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Structure and classification of almost simple algebraic groups

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May 17. Structure and classification of almost simple algebraic groups

k arbitrary, k_s = sep. closure, $\Gamma = \mathrm{Gal}(k_s/k)$

G semisimple k-group ($\Rightarrow G \times_k k_s$ is Chevalley, Thm. of Grothendieck)

$\cup$

T max k-torus (split over k_s)

$\cup$

S max k-split torus of G

Remark: These are usually different:
Take $G = SL_r(D)$, D/k central division algebra of index $d > 1$

$\Rightarrow G \times k_s \cong SL_{r+1}(M_d(k_s))$ (Thm of Wedderburn)

$\simeq SL_{(r+1)d}(k_s)$

$\Rightarrow \dim T = (r+1)d - 1$

$\dim S = r$

Δ = System of simple roots of $G \times_k k_s$ w.r. to T

$\Delta_0 = \{\alpha \in \Delta \mid \alpha|_S = 0\}$.

Def: G is isotropic if it contains a non trivial k-split torus (i.e., $\dim S > 0$), and anisotropic otherwise (i.e., $\dim S = 0$)

Expls: i) D/k as above, $d > 0$, $n : D^* \to k^*$ reduced norm
$G = SL_1(D)$ = kernel of n | while $SL_n(D)$, $n \geq 2$ is isotropic

ii) q anisotropic quadratic form: $G = SO(q)$.

Anisotropic kernel of arbitrary s.s. G.

Model: quadratic forms: Witt decomposition

q regular $\Rightarrow$

$q \simeq q_{an} \perp \mathbb{H}^i$ $\qquad \mathbb{H}$ = hyperbolic plane $/k$

unique up to isometry

q_{an} anisotropic, $\quad i = \text{index}(q)$.

visualize $Z(S)$

$$\begin{pmatrix} s_1 & 0 & \\ 0 & s_{r+1} & 0 \\ & 0 & DZ(S) \end{pmatrix}$$

Take:

$Z(S) := \text{centralizer}_G(S)$ (reductive!) $\supset T$

$DZ(S) :=$ derived group of $Z(S)$ (semisimple)

$Z_a :=$ max anisotropic subtorus of centre $(Z(S))$

Def: $DZ(S)$: semisimple anisotropic kernel of G

$DZ(S) \cdot Z_a$: reductive anisotropic kernel of G

$\Rightarrow \Delta_0$ = set of simple roots of $DZ(S)$ w.r. to $T \cap DZ(S)$

Prop. i) The semisimple anisotropic kernels of G are precisely the subgroups occuring as derived group of Levi-k-subgroups (= semisimple parts) of minimal parabolic k-subgroups of G.

Any two such are conjugate under $G(k)$.

ii) the anisotropic kernels of G are anisotropic k-groups

iii) G is quasisplit (i.e. has a k-Borel subgroup) $\Leftrightarrow$ its semisimple anisotropic kernel is trivial.

Tits-Index of G:

$\Gamma = \mathrm{Gal}(k_s/k)$ operates on Δ as follows:

$T \times_k k_s$ split, hence $G \times_k k_s$ is Chevalley

$\Delta \ni \alpha \longleftrightarrow P_{\Delta \setminus \{\alpha\}}$ (max proper parabolic subgroups)
(each represents one of their conjugacy classes)

Γ operates on the set of conjugacy classes of parabolic subgroups, thereby on Δ: $(\gamma, \alpha) \mapsto \gamma^* \alpha$ ($\gamma \in \Gamma$)
("$*$-operation")

If not the same as ${}^{\gamma}\alpha$, this may not be in Δ

hence γ induces a shift of the ordering of Δ or the underlying Weyl chamber, but then:
There is a unique $w \in W$ with $w({}^{\gamma}\Delta) = \Delta$, as W operates simply transitively on the Weyl chambers,

hence $\gamma^* \alpha = w\, {}^{\gamma}\alpha$.

Def: G is of inner type if $*$-op trivial
— outer — if not.

Def: The Tits index of G is given by
(Δ, Δ_0) together with $*$-operation (leaving Δ_0 invariant)

Pre-Classification theorem ("Witt-Type" theorem):

G is uniquely determined (up to strict isogeny if only ss anisotropic kernel $= 1$)
by its Tits index and by its anisotropic kernel.

Pre-structural theorem:

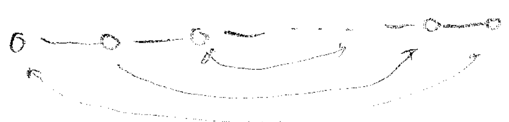
Let P be a minimal parabolic subgroup of G.

Then $$G = \bigvee_{w \in W_{\Delta \setminus \Delta_0}} P w P$$

All these results say NOTHING about anisotropic groups or about anisotropic kernels of arbitrary semisimple gps.

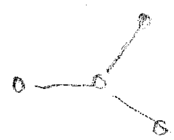
Examples:

Only the following Dynkin diagrams admit Automorphisms:

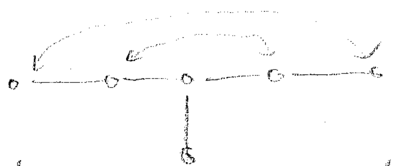


A_n (order 2)

D_n ($n \geq 5$: order 2)



D_4 (order 6)



E_6 (order 2)

Only those groups may have "outer type".
All other groups are a priori of "inner type".

Tits index of G

graphically represented by an annotation of $\mathcal{D}(G)$

Roots repres. nontrivial characters over k are marked ○ "Distinguished vertices"

the others are marked |

Orbits of the $*$-operation are visualized by drawing their members close and encircling orbits not in Δ_0 "Distinguished orbits"

The Type of G is denoted by

$$ {}^{g}X^{t}_{n,r} $$

where X is the type letter (one of $A, B, \ldots, G$)

n = absolute rank

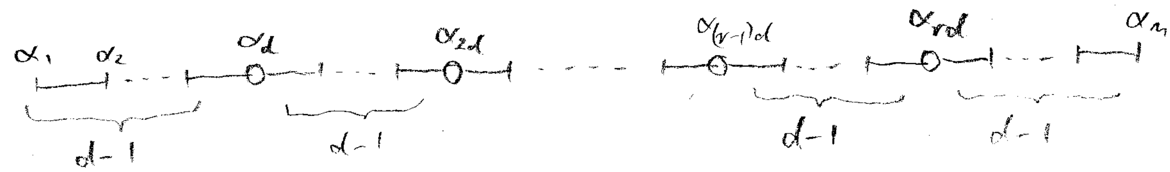
r = k-rank

g = order of quotient automorphism (left out if a priori 1)

$t = \begin{cases} \text{either index of the underlying (the part in ()) Division algebra} \\ \text{or dimension of anisotropic kernel} \end{cases}$

Examples:

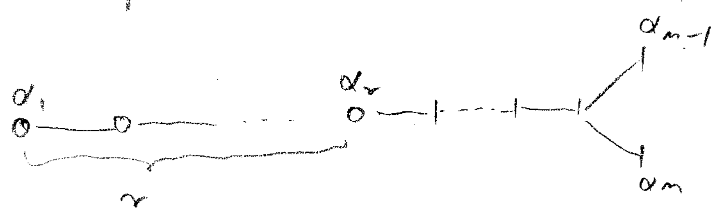
Type ${}^1A^{(d)}_{n,r}$: Group: $SL_{r+1}(D)$, D/k central div alg of degree d



Conditions: $d\cdot(r+1) = n+1$

Distinguished vert: $\alpha_d, \alpha_{2d}, \dots \alpha_{rd}$

Type ${}^1D^{(1)}_{n,r}$

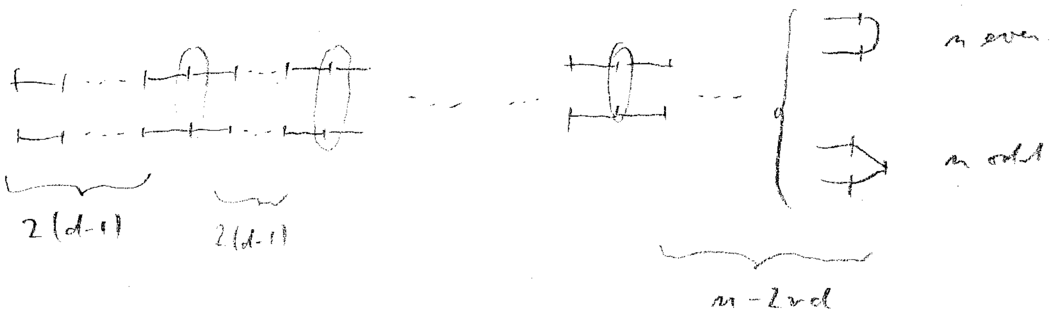


group: Special orthogonal group of q.f. of <u>even</u> dimension with <u>trivial discriminant</u>

Type ${}^2D^{(1)}_{n,r}$

group: special orthogonal group of q.f of even dimension with <u>nontrivial discriminant</u>

Type ${}^2A^{(d)}_{n,r}$



Group: special unitary group $SU_{(n+1)/d}(D, h)$

with D/k' div. alg. of degree d, k'/k quadratic — with involution of second kind 1

h non deg. hermitian form of index r relative to $\sigma : k' \to k' \in Gal(k'/k)$.