The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

SMR/1840-26

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Algebraic Cobordism. Landweber-Novikov and Steenrod operations. Symmetric operations.

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Lecture 5

Algebraic Cobordism. Landweber-Novikov and Steenrod operations. Symmetric operations.

Let k be a field of characteristic 0, and $(Sm.Q.-P.)/k$ be the category of smooth quasiprojective varieties over k.

The *generalized oriented cohomology theory* is a contravariant functor

$$(Sm.Q.-P.)/k \xrightarrow{A^*} \{\mathbf{Z}-\text{graded rings}\}$$

$$X \mapsto A^*(X)$$

$$(f : X \to Y) \mapsto (f^* : A^*(Y) \to A^*(X))$$

together with the *push-forward morphisms* $f_* : A^*(X) \to A^{*-d}(Y)$ for projective equidimensional maps $f : X \to Y$ of relative dimension d.

All these data should satisfy certain compatibility axioms.

Generalized oriented cohomology theory possesses Chern classes.

Chern classes

Let $\mathcal{L}/X$ be line bundle. Consider the zero section $j : X \hookrightarrow \mathcal{L}$. Then one can assign to $\mathcal{L}$ its first *Chern class* $c_1(\mathcal{L}) := j^* j_*(1_X^A) \in A^1(X)$.

Now, if $\mathcal{U}$ is some vector bundle on X, by the *projective bundle axiom* of the generalized oriented cohomology theory,

$$A^*(\mathbb{P}_X(\mathcal{U}^\vee)) = \oplus_{i=0}^{\dim(\mathcal{U})-1} \rho^i \cdot A^*(X),$$

where $\rho := c_1(\mathcal{O}(1))$, and $\mathcal{U}^\vee$ is the vector bundle dual to $\mathcal{U}$. In particular, there is the unique relation:

$$\rho^{\dim(\mathcal{U})} - \lambda_1 \cdot \rho^{\dim(\mathcal{U})-1} + \lambda_2 \cdot \rho^{\dim(\mathcal{U})-2} - \ldots (-1)^{\dim(\mathcal{U})} \lambda_{\dim(\mathcal{U})},$$

for certain $\lambda_i \in A^i(X)$. These coefficients are called the *Chern classes* of the bundle $\mathcal{U}$: $c_i(\mathcal{U}) := \lambda_i$ (we assume $c_0 = \lambda_0 = 1$). Denote as $c_\bullet(\mathcal{U})$ the *total Chern class* $\sum_{i\geqslant 0} c_i(\mathcal{U})$. These classes satisfy the *Cartan formula*: if $0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to \mathcal{U}_3 \to 0$ is a short exact sequence, then

$$c_\bullet(\mathcal{U}_1) \cdot c_\bullet(\mathcal{U}_3) = c_\bullet(\mathcal{U}_2).$$

Cartan formula permits to define Chern classes on the formal differences $\mathcal{V} - \mathcal{U}$ of vector bundles, that is, on K^0.

Examples (of theories):

1) CH^* - the *Chow groups*.

2) $K^0[\beta, \beta^{-1}]$ - the algebraic K^0 (it is convenient to add the formal invertible parameter β to it).

Among such theories there is the universal one Ω^* called *Algebraic Cobordism*. This theory was constructed by M.Levine and F.Morel (further simplified by M.Levine and R.Pandharipande).

$\Omega^*(X)$ is additively generated by the classes $[v : V \to X]$, where V is smooth and v is projective. One imposes certain relations:

1) *Elementary cobordism relations*

The classes $[v_0 : V_0 \to X]$ and $[v_1 : V_1 \to X]$ are *elementary cobordant*, if there exists projective map $w : W \to X \times \mathbb{P}^1$ from a smooth variety W, which is transversal to $X \times \{0\} \hookrightarrow X \times \mathbb{P}^1$ and $X \times \{1\} \hookrightarrow X \times \mathbb{P}^1$ and $w|_{X\times\{0\}} = v_0$, $w|_{X\times\{1\}} = v_1$.

We recall that the morphisms f, g from the Cartesian square

$$\begin{array}{ccc} B \times_A C & \xrightarrow{g'} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & A \end{array}$$

with A, B, C - smooth are called transversal, if the natural map of tangent bundles $(f')^*T_B \oplus (g')^*T_C \to (f \circ g')^*T_A$ is surjective. Then $B \times_A C$ is smooth, and the sequence

$$0 \to T_{B\times_A C} \to (f')^*T_B \oplus (g')^*T_C \to (f \circ g')^*T_A \to 0$$

is exact. The *transversal cartesian squares* behave especially well with respect to the pull-back and push-forward morphisms, and they are used in the definition of the generalized oriented cohomology theory.

In topology these would be all the relations, but in algebraic geometry one has to impose more.

2) *Double point relations* (following M.Levine - R.Pandharipande) Let $[w : W \to X \times \mathbb{P}^1]$ be such projective map that w is transversal to $X \times \{0\} \to X \times \mathbb{P}^1$, where $w|_{X\times\{0\}} = v_0 : V_0 \to X$, and $w^{-1}(X \times \{1\})$ consists of two smooth components $V_{1,a}$ and $V_{1,b}$ intersecting transversally on W at

$U = V_{1,a} \cap V_{1,b}$. Let $\mathcal{N} = \mathcal{N}_{U \subset V_{1,a}}$ be the normal bundle (it is easy to see that then $\mathcal{N}_{U \subset V_{1,b}} = \mathcal{N}^{-1}$). Then we impose the relation:

$$[v_0 : V_0 \to X] = [v_{1,a} : V_{1,a} \to X] + [v_{1,b} : V_{1,b} \to X] - [\mathbb{P}_U(\mathcal{N} \oplus \mathcal{O}) \to X].$$

Notice, that this relation is symmetric with respect to $a \leftrightarrow b$, since $\mathbb{P}_U(\mathcal{N} \oplus \mathcal{O})$ is isomorphic to $\mathbb{P}_U(\mathcal{O} \oplus \mathcal{N}^{-1})$.

One generates all the relations in Ω^* by applying the push-forward operation f_* with respect to all proper morphisms $f : X \to Y$ to the two types of relations above, where $f_*([v : V \to X]) := [f \circ v : V \to Y]$. As was mentioned, the resulting theory Ω^* is universal oriented generalized cohomology theory. The universality follows from the fact that oriented theories have push-forwards: the canonical map

$$\Omega^*(X) \to A^*(X)$$

is given by

$$[v : V \to X] \mapsto (v_A)_*(1_V^A) \in A^{codim(V \subset X)}(X).$$

Remark: It should be mentioned, that it is quite nontrivial to define the pull-back operations f^* on Ω^*. One can find details in the book of M.Levine and F.Morel.

Properties of Ω^*

(1) $\Omega^*(\mathrm{Spec}(k)) = MU^{2*}(pt) = \mathbb{L}$, where MU is the $\mathbb{C}$-oriented cobordism in topology, and $\mathbb{L}$ is the *Lazard ring* - the coefficient ring of the *universal formal group law.* In particular, we see that the result does not depend on k (it does not matter, if k is algebraically closed, or not).

formal group laws

(commutative, 1-dimensional) formal group law is given by the following data: $(R, F(x, y))$, where R is a coefficient ring (associative, commutative, unital), and $F(x, y) \in R[[x, y]]$ is a power series, satisfying:

(*i*) $F(x, 0) = x$, $F(0, y) = y$;

(*ii*) $F(x, y) = F(y, x)$ - commutativity;

(*iii*) $F(F(x, y), z) = F(x, F(y, z))$ - associativity.

From conditions (i) and (ii) it follows that $F(x,y) = x + y + \sum_{i,j\geqslant 1} a_{i,j}x^i y^j$, with $a_{i,j} = a_{j,i}$.

Examples:

1) *Additive group law*: $F_a(x,y) = x + y$ with R-any ring;

2) *multiplicative group law*: $F_m(x,y) = x + y - \beta \cdot xy$, where $\beta \in R$ is invertible.

Among the group laws there is the universal one $(R_U, F_U(x,y))$ such that there is $1-1$ correspondence

$$\{\,\text{f.g.laws}\,(R, F(x,y))\} \leftrightarrow \{\,\text{ring homomorphisms } R_U \xrightarrow{f_F} R\},$$

where $F(x,y) = f_F(F_U(x,y))$. Clearly, it is sufficient to take

$$R_U := \mathbf{Z}[a_{i,j},\ i,j \geqslant 1]/(assoc., comm.)$$

with the $F_U = x+y+\sum_{i,j\geqslant 1} a_{i,j}x^i y^j$. The coefficient ring R_U of the universal formal group law is called the *Lazard ring* $\mathbb{L}$. The following important result is due to Lazard:

Theorem 0.1

$$\mathbb{L} = \mathbf{Z}[z_1, z_2, \ldots],\ \textit{with } deg(z_l) = l,\ \textit{where } deg(a_{i,j}) = i + j - 1.$$

Projective bundle axiom implies that $A^*(\mathbb{P}^\infty) = A^*[[t]]$, where $A^* := A^*(\mathrm{Spec}(k))$, and $t = c_1(\mathcal{O}(1))$. Consider the *Segre embedding*

$$\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{Segre} \mathbb{P}^\infty.$$

It induces the pull-back homomorphism

$$A^*[[x,y]] \xleftarrow{Segre^*} A^*[[t]].$$

It is easy to check that the pair $(A^*, F_A(x,y))$, where $F_A(x,y) := (Segre)^*(t)$ will be a formal group law. Thus to each generalized cohomology theory one can assign the formal group law

$$A^*(X) \mapsto (A^*, F_A(x,y)).$$

Examples:

1) $\mathrm{CH}^* \mapsto (\mathbf{Z}, F_a(x,y))$;

2) $K^0[\beta, \beta^{-1}] \mapsto (\mathbf{Z}[\beta, \beta^{-1}], F_m(x,y))$.

Due to the result of M.Levine and F.Morel, the theories CH^* and $K^0[\beta, \beta^{-1}]$ are the universal ones among the *additive* and *multiplicative* theories, respectively.

It appears that the formal group law assigned to the Algebraic Cobordism theory Ω^* will be the universal one. That is, $F_\Omega(x,y) = F_U(x,y)$, and $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L}$. In particular, since $\Omega^*(\mathrm{Spec}(k))$ is additively generated by the classes of smooth projective varieties over k, the universal constants $a_{i,j}$ can be interpreted as $\mathbf{Z}$-linear combinations of such classes.

Examples:

1) $a_{1,1} = -[\mathbb{P}^1]$;

2) $a_{2,1} = [\mathbb{P}^1 \times \mathbb{P}^1] - [\mathbb{P}^2]$.

In general, $\dim(a_{i,j}) = i + j - 1$.

(2) We have canonical map of theories $pr : \Omega^* \to \mathrm{CH}^*$ given by

$$[v : V \to X] \mapsto v_*(1_V) \in \mathrm{CH}_{\dim(V)}(X).$$

There is the following important result of M.Levine and F.Morel:

Theorem 0.2

$$\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}^{\leqslant 0} \cdot \Omega^*(X).$$

Thus, CH^* can be computed out of Ω^*.

Remark: The topological counterpart of this statement, as well as the one with the motivic cohomology in place of the Chow groups are false.

Landweber-Novikov operations

Let $R(\sigma_1, \sigma_2, \ldots) \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$ be some polynomial, where we assign grading: $\deg(\sigma_i) = i$. Then one can define *Landweber-Novikov operation*

$$S^R_{L-N} : \Omega^*(X) \to \Omega^{*+\deg(R)}(X)$$

by the rule: $S^R_{L-N}([v : V \to X]) := v_*(R(c_1, c_2, \ldots))$, where $c_i = c_i(\mathcal{N}_v) \in \Omega^i(V)$, and $\mathcal{N}_v := -T_V + v^*T_X$ - the *virtual normal bundle.*

There is another parametrization of Landweber-Novikov operations - the one using *partitions*. Partition is the <u>nonordered</u> set of natural numbers $\overline{a} = (a_1, a_2, \ldots, a_m)$ with $|\overline{a}| = \sum_i a_i$. To each partition $\overline{a}$ one can assign the minimal symmetric polynomial, containing the monomial $\overline{b}^{\overline{a}} = \prod_i b_i^{a_i}$. This polynomial can be expressed in terms of *elementary symmetric polynomials* $\sigma_i(b_1, b_2, \ldots)$ on b_i's. Let $R^{\overline{a}}(\sigma_1, \sigma_2, \ldots)$ be the respective expression. Then one defines $S^{\overline{a}}_{L-N} : \Omega^*(X) \to \Omega^{*+|\overline{a}|}(X)$ as $S^{R^{\overline{a}}}_{L-N}$. Parametrized this way, the Landweber-Novikov operations can be easily organized into the *multiplicative operation*

$$S^{Tot}_{L-N} := \sum_{\overline{a}} \overline{b}^{\overline{a}} \cdot S^{\overline{a}}_{L-N} : \Omega^*(X) \to \Omega^*(X) \otimes_{\mathbf{Z}} \mathbf{Z}[b_1, b_2, \ldots].$$

Multiplicativity property means that

$$S^{Tot}_{L-N}(x \cdot y) = S^{Tot}_{L-N}(x) \cdot S^{Tot}_{L-N}(y).$$

Specializing b_i to some values in $\mathbb{L}$ one gets the multiplicative operations $\Omega^*(X) \to \Omega^*(X)$.

Each multiplicative operation $G : A^*(X) \to B^*(X)$ provides a homomorphism of formal group laws

$$\gamma_G : (A^*, F_A(x, y)) \to (B^*, F_B(x, y)),$$

that is, the ring homomorphism $G : A^* \to B^*$ together with the (change of parameter) power series $\gamma_G(z) \in B^*[[z]]$ such that $G(F_A)(\gamma_G(x), \gamma_G(y)) = \gamma_G(F_B(x, y))$. For such operation to be *stable* (in certain sense) one needs the first coefficient of $\gamma_G(z)$ to be 1 ($\gamma_G(z) = z + b_1 z^2 + b_2 z^3 + \ldots$). The total Landweber-Novikov operation S^{Tot}_{L-N} is the universal multiplicative stable operation - here the coefficients $b_1, b_2, \ldots$ in the change of parameter are just independent variables.

When $R = \sigma_i$ (that is, $\overline{a} = (1, 1, \ldots, 1)$ - i-times), we will denote the respective operations $S^{\sigma_i}_{L-N}$ simply as S^i_{L-N}. One can organize S^i_{L-N} into the multiplicative operation $S^{\bullet}_{L-N} = \sum_i S^i_{L-N} : \Omega^*(X) \to \Omega^*(X)$. Clearly, this is just the specialization of S^{Tot}_{L-N} at $b_1 = 1; b_i = 0, i \geqslant 2$.

Steenrod operations

Let $\overline{pr} : \Omega^*(X) \to \mathrm{CH}^*(X)$ be the projection. The following result is due to P.Brosnan, M.Levine and A.Merkurjev:

Theorem 0.3 *There exists (unique) operation* $S^i : \mathrm{CH}^*(X)/2 \to \mathrm{CH}^{*+i}(X)/2$ *called Steenrod operation making commutative the following diagram:*

$$\begin{array}{ccc} \Omega^* & \xrightarrow{S^i_{L-N}} & \Omega^{*+i} \\ \overline{pr}\downarrow & & \downarrow\overline{pr} \\ \mathrm{CH}^*/2 & \xrightarrow{S^i} & \mathrm{CH}^{*+i}/2. \end{array}$$

Both Steenrod and Landweber-Novikov operations commute with the pull-back morphisms.

In a similar way one can construct *reduced power operations*

$$P^i : \mathrm{CH}^*(X)/l \to \mathrm{CH}^{*+i(l-1)}(X)/l$$

corresponding to other primes l. Here one should use $S^{\overline{a}}_{L-N}$ with $\overline{a} = (l-1, l-1, \ldots, l-1)$ - i-times. In the algebro-geometric context these operations were originally constructed by V.Voevodsky in a more general situation of motivic cohomology.

Remark: Note, that if you choose some arbitrary partition $\overline{a}$ and arbitrary number l, you, in general, will not be able to find any operation $\mathrm{CH}^*/l \to \mathrm{CH}^{*+|\overline{a}|}/l$ making the respective diagram commutative.

Symmetric operations

It follows from the explicite construction of Steenrod operations by P.Brosnan that $S^i|_{\mathrm{CH}^m/2} = 0$, if $i > m$, and $S^m|_{\mathrm{CH}^m/2}$ coinside with the operation square $\square : \mathrm{CH}^m/2 \to \mathrm{CH}^{2m}/2$. It follows from the diagram above that

$$(pr \circ S^i_{L-N})(\Omega^m(X)) \subset 2 \cdot \mathrm{CH}^{m+i}(X), \text{ for } i > m, \text{ and}$$
$$(pr \circ (S^m_{L-N} - \square))(\Omega^m(X)) \subset 2 \cdot \mathrm{CH}^{2m}(X).$$

Thus, up to 2-torsion, we have well defined operations

$$\phi^{t^{i-m}} := \frac{pr \circ S^i_{L-N}}{2} : \Omega^m(X) \to \mathrm{CH}^{m+i}(X)/(2-tors.), \text{ for } i > m, \text{ and}$$
$$\phi^{t^0} := \frac{pr \circ (S^m_{L-N} - \square)}{2} : \Omega^m(X) \to \mathrm{CH}^{2m}(X)/(2-tors.).$$

In reality, this operations can be lifted to some well-defined operations

$$\Phi^{t^j} : \Omega^* \to \Omega^{2*+j}.$$

To construct such operations consider the following objects. Let $W \to X$ be some *smooth morphism* (roughly speaking, all the fibers are smooth varieties) of smooth varieties. Denote as $\square(W/X)$ the relative square $W \times_X W$; as $\widetilde{\square}(W/X)$ the Blow-up variety $Bl_{\Delta(W) \subset \square(W/X)}$, and as $\widetilde{C}^2(W/X)$ the quotient variety of $\widetilde{\square}(W/X)$ under the natural (interchanging of factors) $\mathbf{Z}/2$-action. Notice, that the locus of fixed points on $\widetilde{\square}(W/X)$ under our action will be the smooth (special) divisor of $\widetilde{\square}(W/X)$ - the preimage of the diagonal. Thus, $\widetilde{C}^2(W/X)$ will be a smooth variety. These objects fit into the diagram

$$\begin{array}{ccccc} \mathbb{P}_W(T_{W/X}) & \xrightarrow{j} & \widetilde{\square}(W/X) & \xrightarrow{p} & \widetilde{C}^2(W/X) \\ \varepsilon \downarrow & & \downarrow \pi & & \downarrow \xi \\ W & \xrightarrow[\Delta]{} & \square(W/X) & \longrightarrow & X. \end{array}$$

Variety $\widetilde{C}^2(W/X)$ has natural line bundle $\mathcal{L}$ such that $p^*(\mathcal{L}) = \mathcal{O}(1)$ - the canonical line bundle of the Blow-up variety. Denote $\rho := c_1(\mathcal{L}^{-1}) \in \Omega^1(\widetilde{C}^2(W/X))$. When $X = \operatorname{Spec}(k)$ we will omit X in the respective notations: $\widetilde{\square}(W), \widetilde{C}^2(W)$. Notice, that $\widetilde{C}^2(W)$ is nothing else but $Hilb_2(W)$ - the *Hilbert scheme of the length* 2 *subschemes on* W.

Let $v : V \to X$ be the projective morphism of smooth varieties. We can decompose it in the form $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a *regular embedding*, and f is *smooth projective morphism*. Then we get the following natural diagram:

$$\widetilde{C}^2(V) \xhookrightarrow{\alpha} \widetilde{C}^2(W) \xhookleftarrow{\beta} \widetilde{C}^2(W/X) \xrightarrow{\gamma} X,$$

where all the maps are projective.

Symmetric operations will be parametrized by the power series $q(t) \in \mathbb{L}[[t]]$:

$$\Phi^{q(t)} : \Omega^*(X) \to \Omega^*(X).$$

$$\Phi^{q(t)}([v : V \to X]) := \gamma_* \beta^* \alpha_*(q(\rho)) \in \Omega^*(X).$$

For given variety X we can extend symmetric operations by $\Omega^*(X)$-linearity on $q(t)$, and assume that $q(t) \in \Omega^*(X)[[t]]$.

Properties:

(0)
$$\Phi^{q(t)}(x+y) = \Phi^{q(t)}(x) + \Phi^{q(t)}(y) + q(0)xy$$
In particular $\Phi^{q(t)}$ is linear if $q(0) = 0$.

(1) $\Phi^{q(t)}$ commutes with the pull-back morphisms;

(2) If $f : X \hookrightarrow Y$ is a regular embedding with normal bundle $\mathcal{N}_f$, and $q(t) \in \Omega^*(Y)[[t]]$. Then
$$\Phi^{q(t)}(f_*(x)) = f_*\Phi^{f^*(q(t))\cdot c_\bullet^\Omega(\mathcal{N}_f)(t)}(x),$$
where $c_\bullet^\Omega(\mathcal{V})(t) = \prod_i(\lambda_i -_\Omega t)$, where $\lambda_i \in \Omega^1$ are *roots* of $\mathcal{V}$, and $-_\Omega$ is the subtraction in the sense of the universal formal group law.

(3) $\Phi^{q(t)}$ is trivial on the classes of embeddings. Really, if $v : V \to X$ is an embedding, we can take $W = X$, and then the variety $\widetilde{C}^2(W/X)$ will be empty. Thus, the symmetric operations provide the obstructions for the cobordism class to be represented by the embedding.

(4) $2 \cdot (pr \circ \Phi^{t^r})|_{\Omega^m} = (-1)^r \cdot (pr \circ S_{L-N}^{r+m})$. Thus, with the help of the symmetric operations one can get cycles twice as small as with the help of the Landweber-Novikov operations. This difference can be crusial if one works with the varieties where the effect related to prime 2 are important (like quadrics, for example).

Remark: Actually, the properties $(0)-(3)$ determine the operations $\Phi^{q(t)}$ uniquely up to renormalization $q(t) \mapsto q(t)\cdot r(t)$, where $r(t) \in \mathbb{L}[[t]], r(0) = 1$.

The most interesting symmetric operations are not expressible in terms of Landweber-Novikov operations, and can not be organized into the multiplicative operation. Nevetheless, some of them are, and these operations are related to the *Steenrod operations in Cobordism theory.*