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K-theoretic results for Chevalley groups

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May 17: 4. K-theoretic results related to Chevalley groups

(simply connected (i.e., having no proper algebraic central ext.)

Let G be a Chevalley group$/k$, T = max split torus

Σ = set of roots for G w.r.t. T.

For each $\alpha \in \Sigma$ define

$$w_\alpha(x) = u_\alpha(x)\, u_{-\alpha}(-x^{-1})\, u_\alpha(x) \qquad x \in k^*$$

$$h_\alpha(x) = w_\alpha(x)\, w_\alpha(-1)$$

Steinberg relations:

(A) $u_\alpha(x+y) = u_\alpha(x)\, u_\alpha(y) \qquad (\alpha\in\Sigma,\ x,y\in k)$

(B) $[u_\alpha(x), u_\beta(y)] = \prod_{\substack{i,j>0\\ i\alpha+j\beta\in\Sigma}} u_{i\alpha+j\beta}(c_{ij\alpha\beta}\, x^i y^j) \qquad (\alpha,\beta\in\Sigma,\ \alpha+\beta\neq 0)$

(Product taken in some lexicographical order, $c_{ij\alpha\beta}\in\mathbb{Z}$ independent of x,y, only dep. on Dynkin(G) and ...

(B') $w_\alpha(t)\, u_\alpha(x)\, w_\alpha(t)^{-1} = u_{-\alpha}(-t^{-2}x) \qquad (\alpha\in\Sigma,\ t\in k^*, x\in k)$

(C) $h_\alpha(xy) = h_\alpha(x)\, h_\alpha(y) \qquad (\alpha\in\Sigma,\ x,y\in k^*)$

Extension of Thm 1 to Chevalley groups

Thm 1' (Steinberg) ii) The group $\tilde{G}$ presented by (A), (B) in case rank $G \geq 2$ (resp. by (A), (B') in case rank $G=1$) is a central covering of $G(k)$.

i) The group presented by (A), (B), (C) (resp. (A), (B'), (C) if rank $G=1$) is the group $\hat{G}(k)$ of rational points of the simply connected covering $\hat{G}$ of G (hence strictly isogeneous to G).

(explain! see above)

$\tilde{G}$ is the universal central covering of G if $|k|>4$ for rank $G>1$ and $|k|\neq 4, 9$ for rank $G=1$.

Remarks 1.) The roots occuring on the right of (B) can be read off the two dimensional root systems, since $\{\alpha,\beta\}$ generate a subsystem of rank 2:

e.g., for $G = G_2$, one may have

$$[x_\alpha(u), x_\beta(v)] = x_{\alpha+\beta}(uv)x_{2\alpha+\beta}(-u^2v)\,x_{3\alpha+\beta}(-u^3v)\,x_{3\alpha+2\beta}(2u^3v^2)$$

and this is the longest product which might occur.

For groups of type different from G_2, at most two factors do occur on the right.

2.) Each relation involves only generators of some almost simple rank 2 subgroup (generated by the $u_\alpha(x)$, $u_\beta(y)$ involved) hence the theorem implies, that:

<u>$G(k)$ is an amalgamated product of its almost simple rank 2 subgroups</u>.

For $\alpha \in \Sigma$ define $c_\alpha(x,y) := h_\alpha(xy) h_\alpha(x)^{-1} h_\alpha(y)^{-1} \in \tilde{G}$

Thm 2[e] (Moore/Matsumoto): One has $c_\alpha(x,y) = c_{-\alpha}(y,x)$, and if rank $G \geq 2$ then, for long roots α, c_α is independent of α; its values $c.(x,y) := c_\alpha(x,y)$ generate the kernel of $\tilde{G} \to \hat{G}(k)$. The kernel is presented by:

$$\left.\begin{array}{ll} (S1) & c(x,y)\,c(xy,z) = c(x,yz)\,c(y,z) \\ (S2) & c(1,1) = 1, \quad c(x,y) = c(x^{-1}, y^{-1}) \\ (S3) & c(x,y) = c(x,(1-x)y) \qquad \text{if } x \neq 1 \end{array}\right\} \begin{array}{l} \text{rank } G = 1 \\ \text{or} \\ G \text{ symplectic} \end{array}$$

resp.

$$\left.\begin{array}{ll} (S^01) & c(x,yz) = c(x,y)\,c(x,z) \\ (S^02) & c(xy,z) = c(x,z)\,c(y,z) \\ (S^03) & c(x,1-x) = 1 \qquad x \neq 1 \end{array}\right\} G \text{ not symplectic}$$

In every case, the symbol $c^\#$ def. by

$$c^\#(x,y) = c(x,y^2)$$

fulfills $(S^01), (S^02), (S^03)$

Hence we have two groups:

$K_2^{sym}(k)$, defined by $(S1), (S2), (S3)$

$K_2(k)$, defined by $(S^01), (S^02), (S^03)$

And homomorphisms

$$K_2^{sym}(k) \twoheadrightarrow K_2(k) \quad \big(c(x,y) \mapsto c(x,y) \big)$$

$$K_2(k) \longrightarrow K_2^{sym}(k) \quad \big(c(x,y) \longmapsto c^2(x,y) \\ = c(x,y^2) \big)$$

<u>Suslin</u>: There is a hom $K_2^{sym}(k) \longrightarrow I^2(k)$

$$c(x,y) \longmapsto \langle\langle x,y \rangle\rangle$$

which maps the kernel of $K_2^{sym}(k) \longrightarrow K_2(k)$ isomorphically to $I^3(k)$.

Hence we obtain the following exact commut. diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & K_2(k)/c(k^*,-1) \arrow[r, "2"] \arrow[d] & K_2(k)^2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I^3(k) \arrow[r] \arrow[d, equal, "\wr"] & K_2^{sym}(k) \arrow[r] \arrow[d] & K_2(k) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I^3(k) \arrow[r] & I^2(k) \arrow[r] \arrow[d] & \overset{H^2(k)}{\mathrm{Br}_2(k)} \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Moreover, we find, since $K_2(2,k) = K_2^{sym}(k)$:

Exact:

$$1 \longrightarrow I^3(k) \longrightarrow K_2(2,k) \longrightarrow K_2(k) \longrightarrow 1$$