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Root systems and Chevalley groups

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May 16: 3. Root-Systems, Chevalley groups

<u>Root system:</u> $V/\mathbb{R}$ finite dim vector space with pos scalar product

$\Sigma \subset V \setminus \{0\}$ is a <u>root system</u> if

i) Σ finite, $-\Sigma = \Sigma$

ii) for $\alpha \in \Sigma$, $s_\alpha : V \to V$, $s_\alpha(v) := v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha$ Reflection on $H_\alpha = \mathbb{R}\alpha^\perp \subseteq V$

we have $s_\alpha(\Sigma) = \Sigma$

iii) for $\alpha, \beta \in \Sigma$, $n_{\beta,\alpha} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ ("Cartan integers")

$W(\Sigma) = \langle s_\alpha \mid \alpha \in \Sigma \rangle$ = "Weyl group of Σ"

root system

Σ is <u>reducible</u> if $V = V' + V''$, $\Sigma = (\Sigma \cap V') \cup (\Sigma \cap V'')$
$V' \neq 0 \neq V''$

otherwise <u>irreducible</u>

Σ is <u>reduced</u> if $\mathbb{R}\alpha \cap \Sigma = \{\pm\alpha\}$ $\forall \alpha \in \Sigma$

(in general: $\mathbb{R}\alpha \cap \Sigma \subseteq \{\pm\alpha, \pm\frac{\alpha}{2}, \pm 2\alpha\}$)

A <u>Weyl chamber</u> C is a connected component of $V \setminus \bigcup_{\alpha\in\Sigma} H_\alpha$

$W(\Sigma)$ acts simply transitive on all Weyl chambers

Each C defines an ordering of roots:

$\alpha > 0$ if $(\alpha, v) > 0$ for every $v \in C$

$\alpha \in \Sigma$ is <u>simple</u> (Dynkin) (relative to an ordering)

if it is <u>not</u> the sum of 2 positive roots

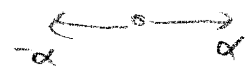
Every $\alpha \in \Sigma$ is an integral sum of simple roots with coeff of same sign

l = dim V = # simple roots

Dynkin diagram

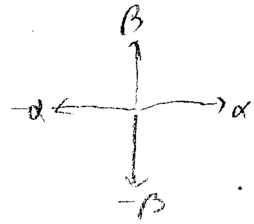
Reduced root systems of rank ≤ 2

rank 1:



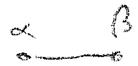
$A_1 \quad W(A_1) = \mathbb{Z}_2$

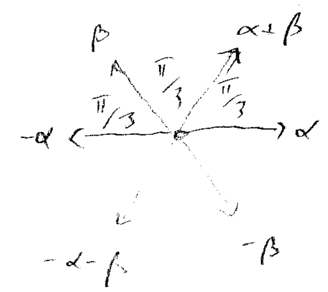
rank 2:



reducible: $A_1 \times A_1 \quad n_{\beta,\alpha} = 0$

$W(A_1 \times A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$





A_2

$n_{\beta\alpha} = -1$

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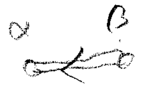
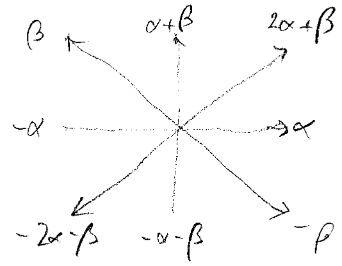
$W = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ (s_\alpha s_\beta)^3 = 1 \rangle$





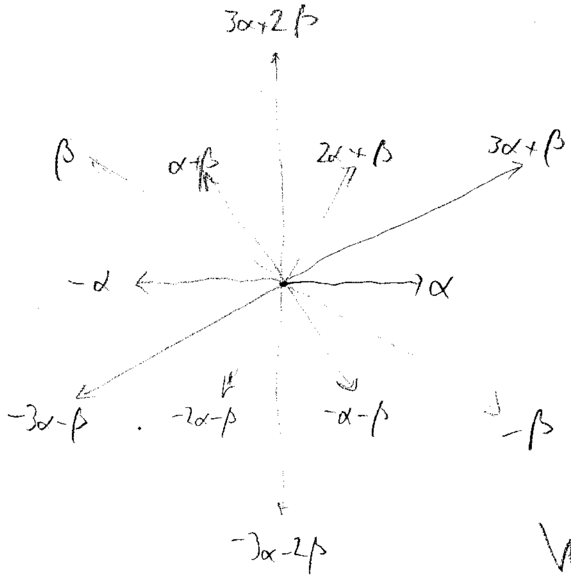
$B_2 = C_2$

$n_{\beta\alpha} = -2$

$n_{\alpha\beta} = -1$

$n_{\alpha+\beta\,\beta} = 0$ etc.

$W = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^4 = 1 \rangle$



G_2

$n_{\beta\alpha} = -3$

$n_{\alpha\beta} = -1$

$n_{\beta+\alpha\,\alpha} = -1$

$n_{2\beta+3\alpha\,\alpha} = 0$ etc.

$W = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^6 =$

$$n_{\beta\alpha}\, n_{\alpha\beta} = 4\,\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$$

Roots of a semisimple G w.r. to a torus $S \subseteq G$

G operates on its Lie alg $\mathfrak{g}$: $Ad_{\mathfrak{g}}: G \to Aut\,\mathfrak{g}$

S contains (only) semi-simple elements $\Rightarrow$ $Ad_{\mathfrak{g}}(S)$ diagon.

$\Rightarrow \mathfrak{g} = \mathfrak{g}_0^S \oplus \bigoplus_\alpha \mathfrak{g}_\alpha^S$, $\mathfrak{g}_\alpha^S = \{x \in \mathfrak{g} \mid Ad_{\mathfrak{g}}\, s(x) = \alpha(s)x\}$

for some charact. $\alpha \in X(S) = Hom(S, \mathbb{G}_m)$

If $T \subset G$ is a split maximal torus

then $\phi(G) = \phi(G,T)$ is "the" set of roots of G

(Uniqueness because max tori are conjugate!)

<u>The Dynkin diagram of G:</u>

$k = \bar{k}$, $T \subset G$ max torus $: T = \prod \mathbb{G}_m$; $N = Norm_G T \subset G$

$X = X(T) = Hom(T, \mathbb{G}_m) =$ free $\mathbb{Z}$-module of rank $= \dim T$

choose scalar product $(\,,\,)$ in $X \otimes \mathbb{R}$, invariant under $W := N$

Then the set Σ of all roots of G w.r. to T is a root system

Choose an ordering (via Weyl chamber)

Choose a set $\Delta \subseteq \Sigma$ of simple roots

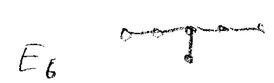
Clearly: $\#\Delta = \dim X \otimes \mathbb{R} = \dim T =:$ rank of G / of Δ

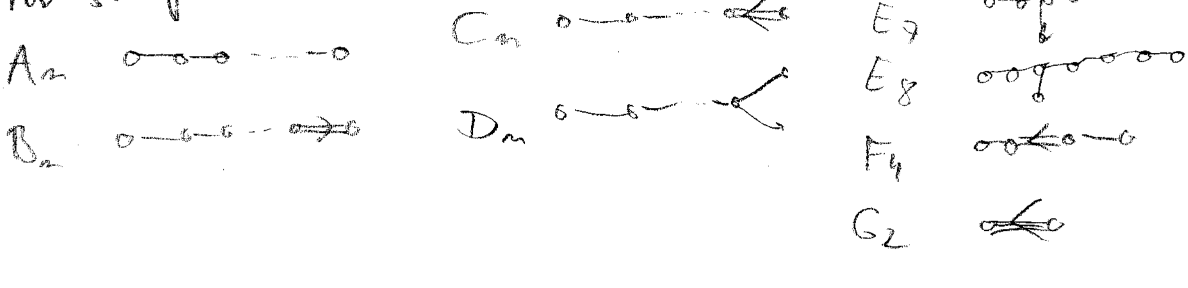
Each pair $\alpha, \beta \in \Delta$ is a sub-root system of rank ≤ 2.

Dynkin diagram $\mathcal{D}$ of G = graph with vertices in Δ

each pair is joined as on page 3.2

For simple G:





Isogenies:

$\varphi : H \longrightarrow G$ is an isogeny

if $\ker\varphi$ finite and φ surjective

an isogeny is central if $\ker\varphi \subset$ center H

G, G' are (strictly) isogeneous

if $\exists$ H and two (central) isogenies $H \to G$, $H \to G'$

(transitive relation)

Ex: i) SL_n, PGL_n are strictly isogeneous

ii) $Spin_n$, SO_n, PSO_n —— " ——

Main theorem on semisimple groups ($k = \bar{k}$)

A semisimple group G is characterized, up to strict isogeny by its Dynkin diagram.

It is almost simple if and only if the D.d. is connected.

Any s.s. group is strictly isogeneous to a direct product of simple groups whose Dd are the connected comp. of the Dd of G. (Proof: Killing 1888, Math Ann.,

Chevalley groups: k arbitrary:

For any Dynkin diagram $\mathcal{D}$, there exists a semisimple k-group G over k (with max Torus $T = \prod G_m$) such that $\mathcal{D}$ is the Dynkin diagram of G w.r. to T

Remark: G exists even over $\mathbb{Z}$ (cf. Chevalley 1959)

Def: A semisimple k-group with a split maximal k-Torus is called a Chevalley group

Structural properties of Chevalley groups:

k arbitrary, G = Chevalley group $/k$, T = max split torus

Σ = set of roots for G w.r.t. T.

Thm: For $\alpha \in \Sigma$ ex $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha \subseteq G$ (unique closed k-subgroup

such that $t\, u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x)$ $(t \in T, x \in k)$ if $k = \bar{k}$

$G(k)$ is generated by $T(k)$ and all $U_\alpha(k)$ $(\alpha \in \Sigma$ without

For every ordering of Σ there is exactly one Borel group

$B \supset T$ of G such that $\alpha > 0 \Leftrightarrow U_\alpha \subset B$, and

$B = T \cdot \prod_{\alpha > 0} U_\alpha$; $R_u(B) = \prod_{\alpha > 0} U_\alpha$

The subgroup $\langle U_{-\alpha}, U_\alpha \rangle$ is isogenous to SL_2 for every $\alpha \in \Sigma$

Bruhat-decomposition:

Let $N = Norm_G T$, then $W = N/T = \text{Weyl}(\Sigma)$

and

$$G = BWB = BNB = \bigcup_{\substack{w \in W \\ \text{mod } T}} BwB \quad \text{(disjoint)}$$

Parabolic subgroups:

There is a 1:1-correspondence of parabolic subgroups $P_\Theta > B$ and the subsets $\Theta \subset \Delta =$ simple roots in Σ w.r. to B.

$$P_\Theta = \langle T, U_\alpha (\alpha \in \Delta), U_{-\alpha} (\alpha \in \Theta) \rangle$$

i.e., $B = P_\emptyset$
$G = P_\Delta$

Let $W_\Theta = \langle s_\alpha \in W \mid \alpha \in \Theta \rangle$

$$\Rightarrow P_\Theta = \bigcup_{w \in W_\Theta} BwB \quad (\text{Bruhat dec.})$$

$$R_u(P_\Theta) = \langle U_\alpha \mid \alpha > 0, \alpha \notin \sum_{\gamma \in \Theta} \mathbb{R}\gamma \rangle$$

$$P_\Theta = L_\Theta \ltimes R_u(P_\Theta), \quad L_\Theta = Z_G(S_\Theta)$$

where $S_\Theta = \bigcap_{\alpha \in \Theta} (\operatorname{Ker} \alpha)^0$

"Levi decomposition":

L_Θ reductive, connected

$B_\Theta = B \cap L_\Theta$

Remark: $(B, N, \dots)$ fulfills the axioms of "BN-pairs
= Tits systems
= combinatorial foundation of "buildings