathbb{Z}/n) \;\to\; K_{top}^{-i}(\star, \mathbb{Z}/n)$$

is an isomorphism.

How can the preceding theorem be possibly correct? The point is that $K_{2i-1}(\operatorname{Spec}\mathbb{C})$ is a divisible group with torsion subgroup $\mathbb{Q}/\mathbb{Z}$. Then, we see that this $\mathbb{Q}/\mathbb{Z}$ is odd degree integral homotopy determines a $\mathbb{Z}/n$ in even degree mod-n homotopy. This is exactly what $K_{top}^{-*}(\star)$ determines in even mod-n homotopy degree.

The K-groups modulo n are defined to be the homotopy groups modulo n of the K-theory space (or spectrum).

Definition 5.15. For positive integers $i, n > 1$, let $M(i, \mathbb{Z}/n)$ denote the C.W. complex obtained by attaching an i-cell D^i to S^{i-1} via the map $\partial(D^i) = S^{i-1} \to S^{i-1}$ given by multiplication by n.

For any connected C.W. complex, we define

$$\pi_i(X, \mathbb{Z}/n) \equiv [M(i, \mathbb{Z}/n), X], \quad i, n > 1.$$

If $X = \Omega^2 Y$, we define

$$\pi_i(X, \mathbb{Z}/n) \equiv [M(i+2, \mathbb{Z}/n), Y], \quad i \geq 0, n > 1.$$

Since $S^{i-1} \to M(i, \mathbb{Z}/n)$ is the cone on the multiplication by n map $S^{i-1} \xrightarrow{n} S^{i-1}$, we have long exact sequences

$$\cdots \to \pi_i(X) \xrightarrow{n} \pi_i(X) \to \pi_i(X/\mathbb{Z}/n) \to \pi_{i-1}(X) \to \cdots$$

Perhaps this is sufficient to motivate our next conjecture, which we might call the Quillen-Lichtenbaum Conjecture for smooth complex algebraic varieties. The special case in which $X = \operatorname{Spec}\mathbb{C}$ is the theorem of Suslin quoted above.

Conjecture 5.16. *(Q-L for smooth* **C** *varieties) If X is a smooth complex variety of dimension d, then is the natural map*

$$K_i(X, \mathbb{Z}/n) \to K_i^{top}(X^{an}, \mathbb{Z}/n)$$

an isomorphism provided that $i \geq d - 1 \geq 0$?

Remark In "low" degrees, $K_*(X, \mathbb{Z}/n)$ should be more interesting and will not be periodic. For example, $K_{ev}^{top}(X, \mathbb{Z}/n)$ has a contribution from the Brauer group of X whereas $K_0(X, \mathbb{Z}/n)$ does not.

5.7. **Etale K-theory.** It is natural to try to find a good "topological model" for the mod-n algebraic K-theory of varieties over fields other than the complex numbers. Suslin's Theorem in its full generality can be formulated as follows

Theorem 5.17. *If k is an algebraically closed field of characteristic $p \geq 0$, then there is a natural isomorphism*

$$K_*(k, \mathbb{Z}/n) \xrightarrow{\simeq} K_*^{et}(\operatorname{Spec} k, \mathbb{Z}/n), \quad (n, p) = 1.$$

Moreover, if the characteristic of k is a positive integer p, then $K_i(k, \mathbb{Z}/p) = 0$, for all $i > 0$.

We have stated the previous theorem in terms of *etale K-theory* although this is not the way Suslin formulated his theorem. We did this in order to introduce the etale topology, a Grothendieck topology associated to the etale site. For this site, the distinguished morphisms $\mathcal{E}$ are etale morphisms of schemes. A map of schemes $f : U \to V$ is said to be etale (or "smooth of relative dimension 0) if there exist affine open coverings $\{U_i\}$ of U, $\{V_j\}$ of V such that the restriction to U_i of f lies in some V_j and such that the corresponding map of commutative rings $A_i \leftarrow R_j$ is unramified (i.e., for all homomorphisms from R to a field k, $A \otimes_R k \leftarrow k$ is a finite separable k algebra) and flat.

The etale topology was introduced by Grothendieck partly to reinterpret Galois cohomology of fields and partly to algebraically realize singular cohomology of complex algebraic varieties. The following "comparison theorem" proved by Michael Artin and Alexander Grothendieck is an important property of the etale topology. (See also Lecture 6.)

Theorem 5.18. *(Artin, Grothendieck) If X is a complex algebraic variety, then*

$$H_{et}^*(X, \mathbb{Z}/n) \simeq H_{sing}^*(X^{an}, \mathbb{Z}/n).$$

Here, $H_{et}^*(X, \mathbb{Z}/n)$ denotes the derived functors of the global section functor applied to the constant sheaf $\mathbb{Z}/n$ on the etale site.

The etale topology not only enables us to define etale cohomological groups, but also etale homotopy types. Using the etale homotopy type, etale K-theory (defined by Bill Dwyer and myself) can be defined in a manner similar to topological K-theory.

For this theory, there is an Atiyah-Hirzeburch spectral sequence

$$E_2^{p,q} = H^p_{et}(X, K^q_{et}(\star)) \Rightarrow K^{p+q}_{et}(X, \mathbb{Z}/n)$$

provided that $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}[1/n]$-modules. If we let μ_n denote the etale sheaf of n-th roots of unity and let $\mu_n^{\otimes q/2}$ denote $\mu_n^{\otimes j}$ if $q = 2j$ and 0 if j is odd, then this spectral sequence can be rewritten

$$E_2^{p,q} = H^p_{et}(X, \mu^{\otimes q/2}) \Rightarrow K^{et}_{q-p}(X, \mathbb{Z}/n).$$

Using etale K-theory, we can reformulate and generalize the Quillen-Lichtenbaum Conjecture (originally stated for $\operatorname{Spec} K$, where K is a number field), putting this conjecture in a quite general context.

Conjecture 5.19. *(Quillen-Lichtenbaum) Let X be a smooth scheme of finite type over a field k, and assume that n is a positive integer with $1/n$ in k or A. Then the natural map*

$$K_i(X, \mathbb{Z}/n) \to K_i^{et}(X, \mathbb{Z}/n)$$

is an isomorphism for $i - 1$ greater or equal to the mod-n etale cohomological dimension of X.

This conjecture appears to be proven, or near-proven, thanks to the work of Rost and Voevodsky on the Bloch-Kato Conjecture.

5.8. **Integral conjectures.** There has been much progress in understanding K-theory with finite coefficients, but much less is known about the result of tensoring the algebraic K-groups with $\mathbb{Q}$.

The following theorem of Soué is proved by investigating the group homology of general linear groups over fields. Soulé proves a vanishing theorem for more general rings R with a range depending upon the "stable range" of R.

Theorem 5.20. *(Soulé) For any field F,*

$$K_n(F)^{(s)}_{\mathbb{Q}} = 0, \quad s > n.$$

Here $K_n(F)^{(s)}_{\mathbb{Q}}$ is the s-eigenspace with respect to the action of the Adams operations on $K_n(F)$.

This motivates the following Beilinson-Souè vanishing conjecture, part of the Beilinson Conjectures discussed in the next lecture. This conjecture is now known if we replace the coefficients $\mathbb{Z}(n)$ by their finite coefficients analgoue $\mathbb{Z}/\ell(n)$.

Conjecture 5.21. *(Beilinson-Soulé) For any field F, the motivic cohomology groups $H^p(\operatorname{Spec} F, \mathbb{Z}(n))$ equal 0 for $p < 0$.*

Yet another "auxillary K-theory has been developed to investigate K-theory of complex varieties, especially some aspects involving rational coefficients.

Theorem 5.22. *(Friedlander-Walker) Let X be a complex quasi-projective variety. The map from the algebraic K-theory spectrum $\mathcal{K}(X)$ to the topological K-theory spectrum $\mathcal{K}_{top}(X^{an})$ factors through the "semi-topological K-theory spectrum $\mathcal{K}^{sst}(X)$.*

$$\mathcal{K}(X) \;\to\; \mathcal{K}^{sst}(X) \;\to\; \mathcal{K}_{top}(X^{an}).$$

The first map induces an isomorphism in homotopy groups modulo n, whereas the second map induces an isomorphism for certain special varieties and typically induces an isomorphism after "inverting the Bott element."

This semi-topological K-theory is related to cycles modulo algebraic equivalence is much the same way as usual algebraic K-theory is related to Chow groups (cycles modulo rational equivalence).

One important aspect of this semi-topological K-theory is that leads to conjectures which are "integral" whose reduction modulo n give the familiar Quillen-Lichtenbaum Conjecture.

We state one precise form of such a conjecture, essentially due to Suslin.

Conjecture 5.23. *Let X be a smooth, quasi-projective complex variety. Then the natural map*

$$K_i^{sst}(X) \;\to\; K_{top}^{-i}(X^{an})$$

is an isomorphism for $i \geq dim(X) - 1$ and a monomorphism for $i = dim(X) - 2$.

Now, we also have a "good semi-topological model" for the K-theory of quasi-projective varieties over $\mathbb{R}$, the real numbers. This is closely related to "Atiyah Real K-theory rather than the topological K-theory we have discussed at several points in these lectures.

Challenge 5.24. *Develop a semi-topological K-theory for varieties over an arbitrary field.*

5.9. **K-theory and Quadratic Forms.** another topic of considerable interest is *Hermetian K-theory* in which we take into account the presence of quadratic forms. Perhaps this topic is best left to Professor Vishik!