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Generic discrete and elementary discrete invariants of quadrics

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Lecture 4

Generic discrete and elementary discrete invariants of quadrics

Last time were introduced two discrete invariants of quadrics: the *motivic decomposition type* and the *splitting pattern*. We will show that both these invariants live inside some big discrete invariant of geometric origin as (rather small) faces. The idea here is, instead of studying the faces, to study the whole invariant, since it should posses more structure. Let us start with $MDT(Q)$. This invariant measures what are possible decompositions of $M(Q)$, that is, what kind of projectors we have in $\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q))$.

Rost Nilpotence Theorem

The following result of M.Rost is central here:

Theorem 0.1 (RNT)

$$\mathrm{Ker}(\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q)) \stackrel{\overline{ac}}{\to} \mathrm{End}_{\mathrm{Chow}^{eff}(\overline{k})}(M(Q|_{\overline{k}}))$$

consists of nilpotents.

This implies that any projector in the image of $\overline{ac}$ can be lifted to a projector in $\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q))$, and two such liftings produce direct summands which are isomorphic as objects of $\mathrm{Chow}^{eff}(k)$. So, to know the decomposition of $M(Q)$ it is sufficient to know the

$$image(\overline{ac}) = image(\mathrm{CH}^{\dim(Q)}(Q \times Q) \to \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})).$$

Consider for simplicity the case $\dim(Q)$-odd (the other one can be done similarly). Then $2 \cdot \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}}) \subset image(\overline{ac})$, since $\mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})$ is additively generated by $[l_i \times h^i]$, and $2 \cdot l_i = h^{\dim(Q)-i}$, which implies that $[h^{\dim(Q)-i} \times l_i] \in image$. Thus, after all, we need to know only the

$$image(\mathrm{CH}^{\dim(Q)}(Q \times Q)/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})/2).$$

Example: Let $\dim(Q)$ is odd. Then $M(Q)$ is indecomposable if and only if the image above consists of just $\mathbf{Z}/2 \cdot [\Delta_Q]$.

Aside: RNT shows that $M(Q)$ does not contain *phantom* direct summands. That is, if N is a direct summand, and $N|_{\overline{k}} = 0$, then $N = 0$.

RNT is generalized to the case of arbitrary projective homogeneous variety by V.Chernousov, A.Merkurjev and S.Gille. So, the motives of these varieties also have no phantom direct summands.

Hypothetically, NT should hold for arbitrary smooth projective variety, and so there should be no phantom objects in $\mathrm{Chow}^{eff}(k)$ at all. But this is a very strong and complicated Conjecture (related to the Conjecture of S.Bloch). Notice, that in $DM_{-}^{eff}(k)$ there is plenty of phantom objects, and many of these were successfully used (most notably, by V.Voevodsky), but they are *infinite dimensional* and do not live in $\mathrm{Chow}^{eff}(k)$.

Definition 0.2 *Consider the following invariant of quadrics:*

$$Q \mapsto image(\mathrm{CH}^*(Q^{\times N})/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(Q^{\times N}|_{\overline{k}})/2),\ for\ all\, N.$$

We call it Generic discrete invariant of quadrics (in noncompact form).

This invariant clearly contains $MDT(Q)$. The disadvantage here is that one has to consider infinitely many objects. But the invariant can be "compactified", and the above problem disappears.

To each smooth projective quadric Q one can assign the respective *quadratic Grassmannians*:

$$Q \mapsto G(i,Q)\ -\ \text{Grassmanian of } i-\text{dim. planes on}\, Q.$$

This is smooth projective (homogeneous) variety, and E-rational points of $G(i,Q)$ are i-dimensional planes $l_i \subset Q|_E$.

We get varieties:

$$Q = G(0,Q),\ G(1,Q), \ldots, G(d,Q), \text{ where } d = \left[\frac{\dim(Q)}{2}\right].$$

Examples:

1) $\dim(q) = 4$, $q = \langle a,b,c,d\rangle$. Then $G(1,Q) = C_{\{-ab,-ac\}} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k\sqrt{abcd})$ - the conic over the quadratic extension. So, $G(1,Q)|_{\overline{k}} = \mathbb{P}^1 \coprod \mathbb{P}^1$.

2) $\dim(q) = 5$, $q = \langle a, b, c, d, e\rangle$. Consider the auxiliary form $p = q \perp \langle -det(q)\rangle = \langle a, b, c, d, e, -abcde\rangle$. Then $\exists\lambda$ (for example $= abc$), such that $\lambda{\cdot}p = \langle A, B, -AB, -C, -D, CD\rangle$ is an Albert form, corresponding to the biquaternion algebra $Al = Quat(\{A, B\}, k) \otimes_k Quat(\{C, D\}, k)$. Then $G(1, Q) = SB(Al)$ is a Severi-Brauer variety for the algebra Al. In particular, $G(1, Q)|_{\overline{k}} = \mathbb{P}^3$.

3) Let q_α be the 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$. Then $G(3, Q_\alpha) = Q_\alpha \coprod Q_\alpha$.

It appears that $M(Q^{\times N})$ can be decomposed into the direct sum of the motives of $G(i, Q)$ with various Tate-shifts.

Example:

$$M(Q{\times}Q) = M(Q){\oplus}(M(G(1, Q)){\oplus}M(G(1, Q))(1)[2]){\oplus}M(Q)(\dim(Q))[2\dim(Q)].$$

Consequently, to know

$$image(\mathrm{CH}^*(Q^{\times N})/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(Q^{\times N}|_{\overline{k}})/2), \text{ for all } N$$

is the same as to know

$$image(\mathrm{CH}^*(G(i, Q))/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(G(i, Q)|_{\overline{k}})/2), \text{ for } 0 \leqslant i \leqslant d = \left[\frac{\dim(Q)}{2}\right]$$

Definition 0.3 *This invariant is called Generic discrete invariant (in compact form) $GDI(Q)$.*

It contains not just $MDT(Q)$, but the $SP(Q)$ as well. Recall, that the Splitting Pattern of Q measures what are possible *Witt-indices* of $q|_E$ for all possible field extensions E/k. It follows from the *Specialization theory* of M.Knebusch, that it is sufficient to consider only the fields $E = k(G(i, Q)), 0 \leqslant i \leqslant d$ - the generic points of quadratic Grassmannians. In the end, one needs only to know, for which i there is a rational map $G(i, Q) \dashrightarrow G(i + 1, Q)$, or, which is the same, the rational section of the projection $F(i, i+1, Q) \to G(i, Q)$ (from the variety of flags $(l_i \subset l_{i+1})$ to the Grassmannian of i-planes on Q). Due to the Theorem of Springer (claiming that Q is isotropic $\Leftrightarrow$ it has a zero-cycle of degree 1) this can be reduced to the existence of cycles of certain type in $\mathrm{CH}^*(F(i, i+1; Q))/2$. But $F(i, i+1; Q)$ is a projective bundle over $G(i + 1, Q)$ and, consequently, $M(F(i, i + 1; Q))$

is a direct sum of $M(G(i+1,Q))$ with various Tate-shifts. Thus, $GDI(Q)$ contains $SP(Q)$.

Varieties $G(i,Q)$ are *geometrically cellular*, that is, can be "cut" into pieces isomorphic to affine spaces $\mathbb{A}^{r_j}$ - *Schubert cells* (to define such a cell, fix a complete flag $\pi_0 \subset \pi_1 \subset \ldots \subset \pi_d$, and natural numbers $n_0, \ldots, n_d$, then the $Cell(n_0, \ldots, n_d)$ is given by the locus of those i-planes l_i that $\dim(l_i \cap \pi_j) = n_j$). Thus, $M(G(i,Q)|_{\overline{k}})$ is (canonically!) a sum of Tate-motives, and $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})$ is a free abelian group with the canonical basis corresponding to Schubert cells

$$Cell \mapsto [\overline{Cell}] \in \mathrm{CH}^*(G(i,Q)|_{\overline{k}}).$$

The Schubert cells are parametrized by some sort of *Young diagrams*, and this way the ring $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ appears as quite combinatorial object. $GDI(Q,i)$ is the subring of $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ consisting of elements defined over k. But the ring $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ is still rather large. For example, for $i=d$ it has the rank $=2^{d+1}$. Need something handier. For this purpose there is $EDI(Q)$ - *Elementary discrete invariant of* Q. It does not determine the whole $image(\overline{ac})$, but just checks if some particular good classes are in the image, or not. These classes are *elementary classes*. To define them , start with the Grassmannian of 0-dimensional planes $G(0,Q)$, that is, with the quadric Q itself. Elementary classes on Q are just the classes $l_0, l_1, \ldots, l_d$ in $\mathrm{CH}^*(Q|_{\overline{k}})/2$ - these are the only interesting classes there (their k-rationality measures only the Witt-index of q). Now, the elementary classes on other Grassmannians can be produced from that on Q. Namely, we have natural projections:

$$Q \xleftarrow{\alpha_i} F(0,i;Q) \xrightarrow{\beta_i} G(i,Q)$$

Definition 0.4 *Define the elementary classes*

$$y_{i,j} := (\beta_i)_*(\alpha_i)^*(l_j) \in \mathrm{CH}^{\dim(Q)-i-j}(G(i,Q)|_{\overline{k}})/2.$$

$EDI(Q)$ measures which of $y_{i,j}$ are defined over k.

Our elementary classes are numbered by $0 \leqslant i,j \leqslant d$, so $EDI(Q)$ can be visualized as $d \times d$ square, where integral node is marked iff the respective class $y_{i,j}$ is defined over k.

$$\begin{array}{c|cccccc}
 & \bullet & \bullet & \circ & \cdots & \circ & \bullet \\
 & \circ & \bullet & \circ & \cdots & \circ & \circ \\
 & \circ & \circ & \circ & \cdots & \circ & \circ \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\uparrow & \circ & \circ & \circ & \cdots & \circ & \circ \\
i & \circ & \circ & \circ & \cdots & \circ & \circ \\
\hline
 & \xrightarrow[j]{} & & & & &
\end{array}$$

Here each row corresponds to a particular Grassmannian, and codimension decreases up and right. SW corner is marked $\Leftrightarrow Q$ is isotropic; SE corner is marked $\Leftrightarrow$ it is completely split.

Examples:

1) q-generic $(\langle a_1, \ldots, a_n\rangle/k = F(a_1, \ldots, a_n))$. Then $EDI(Q)$ is empty.

2) q-completely split $\Rightarrow$ everything is marked.

3) q_α is (anisotropic) n-fold Pfister form. Then the marked points will be exactly those which live strictly above the main (NW-SE) diagonal. In the case of $n = 3$ we get:

$$\begin{array}{cccc}
\circ & \bullet & \bullet & \bullet \\
\circ & \circ & \bullet & \bullet \\
\circ & \circ & \circ & \bullet \\
\circ & \circ & \circ & \circ
\end{array}$$

4) The $EDI(Q)$ for the 10-dimensional excellent form looks as:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \circ \\
\bullet & \bullet & \circ & \bullet & \circ \\
\bullet & \bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}$$

5) Let $q = \langle a, b, -ab, -c, -d, cd\rangle$ be an anisotopic Albert form. Then $EDI(Q)$ is

$$\begin{array}{ccc} \circ & \circ & \bullet \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{array}$$

The serious constraint on such marking is provided by the following:

<u>Rule</u>: $\begin{tikzcd} \bullet & \bullet \\ & \bullet \arrow[ul] \arrow[u] \end{tikzcd}$

Usually, one can not reconstruct GDI from EDI, but for $i = d$ these two invariants carry the same information due to the following result:

Theorem 0.5 *$GDI(Q, d)$ is generated as a ring by elementary classes contained in it.*

So, instead of studying the subrings of the ring of rank 2^{d+1} it is sufficient to study the subset of the set of $(d+1)$ elements $(0, 1, \ldots, d)$, where $j \leftrightarrow y_{d,j}$. Moreover, the action of the Steenrod operations on $GDI(Q, d)$ preserves the elementary classes, and so, provides the action on $EDI(Q, d)$. Hypothetically, the restrictions coming from this action (j-defined, $\binom{j}{r}$-odd $\Rightarrow$ $(j + r)$-defined) are the only restrictions on the possible subsets.

For other Grasmannians nothing of this sort is true. In particular, the elementary classes do not determine GDI, and the rigidity structure on GDI should involve all the Grassmannians simultaneously (in the contrast to the last Grassmannian being "self-sufficient").

It is an interesting task to translate EDI into the classical quadratic form theory language. Here the dots living below the auxiliary (SW-NE) diagonal are better understood. For such classes ($i \leqslant j + 1$) the k-rationality can be hypothetically expressed in terms of *dimensions of B.Kahn*. This important discrete inavriant of quadrics is defined as follows:

Definition 0.6

$$\dim_{I^n}(q) = \min(\dim(p) \,|\, q \perp -p \in I^n).$$

This invariant measures how far is our form from the given power of the fundamental ideal of even-dimensional forms.

Conjecture 0.7 *For $i \leqslant j+1$, the following conditions are equivalent:*

$$y_{i,j} \text{ is } k-\text{rational} \Leftrightarrow \dim_{I^r}(q) \leqslant c,$$

where $c = \dim(Q) - 2j$, *and* $r = [log_2(\dim(Q) - i - j + 1)] + 1$.

Notice, that the dimensions of B.Kahn one encounters here are all in the *stable range* $< 2^{n-1}$ (for such dimensions the closest point in I^n (and the form p above) is unique - follows from the Arason-Pfister Hauptsatz, claiming that the dimensions of anisotropic forms in I^n are either 0, or $\geqslant 2^n$). It is expected that *unstable dimensions of B.Kahn* should appear when one considers invariant similar to GDI, but with $\mathrm{CH}^*/2$ substituted by the ring of Algebraic Cobordism Ω^* (see the next lecture).

The other half of $EDI(Q)$ ($i > j+1$) is substantially less understood, and in the known examples the description here involves invariants similar to the Kahn's dimensions, but of more complicated nature.