Lecture 6

u-invariants of fields.

In this lecture we will demonstrate the applications of the technique discussed earlier to the u-invariants of fields.

Let k be a field. Define the u-invariant of k as

$$u(k) := \max(\dim(q) | q \ - \ \text{anisotropic form over } k).$$

Examples:

(1) k-algebraically closed, then $u(k) = 1$;

(2) $u(\mathbb{R}) = \infty$;

(3) k-finite, then $u(k) = 2$;

(4) k-local, then $u(k) = 4$;

(5) k-global, then $u(k) = \begin{cases} \infty, & \text{if there are real embeddings } k \subset \mathbb{R}; \\ 4, & \text{otherwise} \end{cases}$.

(6) $k = F[[t_1, \ldots, t_n]]$, where F-algebraically closed, then $u(k) = 2^n$.

So, in a certain sense, the u-invariant gives some idea how far our field is from being algebraically closed (of course, it can see only one of the projections of such a distance).

The natural question arises: what are the possible values of this invariant?

It is easy to see that $u(k)$ can not take values $3, 5$, and 7.

Example: Let us show that $u(k) \neq 3$. Really, if $u(k)$ would be 3, then all the forms of dimension $\geqslant 4$ over k would be isotropic, and some form of dimension 3 would be anisotropic. Up to a scalar, such form is $\langle 1, -a, -b \rangle$ and the respective projective quadric is conic $C_{\{a,b\}}$. But as we saw in Lecture 3, such conic is isotropic if and only if the respective 2-dimensional 2-fold Pfister quadric $Q_{\{a,b\}}$ is (for example, because $Q_{\{a,b\}} = C_{\{a,b\}} \times C_{\{a,b\}}$). This gives a contradiction, since $\dim(\langle\langle a, b \rangle\rangle) = 4$.

"Conjecture" of Kaplansky (1953) predicted that the only possible values are powers of two.

It was disproved by A.Merkurjev (1989), who constructed fields with all even u-invariants. Further disproved by O.Izhboldin (1999), who constructed the field k with $u(k) = 9$ - the first odd value (> 1).

The basic ingredient of the construction is the

Merkurjev tower of fields

Let F be some field, and $M \in \mathbb{N}$. We want to construct some extension of F, where all forms of dimension $> M$ will be isotropic. Suppose we have just one form q over F. There are many extensions of F making q isotropic. For example $\overline{F}$ - the algebraic closure of F. But we want the one which would behave in a most gentle way with respect to everything. Such field is, of course, $F(Q)$ - the generic point of the quadric Q - any other field making q isotropic will be a specialization of this one. If we want to make two forms q_1, q_2 isotropic, when we should use the field $F(Q_1 \times Q_2)$, etc. ... .

Denote as J the set of all forms of dimension $> M$ over F, and define the new field F' by the formula

$$\lim_{\to I \subset J} F(\times_{i \in I} Q_i),$$

where I runs over all finite subsets of J. This field has the property, that any form q of dimension $> M$ defined over F is isotropic over F'. And it is universal one among the extensions E/F with such property.

Starting with some field k, consider the sequence of fields

$$k = k_0 \hookrightarrow k_1 \hookrightarrow k_2 \hookrightarrow \ldots,$$

where $k_{i+1} := (k_i)'$. Denote $k_\infty := \lim_{\to i} k_i$. Then all the forms of dimension $> M$ defined over k_∞ are isotropic (since any such form is defined on some finite level k_i, and, thus becomes isotropic over k_{i+1}). In other words, $u(k_\infty) \leqslant M$. But we would want the equality. For this we need some anisotropic form p of dimension M over k_∞. Better to have it already over k, and then check that it stays anisotropic over k_∞. Of course, to be able to control this, one needs to know something interesting about p (not just the fact that it is anisotropic). Formalizing, we need a form p of dimension M over k, and two properties A and B on the set of field extensions E/k, where

$$A(E) \text{ is satisfied } \Leftrightarrow \; p|_E \text{ is anisotropic},$$

and A and B satisfy the following axioms:

(1) $B \Rightarrow A$;

(2) $B(k)$ is satisfied;

(3) $B(F)$ is satisfied, $\dim(q) > M \Rightarrow B(F(Q))$ is satisfied;

(4) $B(F_j)$, for directed system of fields is satisfied $\Rightarrow B(\lim_{\rightarrow_j} F_j)$ is satisfied.

In this case, $A(k_\infty)$ is satisfied, and $u(k_\infty) = M$.

So, we need only to choose the form p and the right property B.

Choice of Merkurjev:

To each quadratic form p one can assign its *Clifford algebra* $C(q)$ defined as $T_k(V_p)/(v^2 - p(v), \forall v \in V_p)$ - the quotient of the tensor algebra of the underlying vector space by the explicite relations. This algebra has a natural $\mathbf{Z}/2$-grading, and it "is not far" from being a central simple algebra. We will be interested only in the case, where $p \in I^2$, that is, $\dim(p)$ is even and $\det_{\pm}(p) = 1$. In such a case, $C(p) = Mat_{2\times 2}(k) \otimes_k C'(p)$, where C' is a central simple algebra over k. In the case $M = 2n$ - even, Merkurjev have chosen the following property:

$$B(E) \text{ is satisfied } \Leftrightarrow C'(p|_E) \text{ is a division algebra.}$$

One should start with the *generic* quadratic form of dimension $2n$ from I^2 - that is, the form $\langle a_1, \ldots, a_{2n-1}, \prod_{i=1}^{2n-1} a_i \rangle$ over the field $k = F(a_1, \ldots, a_{2n-1})$. Let us check the axioms:

1) $B \Rightarrow A$, since $p = \mathbb{H} \perp r \Rightarrow C'(p) = Mat_{2\times 2}(k) \otimes_k C'(r)$.

2) $B(k)$ is satisfied, since the C' of the generic form as above is division.

4) Clear, since zero divisors are defined on the finite level.

3) This is the only nontrivial part. The proof here is based on the *Index reduction formula of Merkurjev.* This formula describes what happens to the index of the central simple algebra over the generic point of a quadric. It says that the index of the division algebra C over $k(Q)$ can drop at most by the factor 2, and the latter happens if and only if there is a k-algebra homomorphism $C_0(q) \rightarrow C$, where $C_0(q)$ is the *even Clifford algebra of* q (the degree zero part of $C(q)$).

Notice, that $C_0(q)$ is either a simple algebra, or a product of two isomorphic simple algebras, and if $\dim(p) = 2n$ is even, and $\dim(q) > \dim(p)$, then the size of each simple factor in $C_0(q)$ will be bigger than the size of $C'(p)$, so we do not have maps $C_0(q) \to C'(p)$. Thus, the condition (3) is fulfilled, and $u(k_\infty) = 2n$.

Another choice for even M

Let us make another choice of the property B. We will choose one based on the EDI - the elementary discrete invariant of quadrics (see Lecture 4). Namely, we start with the generic form $p = \langle a_1, \ldots, a_{2n} \rangle$ over $k = F(a_1, \ldots, a_{2n})$, and the property:

$$B(E) \text{ is satisfied } \Leftrightarrow y_{d,0}(p|_E) \text{ is not defined over } k,$$

where $d = [\dim(P)/2] = n - 1$.

In other words, $EDI(p|_E)$ should have the form

$$\begin{array}{ccccc} \circ & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ? & ? & ? & \cdots & ? \end{array}$$

1) $B \Rightarrow A$ since $A(E)$ is satisfied if and only if $y_{0,0}$ is not defined over E (we remind, that $y_{0,0}$ is just the class of a rational point on $P|_{\overline{E}}$), and $y_{i,j}$ is defined implies $y_{l,j}$ is defined for any $l > i$.

2) $B(k)$ is satisfied, since EDI of the generic form is empty.

4) Follows, since for any X/k, $CH^*(X|_{\lim_{\to j} F_j}) = \lim_{\to j} CH^*(X|_{F_j})$ (with any coefficients).

3) This is again the only nontrivial part, and it follows from the following:

Theorem 0.1 *Let Y be smooth quasiprojective variety over some field k of characteristic zero. Let Q be smooth projective quadric over k, and $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})/2$ be some element. Suppose $2m < \dim(Q)$. Then*

$$\overline{y} \text{ is defined over } k \Leftrightarrow \overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q).$$

Indeed, one just needs to take $\overline{y} = y_{d,0}$. Then $m = \dim(P) - d = d < \dim(Q)/2$ for any q bigger than p, and the Theorem implies what we need.

Shortly, we succeded by controling not the class $y_{0,0}$, but the smaller codimensional (!) class $y_{d,0}$. The point, of course, is: the smaller is the codimension of the cycle, the easier it is to control its rationality.

Notice, that the bound $2m < \dim(Q)$ is optimal: for any pair $\dim(Q), m$ not satisfying the inequality, one can find variety Y, cycle $\overline{y}$, and a quadric Q of needed codimension and dimension, so that $\overline{y}|_{k(Q)}$ is defined over $k(Q)$, but $\overline{y}$ is not defined over k. Just take Q generic, and $\overline{y} = y_{d,0} \times pt$ on $G(d,Q) \times \mathbb{P}^{m-d}$.

The proof of the above Theorem uses the *Symmetric operations* in Algebraic Cobordism (see Lecture 5). If $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$, then we lift the respective element first to $\mathrm{CH}^*(Y \times Q)/2$, and, finally, to $\Omega^*(Y \times Q)$, then we restrict it to $Y \times Q_s$ for the subquadrics $e_s : Q_s \to Q$ of different dimensions, and apply the composition of the various symmetric operations with the projection $(\pi_s)_*$, after which we map the results to $\mathrm{CH}^*(Y)/2$, and add them with certain coefficients.

$$\begin{tikzcd}
\mathrm{CH}^*(Y|_{k(Q)})/2 & \mathrm{CH}^*(Y\times Q)/2 \arrow[l, two heads] & \Omega^*(Y\times Q) \arrow[l, two heads] \arrow[r, "e_s^*"] & \Omega^*(Y\times Q_s) \arrow[d, "(\pi_s)_*"] \\
& \mathrm{CH}^*(Y)/2 \arrow[u] & & \Omega^*(Y) \arrow[ll]
\end{tikzcd}$$

It appears, that if $2m < \dim(Q)$, one can choose the coefficients in such a way that all the choices we made will be cancelled, and the result will be equal to $\overline{y}$ when restricted to $\overline{k}$.

Let me demonstrate the usefulness of our Theorem on the following:

Example: *EDI* of a Pfister forms. Let $\alpha \in \mathrm{K}^M_n(k)/2$ be nonzero pure symbol, and $q_\alpha = \langle\langle \alpha \rangle\rangle$ be the respective anisotropic Pfister form. Then in $EDI(Q_\alpha)$ the marked points will be the ones strictly above the Main (NW-SE) diagonal. Indeed, consider $Q = Q_\alpha$, $Y = G(i, Q_\alpha)$, $\overline{y} = y_{i,j} \in \mathrm{CH}^{\dim(Q_\alpha)-i-j}(Y)/2$. Since $Q_\alpha|_{k(Q_\alpha)}$ is complitely split, all elementary classes on Q_α are defined over this field. But then, by the Theorem, those ones which are of sufficiently small codimension, i.e., exactly the ones living strictly above the Main diagonal, are defined already over the base field k. It remains to see that the other ones are not defined. Because of the rule $\begin{tikzcd}[cramped] \bullet & \bullet \\ & \bullet \arrow[ul] \arrow[u] \end{tikzcd}$ it is sufficient to check that the NW-corner ($y_{d,0}$) is not defined over k. But if it would be defined, all the elementary classes $y_{d,j}$ on the last Grassmannian

$G(d, Q_\alpha)$ would be defined. But the product $\prod_{j=0}^{d} y_{d,j}$ of these classes is equal to the class of rational point on $G(d, Q_\alpha)|_{\overline{k}}$. So, this would imply that Q_α is complitely split (we use the Theorem of Springer here, claiming that the quadric has a rational point, if it has one of odd degree). Thus, $y_{d,0}$ is not defined over k, and $EDI(Q_\alpha)$ is as we described:

$$
\begin{array}{cccccc}
\circ & \bullet & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \circ & \cdots & \bullet & \bullet \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \circ & \circ & \cdots & \circ & \bullet \\
\circ & \circ & \circ & \cdots & \circ & \circ
\end{array}
$$

<u>u-invariants $2^r + 1$, $r \geqslant 3$</u>

The same ideas can be used to construct the fields with some odd u-invariants. These values are 2^r+1, $r \geqslant 3$. In the case of u-invariant 9 we get method different from that of O.Izhboldin, and for $r > 3$ we get the values not known before.

For odd dimensional form p we can not use the class $y_{d,0}$ anymore, since for $q = p \perp \langle \det_{\pm}(p) \rangle$, the class $y_{d,0}(p|_{k(Q)})$ will always be defined, although $\dim(q) > \dim(p)$. So, our condition should involve somehow the classes $y_{i,0}$ $i < d$, since these are the only ones which are defined as soon as P has a rational point.

We have the following:

Theorem 0.2 *Let* $\dim(p) = 2^r + 1$, $r \geqslant 3$, *and* $EDI(P)$ *looks as*

$$
\begin{array}{ccccc}
? & \circ & \circ & \cdots & \circ \\
\circ & \circ & \circ & \cdots & \circ \\
\circ & \circ & \circ & \cdots & \circ \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \circ & \circ & \cdots & \circ
\end{array}
$$

Suppose $\dim(q) > \dim(p)$. *Then* $EDI(P|_{k(Q)})$ *has the same property.*

The above theorem immediately implies that the property:

$$B(E) \text{ is satisfied } \Leftrightarrow \ EDI(p|_E) \text{ is as above}$$

satisfies the axiom (3). Let us take the generic form p of dimension $2^r + 1$, then all the other axioms will be readily fulfilled as well, and $u(k_\infty) = 2^r + 1$.

The proof of Theorem 0.2 uses certain extensions of Theorem 0.1, the knowledge of action of the Steenrod operations on the elementary classes and the fact that on the last Grassmannian the subring of k-rational cycles is always generated by the k-rational elementary classes. So, it involves a bit more than the case of even u-invariants.

In the end, let me mention some useful literature:

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