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Pure motives I: constructions

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Pure motives: Part I

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Outline:

- Pre-history of motives
- Adequate equivalence relations
- Weil cohomology
- Grothendieck motives

Pre-history of motives

Part I: Algebraic cycles

X: a scheme of finite type over a field k.

An *algebraic cycle* on X is $Z = \sum_{i=1}^{m} n_i Z_i$, $n_i \in \mathbb{Z}$, $Z_i \subset X$ integral closed subschemes.

$\mathcal{Z}(X) :=$ the group of algebraic cycles on X.

$\mathcal{Z}(X) = \mathcal{Z}_*(X) := \oplus_{r \geq 0} \mathcal{Z}_r(X)$ graded by dimension.

$\mathcal{Z}(X) = \mathcal{Z}^*(X) := \oplus_{r \geq 0} \mathcal{Z}^r(X)$ graded by codimension (for X equi-dimensional).

Functoriality $X \mapsto \mathcal{Z}_*(X)$ is a covariant functor for proper maps $f : X \to Y$:

$$f_*(Z) := \begin{cases} 0 & \text{if } \dim_k f(Z) < \dim_k Z \\ [k(Z) : k(f(Z))] \cdot f(Z) & \text{if } \dim_k f(Z) = \dim_k Z. \end{cases}$$

For $p : X \to \operatorname{Spec} k$ projective over k, have $\deg : \mathcal{Z}_0(X) \to \mathbb{Z}$ by

$$\deg(z) := p_*(z) \in \mathcal{Z}_0(\operatorname{Spec} k) = \mathbb{Z} \cdot [\operatorname{Spec} k] \cong \mathbb{Z}.$$

$X \mapsto \mathcal{Z}^*(X)$ is a contravariant functor for flat maps $f : Y \to X$:

$$f^*(Z) := \operatorname{cyc}(f^{-1}(Z)) := \sum_{T \subset f^{-1}(Z)} \ell_{\mathcal{O}_{Y,T}}(\mathcal{O}_{Z,T}) \cdot T;$$

sum over irreducible components T of $f^{-1}(Z)$.

Intersection theory Take X smooth, $Z, W \subset X$ irreducible.

Z and W *intersect properly* on X: each irreducible component T of $Z \cap W$ has

$$\mathrm{codim}_X T = \mathrm{codim}_X Z + \mathrm{codim}_X W.$$

The *intersection product* is

$$Z \cdot_X W := \sum_T m(T; Z \cdot_X W) \cdot T.$$

$m(T; Z \cdot_X W)$ is *Serre's intersection multiplicity*:

$$m(T; Z \cdot_X W) := \sum_i (-1)^i \ell_{\mathcal{O}_{X,T}}(\mathrm{Tor}_i^{\mathcal{O}_{X,T}}(\mathcal{O}_{Z,T}, \mathcal{O}_{W,T})).$$

Extend to cycles $Z = \sum_i n_i Z_i$, $W = \sum_j m_j W_j$ of pure codimension by linearity.

Contravariant functoriality

Intersection theory extends flat pull-back to a partially defined pull-back for $f : Y \to X$ in $\mathbf{Sm}/k$:

$$f^*(Z) := p_{1*}(\Gamma_f \cdot p_2^*(Z))$$

$\Gamma_f \subset Y \times X$ the graph of f, $p_1 : \Gamma_f \to Y$, $p_2 : Y \times X \to X$ the projections.

And: a partially defined associative, commutative, unital graded ring structure on $\mathcal{Z}^*(X)$ with (when defined)

$$f^*(a \cdot b) = f^*(a) \cdot f^*(b)$$

and (the *projection formula*)

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b)$$

for f projective.

Example: the zeta-function

X: smooth projective over $\mathbb{F}_q$.

$$Z_X(t) := \exp\Big(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} \cdot t^n\Big).$$

Note that

$$\#X(\mathbb{F}_{q^n}) = \deg(\Delta_X \cdot \Gamma_{Fr_X^n})$$

$\Delta_X \subset X \times X$ the diagonal, Fr_X the *Frobenius*

$$Fr_X^*(h) := h^q.$$

Part II: cohomology

Weil: the singular cohomology of varieties over $\mathbb{C}$ should admit a purely algebraic version, suitable also for varieties over $\mathbb{F}_q$.

Grothendieck *et al.*: *étale cohomology* with $\mathbb{Q}_\ell$ coefficients ($\ell \neq \operatorname{char}(k)$, $k = \bar{k}$) works.

Example. The Lefschetz trace formula $\Longrightarrow$

$$\deg(\Delta_X \cdot \Gamma_{Fr_X^n}) = \sum_{i=0}^{2d_X} (-1)^i Tr(Fr_X^{n*}|_{H^i(\bar{X},\mathbb{Q}_\ell)})$$

$$Z_X(t) = \frac{\det(1 - tFr_X^*|_{H^-(\bar{X},\mathbb{Q}_\ell)})}{\det(1 - tFr_X^*|_{H^+(\bar{X},\mathbb{Q}_\ell)})}$$

Thus: $Z_X(t)$ is a *rational function* with $\mathbb{Q}$-coefficients.

A mystery

In fact, by the Weil conjectures, the characteristic polymomial $\det(1 - tFr_X^*|_{H^i(\bar{X},\mathbb{Q}_\ell)})$ has $\mathbb{Q}$ (in fact $\mathbb{Z}$) coefficients, *independent of ℓ*.

However: Serre's example of an elliptic curve E over $\mathbb{F}_{p^2}$ with $\mathrm{End}(E)_{\mathbb{Q}}$ a quaternion algebra shows: there is no "good" cohomology over $\bar{\mathbb{F}}_p$ with $\mathbb{Q}$-coefficients.

An “answer”

Grothendieck suggested: there is a $\mathbb{Q}$-linear category of “motives” over k which has the properties of a universal cohomology theory for smooth projective varieties over k.

This category would explain why the étale cohomology $H^*(-, \mathbb{Q}_\ell)$ for different ℓ all yield the same data.

Grothendieck’s idea: make a cohomology theory purely out of algebraic cycles.

Adequate equivalence relations

To make cycles into cohomology, we need to make the pull-back and intersection product everywhere defined.

Consider an equivalence relation $\sim$ on $\mathcal{Z}^*$ for smooth projective varieties: for each $X \in \mathbf{SmProj}/k$ a graded quotient $\mathcal{Z}^*(X) \twoheadrightarrow \mathcal{Z}^*_\sim(X)$.

Definition $\sim$ is an *adequate equivalence relation* if, for all $X, Y \in \mathbf{SmProj}/k$:

1. Given $a, b \in \mathcal{Z}^*(X)$ there is $a' \sim a$ such that a' and b intersect properly on X

2. Given $a \in \mathcal{Z}^*(X)$, $b \in \mathcal{Z}^*(X \times Y)$ such that $p_1^*(a)$ intersects b properly. Then

$$a \sim 0 \Longrightarrow p_{2*}(p_1^*(a) \cdot b) \sim 0.$$

For a field F (usually $\mathbb{Q}$) make the same definition with $\mathcal{Z}^*(X)_F$ replacing $\mathcal{Z}^*(X)$.

Functoriality

(1) and (2) imply:

• The partially defined intersection product on $\mathcal{Z}^*(X)$ descend to a well-defined product on $\mathcal{Z}^*_\sim(X)$.

• Push-forward for projective $f : Y \to X$ descends to $f_* : \mathcal{Z}_\sim(Y) \to \mathcal{Z}_\sim(X)$

• Partially defined pull-back for $f : Y \to X$ descends a well-defined $f^* : \mathcal{Z}^*_\sim(X) \to \mathcal{Z}^*_\sim(Y)$.

Order adequate equivalence relations by $\sim_1 \succ \sim_2$ if $Z \sim_1 0 \implies Z \sim_2 0$: $\sim_1$ is *finer than* $\sim_2$.

Geometric examples Take $Z \in \mathcal{Z}^n(X)$.

1. $Z \sim_{\text{rat}} 0$ if there is a $W \in \mathcal{Z}^*(X \times \mathbb{P}^1)$ with

$$p_{1*}[(X \times 0 - X \times \infty) \cdot W] = Z.$$

2. $Z \sim_{\text{alg}} 0$ if there is a smooth projective curve C with k-points c, c' and $W \in \mathcal{Z}^*(X \times C)$ with

$$p_{1*}[(X \times c - X \times c') \cdot W] = Z.$$

3. $Z \sim_{\text{num}} 0$ if for $W \in \mathcal{Z}^{d_X - n}(X)$ with $W \cdot_X Z$ defined,

$$\deg(W \cdot_X Z) = 0.$$

$$\sim_{\text{rat}} \succ \sim_{\text{alg}} \succ \sim_{\text{num}}$$

Write $\mathrm{CH}^*(X) := \mathcal{Z}^*_{\sim_{\text{rat}}}(X) = \mathcal{Z}^*_{\text{rat}}(X)$: the *Chow ring* of X. Write $\mathcal{Z}_{\text{num}} := \mathcal{Z}_{\sim_{\text{num}}}$, etc.

Remark $\sim_{\text{rat}}$ is the *finest* adequate equivalence relation $\sim$:

i. $[0] \sim \sum_i n_i[t_i]$ with $t_i \neq 0$ all i by (1).

ii. Let $f(x) = 1 - \prod_i f_{t_i}(x)$,

$f_{t_i}(x) \in k[x/(x-1)]$ minimal polynomial of t_i, normalized by $f_{t_i}(0) = 1$.

Then $f(0) = 0$ $f(t_i) = 1$, so

$$f_*([0] - \sum_i n_i[t_i]) = [0] - (\sum_i n_i')[1] \sim 0,$$

by (2), where $n_i' = [k(t_i) : k]n_i$.

iii. Send $x \mapsto 1/x$, get $[\infty] - (\sum_i n_i')[1] \sim 0$, so $[0] \sim [\infty]$ by (2).

iv. $\sim_{\text{rat}} \succ \sim$ follows from (2).

Remark $\sim_{\mathrm{num}}$ is the coarsest non-zero adequate equivalence relation $\sim$ (with fixed coefficient field $F \supset \mathbb{Q}$).

If $\sim \neq 0$, then $F = \mathcal{Z}^0(\mathrm{Spec}\, k)_F \to \mathcal{Z}^0_\sim(\mathrm{Spec}\, k)_F$ is an isomorphism: if not, $\mathcal{Z}^0_\sim(\mathrm{Spec}\, k)_F = 0$ so

$$[X]_\sim = p_X^*([\mathrm{Spec}\, k]_\sim) = 0$$

for all $X \in \mathbf{SmProj}/k$. But $? \cdot [X]_\sim$ acts as id on $\mathcal{Z}_\sim(X)_F$.

If $Z \sim 0$, $Z \in \mathrm{CH}^r(X)_F$ and W is in $\mathrm{CH}^{d_X - r}(X)$ then $Z \cdot W \sim 0$ so

$$0 = p_{X*}(Z \cdot W) \in \mathcal{Z}^0_\sim(\mathrm{Spec}\, k)_F = \mathcal{Z}^0_{\mathrm{num}}(\mathrm{Spec}\, k)_F$$

i.e. $Z \sim_{\mathrm{num}} 0$.

Weil cohomology

$\mathbf{SmProj}/k :=$ smooth projective varieties over k.

$\mathbf{SmProj}/k$ is a symmetric monoidal category with product $= \times_k$ and symmetry the exchange of factors $t : X \times_k Y \to Y \times_k X$.

$\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ is the tensor category of graded finite dimensional K vector spaces $V = \oplus_{r \geq 0} V^r$.

$\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ has tensor $\otimes_K$ and symmetry

$$\tau(v \otimes w) := (-1)^{\deg v \deg w} w \otimes v$$

for homogeneous elements v, w.

Definition A *Weil cohomology theory* over k is a symmetric monoidal functor

$$H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K,$$

K is a field of characteristic 0, satisfying some axioms.

Note: H^* monoidal means: $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$. Using

$$\delta_X^* : H^*(X \times X) \to H^*(X)$$

makes H^* a functor to graded-commutative K-algebras.

The axioms

1. $\dim_K H^2(\mathbb{P}^1) = 1$. Write $V(r)$ for $V \otimes_F H^2(\mathbb{P}^1)^{\otimes -r}$, $r \in \mathbb{Z}$.

2. If X has dimension d_X, then there is an isomorphism

$$Tr_X : H^{2d_X}(X)(d_X) \to K$$

such that $Tr_{X\times Y} = Tr_X \otimes Tr_Y$ and the pairing

$$H^i(X) \otimes H^{2d_X - i}(X)(d_X) \xrightarrow{\cup_X} H^{2d_X}(X)(d_X) \xrightarrow{Tr_X} K$$

is a perfect duality.

3. There is for $X \in \mathbf{SmProj}/k$ a *cycle class homomorphism*

$$\gamma_X^r : \mathrm{CH}^r(X) \to H^{2r}(X)(r)$$

compatible with f^*, $\cdot_X$ and with $Tr_X \circ \gamma_X^{d_X} = \deg$.

Remarks

By (2), $H^i(X) = 0$ for $i > 2d_X$. Also, $H^0(\mathrm{Spec}\, k) = K$ with $1 = \gamma([\mathrm{Spec}\, k])$. $\gamma_X([X])$ is the unit in $H^*(X)$.

Using Poincaré duality (2), we have $f_* : H^*(X)(d_X) \to H^{*+2c}(Y)(d_Y)$ for $f : X \to Y$ projective, $c = 2d_Y - 2d_X$ defined as the dual of f^*. $Tr_X = p_{X*}$

By (3), the cycle class maps γ_X are natural with respect to f_*.

Correspondences

For $a \in \mathsf{CH}^{\dim X + r}(X \times Y)$ define:

$$a_* : H^*(X) \to H^{*+2r}(Y)(r)$$
$$a_*(x) := p_{2*}(p_{1*}(x) \cup \gamma(a))).$$

Example $a = {}^t\Gamma_f$ for $f : Y \to X$ $(r = 0)$. $a_* = f^*$.

$a = \Gamma_g$ for $g : X \to Y$ $(r = \dim Y - \dim X)$. $a_* = f_*$.

Composition law

Given $a \in \mathsf{CH}^{\dim X + r}(X \times Y)$, $b \in \mathsf{CH}^{\dim Y + s}(Y \times Z)$ set

$$b \circ a := p_{13*}(p_{12}^*(a) \cdot p_{23}^*(b)) \in \mathsf{CH}^{\dim X + r + s}(X \times Z).$$

Then

$$(b \circ a)_* = b_* \circ a_*.$$

Lemma $H^1(\mathbb{P}^1) = 0$.

Proof. Set $i := i_0 : \mathrm{Spec}\, k \to \mathbb{P}^1$, $p : \mathbb{P}^1 \to \mathrm{Spec}\, k$.

$\Gamma_{\mathrm{id}_{\mathbb{P}^1}} = \Delta_{\mathbb{P}^1} \sim_{\mathrm{rat}} 0 \times \mathbb{P}^1 + \mathbb{P}^1 \times 0 \Longrightarrow$

$$\begin{aligned}\mathrm{id}_{H^1(\mathbb{P}^1)} &= \Delta_{\mathbb{P}^1 *} \\ &= (0 \times \mathbb{P}^1)_* + (\mathbb{P}^1 \times 0)_* \\ &= p^* i^* + i_* p_*.\end{aligned}$$

But $H^n(\mathrm{Spec}\, k) = 0$ for $n \neq 0$, so

$$i^* : H^1(\mathbb{P}^1) \to H^1(\mathrm{Spec}\, k);\ p_* : H^1(\mathbb{P}^1) \to H^{-1}(\mathrm{Spec}\, k)(-1)$$

are zero.

A Weil cohomology H yields an adequate equivalence relation: $\sim_H$ by

$$Z \sim_H 0 \Longleftrightarrow \gamma(Z) = 0$$

Note: $\sim_{\text{rat}} \succ \sim_H \succ \sim_{\text{num}}$.

Lemma $\sim_{\text{alg}} \succ \sim_H$.

Take $x, y \in C(k)$. $p : C \to \operatorname{Spec} k$. Then $p_* = Tr_C : H^2(C)(1) \to H^0(\operatorname{Spec} k) = K$ is an isomorphism and

$$Tr_C(\gamma_C(x-y)) = \gamma_{\operatorname{Spec} k}(p_*(x-y)) = 0$$

so $\gamma_C(x-y) = 0$. Promote to $\sim_{\text{alg}}$ by naturality of γ.

Conjecture: $\sim_H$ is independent of the choice of Weil cohomology H.

We write $\sim_H$ as $\sim_{\text{hom}}$.

Lefschetz trace formula

$V = \oplus_r V_r$: a graded K-vector space with dual $V^\vee = \oplus_r V^\vee_{-r}$ and duality pairing

$$<,>_V : V \otimes V^\vee \to K.$$

Identify $(V^\vee)^\vee = V$ by $<v^\vee, v>_{V^\vee} := (-1)^{\deg v}<v, v^\vee>$.

$\mathrm{Hom}_{\mathsf{GrVec}}(V,V) \cong \oplus_r V^\vee_{-r} \otimes V_r$ and for $f = v^\vee \otimes v : V \to V$ the graded trace is

$$Tr_V f = <v^\vee, v> = (-1)^{\deg v} v^\vee(v).$$

The graded trace is $(-1)^r$ times the usual trace on V_r.

If $W = \oplus_s W_s$ is another graded K vector space, identify $(V^\vee \otimes W)^\vee = V \otimes W^\vee$ by the pairing

$$<v^\vee \otimes w, v \otimes w^\vee> := (-1)^{\deg w \deg v}<v^\vee, v><w, w^\vee>$$

Given

$$\phi \in \mathsf{Hom}_{\mathsf{GrVec}}(V, W) \subset V^\vee \otimes W$$
$$\psi \in \mathsf{Hom}_{\mathsf{GrVec}}(W, V) \subset W^\vee \otimes V$$

get $\phi \circ \psi : W \to W$.

Let $c : W^\vee \otimes V \to V \otimes W^\vee$ be the exchange isomorphism, giving

$$c(\psi) \in V \otimes W^\vee = (V^\vee \otimes W)^\vee.$$

Checking on decomposable tensors gives the LTF:

$$Tr_W(\phi \circ \psi) = <\phi, c(\psi)>_{V^\vee \otimes W}.$$

Apply the LTF to $V = W = H^*(X)$. We have

$$V^\vee = H^*(X)(d_X)$$
$$\oplus_r V_r \otimes V_{-r}^\vee = \oplus_r H^r(X) \otimes H^{2d_X - r}(X)(d_X) = H^{2d_X}(X \times X)(d_X)$$
$$<,>_V = Tr_X \circ \delta_X^* : H^{2d_X}(X \times X)(d_X) \to K$$

Theorem (Lefschetz trace formula) *Let* $a, b \in \mathcal{Z}^{d_X}(X \times X)$ *be correspondences. Then*

$$\deg(a \cdot {}^t b) = \sum_{i=0}^{2d_X} (-1)^i Tr(a_* \circ b_*)_{|H^i(X)}.$$

Just apply the LTF to $\phi = a_* = H^*(a)$, $\psi = b_* = H^*(b)$ and note: H^* intertwines ^{t} and c and $\deg(a \cdot {}^t b) = <H^*(a), H^*({}^t b)>_{H^*(X)}$.

Taking $b = \Delta_X$ gives the Lefschetz fixed point formula.

Classical Weil cohomology

1. Betti cohomology ($K = \mathbb{Q}$): $\sigma : k \to \mathbb{C} \rightsquigarrow H^*_{\mathfrak{B},\sigma}$

$$H^*_{\mathfrak{B},\sigma}(X) := H^*(X_\sigma(\mathbb{C}), \mathbb{Q})$$

2. De Rham cohomology ($K = k$, for $\operatorname{char} k = 0$):

$$H^*_{dR}(X) := \mathbb{H}^*_{\mathsf{Zar}}(X, \Omega^*_{X/k})$$

3. Étale cohomology ($K = \mathbb{Q}_\ell$, $\ell \neq \operatorname{char} k$):

$$H^*_{\text{ét}}(X)_\ell := H^*_{\text{ét}}(X \times_k k^{sep}, \mathbb{Q}_\ell)$$

In particular: for each k, there exists a Weil cohomology theory on $\mathbf{SmProj}/k$.

An application

Proposition *Let F be a field of characteristic zero. $X \in \mathbf{SmProj}/k$. Then the intersection pairing*

$$\cdot_X : \mathcal{Z}^r_{\mathrm{num}}(X)_F \otimes_F \mathcal{Z}^{d_X-r}_{\mathrm{num}}(X)_F \to F$$

is a perfect pairing for all r.

Proof. May assume $F =$ the coefficient field of a Weil cohomology H^* for k.

$$H^{2r}(X)(r) \hookleftarrow \mathcal{Z}^r_{\mathrm{hom}}(X)_F \twoheadrightarrow \mathcal{Z}^r_{\mathrm{num}}(X)_F$$

so $\dim_F \mathcal{Z}^r_{\mathrm{num}}(X)_F < \infty$.

By definition of $\sim_{\mathrm{num}}$, $\cdot_X$ is non-degenerate; since the factors are finite dimensional, $\cdot_X$ is perfect.

Matsusaka's theorem (weak form)

Proposition $\mathcal{Z}^1_{\mathsf{alg}}(X)_{\mathbb{Q}} = \mathcal{Z}^1_H(X)_{\mathbb{Q}} = \mathcal{Z}^1_{\mathsf{num}}(X)_{\mathbb{Q}}$.

Proof. Matsusaka's theorem is $\mathcal{Z}^1_{\mathsf{alg}\mathbb{Q}} = \mathcal{Z}^1_{\mathsf{num}\mathbb{Q}}$.

But $\sim_{\mathsf{alg}} \succ \sim_H \succ \sim_{\mathsf{num}}$.

Grothendieck motives

How to construct the category of motives for an adequate equivalence relation $\sim$.

Pseudo-abelian categories

An additive category $\mathcal{C}$ is *abelian* if every morphism $f : A \to B$ has a (categorical) kernel and cokernel, and the canonical map $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ is always an isomorphism.

An additive category $\mathcal{C}$ is *pseudo-abelian* if every idempotent endomorphism $p : A \to A$ has a kernel:

$$A \cong \ker p \oplus \ker 1 - p.$$

The pseudo-abelian hull

For an additive category $\mathcal{C}$, there is a universal additive functor to a pseudo-abelian category $\psi : \mathcal{C} \to \mathcal{C}^\natural$.

$\mathcal{C}^\natural$ has objects (A, p) with $p : A \to A$ an idempotent endomorphism,

$$\mathrm{Hom}_{\mathcal{C}^\natural}((A,p),(B,q)) = q\mathrm{Hom}_{\mathcal{C}}(A,B)p.$$

and $\psi(A) := (A, \mathrm{id})$, $\psi(f) = f$.

If $\mathcal{C}, \otimes$ is a tensor category, so is $\mathcal{C}^\natural$ with

$$(A,p) \otimes (B,q) := (A \otimes B, p \otimes q).$$

Correspondences again

The category $\mathrm{Cor}_\sim(k)$ has the same objects as $\mathbf{SmProj}/k$. Morphisms (for X irreducible) are

$$\mathrm{Hom}_{\mathrm{Cor}_\sim}(X,Y) := \mathcal{Z}_\sim^{d_X}(X \times Y)_{\mathbb{Q}}$$

with composition the composition of correspondences.

In general, take the direct sum over the components of X.

Write X (as an object of $\mathrm{Cor}_\sim(k)$) $= h_\sim(X)$ or just $h(X)$. For $f : Y \to X$, set $h(f) := {}^t\Gamma_f$. This gives a functor

$$h_\sim : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Cor}_\sim(k).$$

1. $\mathrm{Cor}_\sim(k)$ is an additive category with $h(X) \oplus h(Y) = h(X \amalg Y)$.

2. $\mathrm{Cor}_\sim(k)$ is a tensor category with $h(X) \otimes h(Y) = h(X \times Y)$. For $a \in \mathcal{Z}_\sim^{d_X}(X \times Y)_\mathbb{Q}$, $b \in \mathcal{Z}_\sim^{d_{X'}}(X' \times Y')_\mathbb{Q}$

$$a \otimes b := t^*(a \times b)$$

with $t : (X \times X') \times (Y \times Y') \to (X \times Y) \times (X' \times Y')$ the exchange.

$h_\sim$ is a symmetric monoidal functor.

Effective pure motives

Definition $M_\sim^{\text{eff}}(k) := \text{Cor}_\sim(k)^\natural$. For a field $F \supset \mathbb{Q}$, set

$$M_\sim^{\text{eff}}(k)_F := [\text{Cor}(k)_F]^\natural$$

Explicitly, $M_\sim^{\text{eff}}(k)$ has objects (X, α) with $X \in \mathbf{SmProj}/k$ and $\alpha \in \mathcal{Z}_\sim^{d_X}(X \times X)_\mathbb{Q}$ with $\alpha^2 = \alpha$ (as correspondence mod $\sim$).

$M_\sim^{\text{eff}}(k)$ is a tensor category with unit $\mathbb{1} = (\text{Spec}\, k, [\text{Spec}\, k])$.

Set $\mathfrak{h}_\sim(X) := (X, \Delta_X)$, for $f : Y \to X$, $\mathfrak{h}_\sim(f) := {}^t\Gamma_f$.

This gives the symmetric monoidal functor

$$\mathfrak{h}_\sim : \mathbf{SmProj}(k)^{\text{op}} \to M_\sim^{\text{eff}}(k).$$

Universal property

Theorem *Let H be a Weil cohomology on $\mathbf{SmProj}/k$. Then the functor $H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ extends to a tensor functor $H^* : M_{\mathrm{hom}}^{\mathrm{eff}}(k) \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ making*

$$\begin{tikzcd} \mathbf{SmProj}/k^{\mathrm{op}} \arrow[d, "\mathfrak{h}"'] \arrow[dr, "H^*"] & \\ M_{\mathrm{hom}}^{\mathrm{eff}}(k) \arrow[r, "H^*"'] & \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K. \end{tikzcd}$$

commute.

Proof. Extend H^* to $\mathrm{Cor}_{\mathrm{hom}}(k)$ by $H^*(a) = a_*$ for each correspondence a. Since $\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ is pseudo-abelian, H^* extends to $M_{\mathrm{hom}}^{\mathrm{eff}}(k) = \mathrm{Cor}_{\mathrm{hom}}(k)^{\natural}$.

Examples 1. $\Delta_{\mathbb{P}^1} \sim \mathbb{P}^1 \otimes 0 + 0 \otimes \mathbb{P}^1$ gives

$$\mathfrak{h}(P^1) = (\mathbb{P}^1, \mathbb{P}^1 \otimes 0) \oplus (\mathbb{P}^1, 0 \times \mathbb{P}^1).$$

The maps $i_0 : \operatorname{Spec} k \to \mathbb{P}^1$, $p : \mathbb{P}^1 \to \operatorname{Spec} k$, give

$$p^* : \mathfrak{h}(\operatorname{Spec} k) \to \mathfrak{h}(\mathbb{P}^1)$$
$$i_0^* : \mathfrak{h}(\mathbb{P}^1) \to \mathfrak{h}(\operatorname{Spec} k)$$

and define an isomorphism

$$\mathbb{1} \cong (\mathbb{P}^1, 0 \times \mathbb{P}^1).$$

The remaining factor $(\mathbb{P}^1, \mathbb{P}^1 \otimes 0)$ is the *Lefschetz motive* $\mathbb{L}$.

2. $\Delta_{\mathbb{P}^n} \sim \sum_{i=0}^{n} \mathbb{P}^i \times \mathbb{P}^{n-i}$. The $\mathbb{P}^i \times \mathbb{P}^{n-i}$ are orthogonal idempotents so

$$\mathfrak{h}(\mathbb{P}^n) = \oplus_{i=0}^{n} (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}).$$

In fact $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong \mathbb{L}^{\otimes i}$ so

$$\mathfrak{h}(\mathbb{P}^n) \cong \oplus_{i=0}^{n} \mathbb{L}^i.$$

3. Let C be a smooth projective curve with a k-point 0. $0 \times C$ and $C \times 0$ are orthogonal idempotents in Cor(C, C). Let $\alpha := \Delta_C - 0 \times C - C \times 0$ so

$$\mathfrak{h}(C) = (C, 0 \times C) + (C, \alpha) + (C, C \times 0) \cong \mathbb{1} \oplus (C, \alpha) \oplus \mathbb{L}$$

Each decomposition of $\mathfrak{h}(X)$ in $M^{\mathrm{eff}}_{\mathrm{hom}}(k)$ gives a corresponding decomposition of $H^*(X)$ by using the action of correspondences on H^*.

1. The decomposition $\mathfrak{h}(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$ decomposes $H^*(\mathbb{P}^1)$ as $H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)$, with $\mathbb{1} \leftrightarrow H^0(\mathbb{P}^1) = K$ and $\mathbb{L} \leftrightarrow H^2(\mathbb{P}^1) = K(-1)$. Set

$$\mathfrak{h}^0_\sim(\mathbb{P}^1) := (\mathbb{P}^1, 0 \times \mathbb{P}^1), \mathfrak{h}^2_\sim(\mathbb{P}^2) := (\mathbb{P}^1, \mathbb{P}^1 \times 0)$$

so $\mathfrak{h}_\sim(\mathbb{P}^1) = \mathfrak{h}^0_\sim(\mathbb{P}^1) \oplus \mathfrak{h}^2_\sim(\mathbb{P}^1)$ and

$$H^*(\mathfrak{h}^i_{\mathrm{hom}}(\mathbb{P}^1)) = H^i(\mathbb{P}^1)$$

2. The factor $(\mathbb{P}^n, \mathbb{P}^{n-i} \times \mathbb{P}^i)$ of $[\mathbb{P}^n]$ acts by

$$(\mathbb{P}^i \times \mathbb{P}^{n-i})_* : H^*(\mathbb{P}^n) \to H^*(\mathbb{P}^n)$$

which is projection onto the summand $H^{2i}(\mathbb{P}^n)$. Since $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong \mathbb{L}^{\otimes i}$ this gives

$$H^{2i}(\mathbb{P}^n) \cong K(-i) \cong H^2(\mathbb{P}^1)^{\otimes i}.$$

Setting $\mathfrak{h}^{2i}_{\sim}(\mathbb{P}^n) := (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i})$ gives

$$\mathfrak{h}_{\sim}(\mathbb{P}^n) = \oplus_{i=0}^n \mathfrak{h}^{2i}_{\sim}(\mathbb{P}^n),$$

with $H^*(\mathfrak{h}^r_{\text{hom}}(\mathbb{P}^n)) = H^r(\mathbb{P}^n)$.

3. The decomposition $\mathfrak{h}_\sim(C) = \mathbb{1} \oplus (C, \alpha) \oplus \mathbb{L}$ gives

$$H^*(C) = H^0(C) \oplus H^1(C) \oplus H^2(C) = K \oplus H^1(C) \oplus K(-1).$$

Thus we write $\mathfrak{h}^1(C) := (C, \alpha)$, $\mathfrak{h}^0_\sim(C) := (C, 0 \times C)$, $\mathfrak{h}^2_\sim(C) := (C, C \times 0)$ and

$$\mathfrak{h}_\sim(C) \cong \mathfrak{h}^0_\sim(C) \oplus \mathfrak{h}^1_\sim(C) \oplus \mathfrak{h}^2_\sim(C).$$

with $H^*(\mathfrak{h}^r_{\mathrm{hom}}(C)) = H^r(C)$.

Note. $\mathfrak{h}^1_\sim(C) \neq 0$ iff $g(C) \geq 1$. It suffices to take $\sim =$ num. Since $\dim C \times C = 2$, it suffices to show $\mathfrak{h}^1_{\mathrm{hom}}(C) \neq 0$ for some classical Weil cohomology. But then $H^1(C) \cong K^{2g}$.

The decompositions in (1) and (2) are canonical. In (3), this depends (for e.g $\sim =$ rat, but not for $\sim =$ hom, num) on the choice of $0 \in C(k)$ (or degree 1 cycle $0 \in \mathrm{CH}_0(C)_{\mathbb{Q}}$).

Grothendieck motives

Definition 1. $\mathrm{Cor}^*_\sim(k)$ has objects $h(X)(r)$, $r \in \mathbb{Z}$ with

$$\mathrm{Hom}_{\mathrm{Cor}^*_\sim(k)}(h(X)(r), h(Y)(s)) := \mathcal{Z}_\sim^{d_X+s-r}(X \times Y)$$

with composition as correspondences.

2. $M_\sim(k) := \mathrm{Cor}^*_\sim(k)^\natural$. For a field $F \supset \mathbb{Q}$, set

$$M_\sim(k)_F := [\mathrm{Cor}^*(k)_F]^\natural$$

Sending X to $\mathfrak{h}(X) := h(X)(0)$, $f : Y \to X$ to ${}^t\Gamma_f$ defines the functor

$$\mathfrak{h}_\sim : \mathbf{SmProj}/k^{\mathrm{op}} \to M_\sim(k).$$

Examples 1. $0 \in \mathcal{Z}^1(\mathbb{P}^1)$ gives a map $i_0 : \mathbb{1}(-1) \to \mathfrak{h}(\mathbb{P}^1)$, identifying

$$\mathbb{1}(-1) \cong \mathbb{L}.$$

2. $\mathbb{1}(-r) \cong \mathbb{L}^{\otimes r}$, so $\mathfrak{h}(\mathbb{P}^n) \cong \oplus_{r=0}^{n} \mathbb{1}(-r)$ and $\mathfrak{h}^{2r}(\mathbb{P}^n) = \mathbb{1}(-r)$

3. For C a curve, $\mathfrak{h}^0(C) = \mathbb{1}$, $\mathfrak{h}^2(C) = \mathbb{1}(-1)$.

4. The objects $\mathfrak{h}(X)(r)$ are *not* in $M_{\sim}^{\mathrm{eff}}(k)$ for $r > 0$.

For $r < 0$ $\mathfrak{h}(X)(r) \cong \mathfrak{h}(X) \otimes \mathbb{L}^{\otimes r}$.

Inverting $\mathbb{L}$

Sending $(X, \alpha) \in M_\sim^{\text{eff}}(k)$ to $(X, 0, \alpha) \in M_\sim(k)$ defines a full embeding

$$i : M_\sim^{\text{eff}}(k) \hookrightarrow M_\sim(k).$$

Since $i(\mathbb{L}) \cong \mathbb{1}(-1)$, the functor $\otimes\mathbb{L}$ on $M_\sim^{\text{eff}}(k)$ has inverse $\otimes\mathbb{1}(1)$ on $M_\sim(k)$.

$(X, r, \alpha) = (X, 0, \alpha) \otimes \mathbb{1}(r) \cong i(X, \alpha) \otimes \mathbb{L}^{\otimes -r}$.

Thus $M_\sim(k) \cong M_\sim^{\text{eff}}(k)[(- \otimes \mathbb{L})^{-1}]$.

Universal property Let GrVec_K be the tensor category of finite dimensional graded K vector spaces.

Theorem *Let H be a Weil cohomology on $\mathbf{SmProj}/k$. Then the functor $H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ extends to a tensor functor $H^* : M_{\mathrm{hom}}(k) \to \mathrm{GrVec}_K$ making*

$$\begin{tikzcd} \mathbf{SmProj}/k^{\mathrm{op}} \arrow[d, "\mathfrak{h}"'] \arrow[dr, "H^*"] & \\ M_{\mathrm{hom}}(k) \arrow[r, "H^*"'] & \mathrm{GrVec}_K. \end{tikzcd}$$

commute.

Proof. Extend H^* to $H^* : \mathrm{Cor}^*_{\mathrm{hom}}(k) \to$ by

$$H^n(X, r) := H^n(X)(r),\ H^*(a) = a_*$$

for each correspondence a. Since GrVec_K is pseudo-abelian, H^* extends to $M_{\mathrm{hom}}(k) = \mathrm{Cor}^*_{\mathrm{hom}}(k)^\natural$.

Duality

Why extend $M^{\mathsf{eff}}(k)$ to $M(k)$? In $M(k)$, each object has a dual:

$$(X, r, \alpha)^{\vee} := (X, d_X - r, {}^t\alpha)$$

The diagonal Δ_X yields

$$\delta_X : \mathbb{1} \to \mathfrak{h}(X \times X)(d_X) = \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^{\vee}$$
$$\epsilon_X : \mathfrak{h}(X)(r)^{\vee} \otimes \mathfrak{h}(X)(r) = \mathfrak{h}(X \times X)(d_X) \to \mathbb{1}$$

with composition

$$\mathfrak{h}(X)(r) = \mathbb{1} \otimes \mathfrak{h}(X) \xrightarrow{\delta \otimes \mathrm{id}} \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^{\vee} \otimes \mathfrak{h}(X)(r)$$
$$\xrightarrow{\mathrm{id} \otimes \epsilon} \mathfrak{h}(X)(r) \otimes \mathbb{1} = \mathfrak{h}(X)$$

the identity.

This yields a natural isomorphism

$$\mathrm{Hom}(A \otimes \mathfrak{h}(X)(r), B) \cong \mathrm{Hom}(A, B \otimes \mathfrak{h}(X)(r)^\vee)$$

by sending $f : A \otimes \mathfrak{h}(X)(r) \to B$ to

$$A = A \otimes \mathbb{1} \xrightarrow{\delta} A \otimes \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^\vee \xrightarrow{f \otimes \mathrm{id}} B \otimes \mathfrak{h}(X)(r)^\vee$$

The inverse is similar, using ϵ.

This extends to objects (X, r, α) by projecting. $A \to (A^\vee)^\vee = A$ is the identity.

Theorem *$M_\sim(k)$ is a* rigid *tensor category. For $\sim=$ hom, the functor H^* is compatible with duals.*

Chow motives and numerical motives

If $\sim \succ \approx$, the surjection $\mathcal{Z}_\sim \to \mathcal{Z}_\approx$ yields functors $\mathrm{Cor}_\sim(k) \to \mathrm{Cor}_\approx(k)$, $\mathrm{Cor}^*_\sim(k) \to \mathrm{Cor}^*_\approx(k)$ and thus

$$M^{\mathrm{eff}}_\sim(k) \to M^{\mathrm{eff}}_\approx(k);\ M_\sim(k) \to M_\approx(k).$$

Thus the category of pure motives with the most information is for the finest equivalence relation $\sim=$ rat. Write

$$CHM(k)_F := M_{\mathrm{rat}}(k)_F$$

For example $\mathrm{Hom}_{CHM(k)}(\mathbb{1}, \mathfrak{h}(X)(r)) = \mathrm{CH}^r(X)$.

The coarsest equivalence is $\sim_{\mathrm{num}}$, so $M_{\mathrm{num}}(k)$ should be the most simple category of motives.

Set $NM(k) := M_{\mathrm{num}}(k)$, $NM(k)_F := M_{\mathrm{num}}(k)_F$.

Jannsen's semi-simplicity theorem

Theorem (Jannsen) *Fix F a field, char $F = 0$. $NM(k)_F$ is a semi-simple abelian category. If $M_\sim(k)_F$ is semi-simple abelian, then* $\sim =$ num.

Proof. We show $\mathrm{End}_{NM(k)_F}(\mathfrak{h}(X)) = \mathcal{Z}_{\mathrm{num}}(X^2)_F$ is a finite dimensional semi-simple F-algebra for all $X \in \mathbf{SmProj}/k$. We may extend F, so can assume $F = K$ is the coefficient field for a Weil cohomology on $\mathbf{SmProj}/k$.

Consider the surjection $\pi : \mathcal{Z}_{\mathrm{hom}}(X^2)_F \to \mathcal{Z}_{\mathrm{num}}(X^2)_F$. $\mathcal{Z}_{\mathrm{hom}}(X^2)_F$ is finite dimensional, so $\mathcal{Z}_{\mathrm{num}}(X^2)_F$ is finite dimensional.

Also, the radical $\mathcal{N}$ of $\mathcal{Z}_{\mathrm{hom}}(X^2)_F$ is nilpotent and it suffices to show that $\pi(\mathcal{N}) = 0$.

Take $f \in \mathcal{N}$. Then $f \circ {}^t g$ is in $\mathcal{N}$ for all $g \in \mathcal{Z}_{\mathrm{hom}}(X^2)_F$, and thus $f \circ {}^t g$ is nilpotent. Therefore

$$Tr(H^+(f \circ {}^t g)) = Tr(H^-(f \circ {}^t g)) = 0.$$

By the LTF

$$\deg(f \cdot g) = Tr(H^+(f \circ {}^t g)) - Tr(H^-(f \circ {}^t g)) = 0$$

hence $f \sim_{\mathrm{num}} 0$.

Chow motives $CHM(k)_F$ has a nice universal property extending the one we have already described:

Theorem *Giving a Weil cohomology theory H^* on $\mathbf{SmProj}/k$ with coefficient field $K \supset F$ is equivalent to giving a tensor functor*

$$H^* : CHM(k)_F \to \mathsf{GrVec}_K$$

with $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2$.

"Weil cohomology" $\rightsquigarrow H^*$ because $\sim_{\mathsf{rat}} \succ \sim_H$.

$H^* \rightsquigarrow$ Weil cohomology: $\mathbb{1}(-1)$ is invertible and $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2 \implies H^2(\mathbb{P}^1) \cong K$.

$\mathfrak{h}(X)^\vee = \mathfrak{h}(X)(d_X) \rightsquigarrow H^*(\mathfrak{h}(X))$ is supported in degrees $[0, 2d_X]$

Rigidity of $CHM(k)_F \rightsquigarrow$ Poincaré duality.

Adequate equivalence relations revisited

Definition Let $\mathcal{C}$ be an additive category. The *Kelly radical* $\mathcal{R}$ is the collection

$$\mathcal{R}(X,Y) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid \forall g \in \mathrm{Hom}_{\mathcal{C}}(Y,X), 1-gf \text{ is invertible}\}$$

$\mathcal{R}$ forms an *ideal* in $\mathcal{C}$ (subgroups of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ closed under $\circ g$, $g\circ$).

Lemma *$\mathcal{C} \to \mathcal{C}/\mathcal{R}$ is conservative, and $\mathcal{R}$ is the largest such ideal.*

Note. If $\mathcal{I} \subset \mathcal{C}$ is an ideal such that $\mathcal{I}(X,X)$ is a nil-ideal in $\mathrm{End}(X)$ for all X, then $\mathcal{I} \subset \mathcal{R}$.

Definition $(\mathcal{C}, \otimes)$ a tensor category. A ideal $\mathcal{I}$ in $\mathcal{C}$ is a $\otimes$ *ideal* if $f \in \mathcal{I}$, $g \in \mathcal{C} \Rightarrow f \otimes g \in \mathcal{I}$.

$\mathcal{C} \to \mathcal{C}/\mathcal{I}$ is a tensor functor iff $\mathcal{I}$ is a tensor ideal. $\mathcal{R}$ is *not* in general a $\otimes$ ideal.

Theorem *There is a 1-1 correspondence between adequate equivalence relations on* $\mathbf{SmProj}/k$ *and proper* $\otimes$ *ideals in* $CHM(k)_F$*:* $M_\sim(k)_F := (CHM(k)_F/\mathcal{I}_\sim)^\natural$.

In particular: Let $\mathcal{N} \subset CHM(k)_\mathbb{Q}$ be the tensor ideal defined by numerical equivalence. Then $\mathcal{N}$ is the largest proper $\otimes$ ideal in $CHM(k)_\mathbb{Q}$.