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## Beilinson's vision, partially fulfilled

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## 6. Beilinson's vision partially fulfilled

6.1. **Motivation.** In this lecture, we will discuss Alexander Beilinson's vision of what algebraic $K$-theory should be for smooth varieties over a field $k$. In particular, we will provide some account of progress towards the solution of these conjectures. Essentially, Beilinson conjectures that algebraic $K$-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type using "motivic complexes" $\mathbf{Z}(n)$ which satisfy various good properties and whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological $K$-theory.

Although our goal is to describe conjectures which would begin to "explain" algebraic $K$-theory, let me start by mentioning one (of many) reasons why algebraic $K$-theory is so interesting to algebraic geometers (and algebraic number theorists). It has been known for some time that there can not be an algebraic theory whose values on complex algebraic varieties is integral (or even rational) singular homology of the associated analytic space. Indeed, Jean-Pierre Serre observed that this is not possible even for smooth projective algebraic curves because some such curves have automorphism groups which do not admit a representation which would be implied by functoriality. On the other hand, algebraic $K$-theory is in some sense integral – we define it without inverting residue characteristics or considering only mod-$n$ coefficients. Thus, if we can formulate a sensible Atiyah-Hirzebruch type spectral sequence converging to algebraic $K$-theory, then the $E_2$-term offers an algebraic formulation of integral cohomology.

Before we launch into a discussion of Beilinson's Conjectures, let us recall two results relating algebraic cycles and algebraic K-theory which precede these conjectures.

The first is the theorem of Grothendieck mentioned earlier relating algebraic $K_0(X)$ to the Chow ring of algebraic cycles modulo algebraic equivalence.

**Theorem 6.1.** *If $X$ is a smooth variety over a field $k$, then the Chern character determines an isomorphism*

$$ch : K_0(X) \otimes \mathbb{Q} \ \simeq \ CH^*(X) \otimes \mathbb{Q}.$$

The second is *Bloch's formula* proved in degree 2 by Bloch and in general by Quillen.

**Theorem 6.2.** *Let $X$ be a smooth variety over a field and let $\underline{K}_i$ denote the Zariski sheaf associated to the presheaf $U \mapsto K_i(U)$ for an open subset $U \subset X$. Then there is a convergent spectral sequence of the form*

$$E_2^{p,q} \ = \ H^p_{Zar}(X, \underline{K}_q) \ \Rightarrow K_{q-p}(X).$$

6.2. **Statement of conjectures.** We now state Beilinson's conjectures and use these conjectures as a framework to discuss much interesting mathematics. It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic $K$-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored.

**Conjecture 6.3.** *(Beilinson's Conjectures) For each $n \geq 0$ there should be complexes $\mathbb{Z}(n), n \geq 0$ of sheaves on the Zariski site of smooth quasi-projective varieties over a field $k$, $(Sm/k)_{Zar}$ which satisfy the following:*

(1) $\mathbb{Z}(0) = \mathbb{Z}, \quad \mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$.
(2) $H^n(\operatorname{Spec} F, \mathbb{Z}(n)) = K_n^{Milnor}(F)$ *for any field $F$ finitely generated over $k$.*
(3) $H^{2n}(X, \mathbb{Z}(n)) = CH^n(X)$ *whenever $X$ is smooth over $k$.*
(4) *Vanishing Conjecture: $\mathbb{Z}(n)$ is acyclic outside of $[0, n]$:*

$$H^p(X, \mathbb{Z}(n)) \;=\; 0, \quad p < 0.$$

(5) *Motivic spectral sequences for $X$ smooth over $k$:*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}/\ell(-q)) \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell), \quad if\ 1/\ell \in k.$$

(6) *Beilinson-Lichtenbaum Conjecture:*

$$\mathbb{Z}(n) \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq n} \mathbf{R}\pi_* \mu_\ell^{\otimes n}, \quad if\ 1/\ell \in k$$

*where $\pi$ : etale site $\to$ Zariski site is the natural "forgetful continuous map" and $\tau_{\leq} n$ indicates truncation.*

(7) $H^i(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \simeq K_{2n-i}(X)_{\mathbb{Q}}^{(n)}$.

In other words, Beilinson conjectures that there should be a bigraded *motivic cohomology* groups $H^p(X, \mathbb{Z}(q))$ computed as the Zariski cohomology of *motivic complexes* $\mathbb{Z}(q)$ of sheaves which satisfy good properties and are related to algebraic K-theory as singular cohomology is related to topological $K$-theory.

6.3. **Status of Conjectures.** Bloch's higher Chow groups $CH^q(X, n)$ serve as motivic cohomology groups which are known to satisfy most of the conjectures, where the correspondence of indexing is as follows:

$$CH^q(X, n) \;\simeq\; H^{2q-n}(X, \mathbb{Z}(q)). \tag{1}$$

Furthermore, Suslin and Voevodsky have formulated complexes $\mathbb{Z}(q),\ q \geq 0$ and Voevodsky has proved that the (hyper-)cohomology groups of these complexes satisfy the relationship to Bloch's higher Chow groups as in (1).

Presumably, these constructions will be discussed in detail in the lectures of Professor Levine. For completeness, I sketch the definitions. Recall that the standard (algebro-geometric) $n$-simplex $\Delta^n$ over a field $F$ (which we leave implicit) is given by $\operatorname{Spec} F[t_0, \ldots, t_n]/\Sigma_i t_i = 1$.

**Definition 6.4.** Let $X$ be a quasi-projective variety over a field. For any $q, n \geq 0$, we define $z^q(X, n)$ to be the free abelian group on the set of cycles $W \subset X \times \Delta^n$ of codimension $q$ which meet all faces $X \times \Delta^i \subset X \times \Delta^n$ properly. This admits the structure of a simplicial abelian group and thus a chain complex with boundary maps given by restrictions to (codimension 1) faces.

The *Bloch higher Chow group* $CH^q(X,n)$ is defined by

$$CH^q(X,n) \ = H^{2q-n}(z^q(X,*)).$$

The values of Bloch's higher Chow groups are "correct", but they are not given as (hyper)-cohomology of complexes of sheaves and they are so directly defined that abstract properties for them are difficult to prove. The Suslin-Voevodsky motivic cohomology groups fit in a good formalism as envisioned by Beilinson and agree with Bloch's higher Chow groups as verified by Voevodsky.

**Definition 6.5.** Let $X$ be a quasi-projective variety over a field. For any $q \geq 0$, we define the complex of sheaves in the cdh topology (the Zariski topology suffices if $X$ is smooth over a field of characteristic 0)

$$\mathbb{Z}(q) \ = \ \underline{C}_*(c_{equi}(\mathbb{P}^n,0)/c_{equi}(\mathbb{P}^{n-1},0))[-2n]$$

defined as the shift $2n$ steps to the right of the complex of sheaves whose value on a Zariski open subset $U \ \subset \ X$ is the complex

$$j \ \mapsto \ c_{equi}(\mathbb{P}^n,0)(\Delta^j)/c_{equi}(\mathbb{P}^{n-1})(U \times \Delta^j)$$

where $c_{equi}(\mathbb{P}^n,0)(U \times \Delta^j)$ is the free abelian group on the cycles on $\mathbb{P}^n \times U \times \Delta^j$ which are equi-dimensional of relative dimension 0 over $U \times \Delta^j$.

Conjecture (1) is essentially a normalization, or it specifies what $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ must be. Bloch verified Conjecture 2 (essentially, a result of Suslin), Conjecture 3, and Conjecture 7 (the latter with help from Levine) for his higher Chow groups. Bloch and Lichtenbaum produced a motivic spectral sequence for $X = \operatorname{Spec} k$; this was generalized to a verification of the full Conjecture (5) by Friedlander and Suslin, and later proofs were given by Levine and then Suslin following work of Grayson.

The Beilinson-Lichtenbaum conjecture in some sense "identifies" mod-$\ell$ motivic cohomology in terms of etale cohomology. Suslin and Voevodsky proved that this Conjecture (6) follows from the following:

**Conjecture 6.6.** *(Bloch-Kato Conjecture) For fields $F$ finitely generated over $k$,*

$$K_n^{Milnor} \otimes \mathbb{Z}/\ell \ \simeq H^n_{et}(\operatorname{Spec} F, \mu_\ell^{\otimes n}).$$

*In particular, the Galois cohomology of the field $F$ is generated mulitplicatively by classes in degree 1.*

For $\ell = 2$, the Bloch-Kato Conjecture is a form of *Milnor's Conjecture* which has been proved by Voevodsky. For $\ell > 2$, a proof of Bloch-Kato Conjecture has apparently been given by Rost and Voevodsky, although not all details have been made available. This conjecture will be the main focus of Professor Weibel's lectures.

This leaves Conjecture (4), one aspect of this is the following Vanishing Conjecture due to Beilinson and Soulé.

**Conjecture 6.7.** *For fields $F$,*

$$K_p(F)^{(q)}_{\mathbb{Q}} \;=\; 0, \quad 2q \leq p, p > 0.$$

*Reindexing according to Conjecture (7), this becomes*

$$H^i(\operatorname{Spec} F, \mathbb{Z}(q)) \;=\; 0, \quad i \leq 0, q \neq 0.$$

The status of this Conjecture (4), and in particular the Beilinson-Soué vanishing conjecture, is up in the air. Experts are not at all convinced that this conjecture should be true for a general field $F$. It is known to be true for a number field.

6.4. **The Meaning of the Conjectures.** Let us begin by looking a bit more closely at the statement

$$\mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$$

of Conjecture (1).

**Convention** If $C^*$ is a cochain complex (i.e., the differential increases degree by 1, $d : C^i \to C^{i+1}$), we define the chain complex $C^*[n]$ for any $n \in \mathbb{Z}$ as the shift of $C^*$ "$n$ places to the right". In other words, $(C^*[n])^j \;=\; C^{*-j}$.

In particular, $\mathcal{O}^*[-1]$ is the complex (of Zariski sheaves) with only one non-zero term, the sheaf $\mathcal{O}^*$ of units, placed in degree -1 (i.e., shifted 1 place to the left). In particular,

$$H^*_{Zar}(X, \mathcal{O}^*[-1]) \;=\; H^{*-1}_{Zar}(X, \mathcal{O}^*);$$

thus,

$$Pic(X) \;=\; H^1_{Zar}(X, \mathcal{O}^*_X) \;=\; H^2(X, \mathbb{Z}(1)).$$

This last equality is a special case of item (3).

Perhaps it would be useful to be explicit about what we mean by the cohomology of a complex $C^*$ of Zariski sheaves on $X$. A quick way to define this is as follows: find a map of complexes $C^* \to I^*$ with each $I^j$ an injective object in the category of sheaves (an injective sheaf) such that the map on cohomology sheaves is an isomorphism; in other words, for each $j$, the map of presheaves

$$ker\{d : C^j \to C^{j+1}\}/im\{d : C^{j-1} \to C^j\}$$
$$\to \; ker\{d : I^j \to I^{j+1}\}/im\{d : I^{j-1} \to I^j\}$$

induces an isomorphism on associated sheaves

$$\mathcal{H}^j(C^*) \;\simeq\; \mathcal{H}^j(I^*)$$

for each $j$. A fundamental property of this cohomology is the existence of "hypercohomology spectral sequences"

$${}'E_1^{p,q} \;=\; H^p(X, C^q) \;\Rightarrow\; H^{p+q}(X, C^*)$$
$$E_2^{p,q} \;=\; H^q(X, \mathcal{H}^j(C^*)) \;\Rightarrow\; H^{p+q}(X, C^*)$$

Conjecture (2) helps to pin down motivic cohomology by specifying what the top dimensional motivic cohomology (thanks to Conjecture (4)) should be for a field. Since Milnor K-theory and algebraic K-theory of the field $k$ are different, this difference must be reflected in the other motivic cohomology groups of the field and tied together with the spectral sequence of Conjecture (5).

Conjecture (2) can be viewed as "arithmetic" for it deals with subtle invariants of the field $k$. Conjecture (3) is "geometric", stating that motivic cohomology reflects global geometric properties of $X$. Observe that since we are taking Zariski cohomology, $H^n(\operatorname{Spec} k, -) = 0$ for $n > 0$ and this item simply says that $CH^0(\operatorname{Spec} k) = \mathbb{Z},\ CH^n(\operatorname{Spec} k) = 0, n > 0$.

Bloch has also proved that the spectral sequence of Conjecture (5) collapses after tensoring with $\mathbb{Q}$; indeed, Conjecture (7) proved by Bloch is a refinement of this "rational collapse". Conjectures (3) and (5) together with this collapsing gives Grothendieck's isomorphism $K_(X)\mathbb{Q} \simeq CH^*(X)$. By simply re-indexing, one can write the spectral sequence of Conjecture (5) in the more familiar "Atiyah-Hirzeburch manner"

$$E_2^{p,q} = H^p(X, \mathbf{Z}(-q/2)) \Rightarrow K_{-p-q}(X)$$

where $\mathbf{Z}(-q/2) = 0$ if $q$ is not an even non-positive integer and $\mathbf{Z}(-q/2) = \mathbf{Z}(i)$ is $-q = 2i \geq 0$.

Let me try to "draw" this spectral sequence, using the notation

$$K_{q-i}^{(q)} \ \equiv \ H^i(X, \mathbb{Z}(q)$$

as in Conjecture (7). This is copied from a picture drawn years ago by Dan Grayson.

$$\begin{array}{ccccccccc}
 & & & & \mathbb{Z} & & & & \\
 & & & 0 & \mathcal{O}^* & Pic(X) & & & \\
 & & 0? & K_3^{(2)} & K_2^{(2)} & \bullet & CH^2(X) & & \\
 & 0? & K_5^{(3)} & K_4^{(3)} & K_3^{(3)} & \bullet & \bullet & CH^3(X) & \\
0? & K_7^{(4)} & K_6^{(4)} & K_5^{(4)} & K_4^{(4)} & \bullet & \bullet & \bullet & CH^4(X)
\end{array}$$

In this picture, the associated graded of $K_0$ is given by the right-most diagonal, then $gr(K_1)$ by the next diagonal to the left, etc. The top horizontal row is the "weight 0" part of $K_*$, the next row down is the "weight 1" part of $K_*$, etc. There is conjectured vanishing at and to the left of the positions with 0? in the picture – i.e., to the left.

6.5. **Etale cohomology.** Our final task is to introduce the etale topology and attempt to give some understanding why Conjecture (6) of the Beilinson Conjectures comparing mod-$\ell$ motivic cohomology with mod-$\ell$ etale cohomology makes motivic cohomology more understandable.

Grothendieck had the insight to realize that one could formulate sheaves and sheaf cohomology in a setting more general than that of topological spaces. What is essential in sheaf theory is the notion of a covering, but such a covering need not consist of open subsets.

**Definition 6.8.** A (Grothendieck) site is the data of a category $\mathcal{C}/X$ of schemes over a given scheme $X$ which is closed under fiber products and a distinguished class of morphisms (e.g., Zariski open embeddings; or etale morphisms) closed under composition, base change and including all isomorphisms. A covering of an object $Y \in \mathcal{C}/X$ for this site is a family of distinguished morphisms $\{g_i : U_i \to Y\}$ with the property that $Y = \cup_i g_i(U_i)$.

The data of the site $\mathcal{C}/X$ together with its associated family of coverings is called a Grothendieck topology on $X$.

**Example 6.9.** Recall that a map $f : U \to X$ of schemes is said to be *etale* if it is flat, unramified, and locally of finite type. Thus, open immersions and covering space maps are examples of etale morphisms. If $f : U \to X$ is etale, then for each point $u \in U$ there exist affine open neighborhoods $SpecA \subset U$ of $u$ and $SpecR \subset X$ of $f(u)$ so that $A$ is isomorphic to $(R[t]/g(t))_h$ for some monic polynomial $g(t)$ and some $h$ so that $g'(t) \in (R[t]/g(t))_h$ is invertible.

The (small) etale site $X_{et}$ has objects which are etale morphisms $Y \to X$ and coverings $\{U_i \to Y\}$ consist of families of etale maps the union of whose images equals $Y$. The big etale site $X_{ET}$ has objects $Y \to X$ which are locally of finite type over $X$ and coverings $\{U_i \to Y\}$ defined as for $X_{et}$ consisting of families of etale maps the union of whose images equals $Y$. If $k$ is a field, we shall also consider the site $(Sm/k)_{et}$ which is the full subcategory of $(\operatorname{Spec} k)_{ET}$ consisting of smooth, quasi-projective varieties $Y$ over $k$.

An instructive example is that of $X = SpecF$ for some field $F$. Then an etale map $Y \to X$ with $Y$ connected is of the form $SpecE \to SpecF$, where $E/F$ is a finite separable field extension.

**Definition 6.10.** A presheaf sets (respectively, groups, abelian groups, rings, etc) on a site $\mathcal{C}/X$ is a contravariant functor from $\mathcal{C}/X$ to $(sets)$ (resp., to groups, abelian groups, rings, etc). A presheaf $P : (\mathcal{C}/X)^{op} \to (sets)$ is said to be a sheaf if for every covering $\{U_i \to Y\}$ in $\mathcal{C}/X$ the following sequence is exact:

$$P(Y) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_X U_j).$$

(Similarly, for presheaves of groups, abelian presheaves, etc.) In other words, if for every $Y$, the data of a section $s \in P(Y)$ is equivalent to the data of sections $s_i \in P(U_i)$ which are compatible in the sense that the restrictions of $s_i, s_j$ to $U_i \times_X U_j$ are equal.

The category of abelian sheaves on a Grothendieck site $\mathcal{C}/X$ is an abelian category with enough injectives, so that we can define sheaf cohomology in the usual way. If $F : \mathcal{C}/X)^{op} \to (Ab)$ is an abelian sheaf, then we define

$$H^i(X_{\mathcal{C}/X}, F) = R^i\Gamma(X, F).$$

Etale cohomology has various important properties. We mention two in the following theorem.

**Theorem 6.11.** *Let $X$ be a quasi-projective, complex variety. Then the etale cohomology of $X$ with coefficients in (constant) sheaf $\mathbb{Z}/n$, $H^*(X_{et}, \mathbb{Z}/n)$, is naturally isomorphic to the singular cohomology of $X^{an}$,*

$$H^*(X_{et}, \mathbb{Z}/n) \;\simeq\; H^*_{sing}(X^{an}, \mathbb{Z}/n).$$

*Let $X = Speck$, the spectrum of a field. Then an abelian sheaf on $X$ for the etale topology is in natural 1-1 correspondence with a (continuous) Galois module for the Galois group $Gal(\overline{k}/k)$. Moreover, the etale cohomology of $X$ with coefficients in such a sheaf $F$ is equivalent to the Galois cohomology of the associated Galois module,*

$$H^*(k_{et}, F) \;\simeq\; H^*(Gal(\overline{F}/F), F(k)).$$

From the point of view of sheaf theory, the essence of a continuous map $g : S \to T$ of topological spaces is a mapping from the category of open subsets of $T$ to the open subsets of $S$. In the context of Grothendieck topologies, we consider a map of sites $g : \mathcal{C}/X \to \mathcal{D}/Y$, a functor from $\mathcal{C}/Y$ to $cC/X$ which takes distinguished morphisms to distinguished morphisms. In particular, For example, Conjecture (6) of Beilinson's Conjectures involves the map of sites

$$\pi : X_{et} \to X_{Zar}, \quad (U \subset X) \mapsto U \to X.$$

Such a map of sites induces a map on sheaf cohomology: if $F : (\mathcal{D}/Y)^{op} \to (Ab)$ is an abelian sheaf on $\mathcal{C}/Y$, then we obtain a map

$$H^*(Y_{\mathcal{D}/Y}, F) \to H^*(X_{\mathcal{C}/X}, g^*F).$$

6.6. **Voevodsky's sites.** We briefly mention two Grothendieck sites introduced by Voevodsky which are central to his approach to motivic cohomology.

**Definition 6.12.** The Nisnevich site on smooth quasi-projective varieties over a field $k$, $(Sm/k)_{Nis}$, is determined by specifying that a covering $\{U_i \to U\}$ of some $U \in (Sm/k)$ is an etale covering with the property that for each point $x \in U$ there exists some $i$ and some point $\tilde{u} \in U_i$ such that the induced map on residue fields $k(u) \to k(\tilde{u})$ is an isomorphism.

**Definition 6.13.** The cdh (or completely decomposed, homotopy) site on smooth quasi-projective varieties over a field $k$, $(Sm/k)_{cdh}$, is determined as the site whose coverings of a smooth variety $X$ are generated by Nisnevich coverings of $X$ and coverings $\{Y \to X,\ X' \ \to \ X\}$ consisting of a closed immersion $i : Y \to X$ and a proper map $g : X' \to X$ with the property that the restriction of $g$ to $g^{-1}(X \backslash i(Y))$ is an isomorphism.