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Further K-theoretic results for simple algebraic groups

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May 21: 6. Further K-theoretic results for simple algebraic groups

Most K-theoretic results for so called "classical groups" -- besides SL_n - Unitary groups for various (skew-)hermitian forms are for quasisplit groups, i.e., for such having a Borel subgroup, and the methods are variations of those used by Steinberg - Matsumoto (cf. Hahn, O'Meara "The classical groups and K-Theory, Springer, Grundlehren 291, 1989)

However, a few results concern groups with nontrivial anisotropic kernel:

<u>On K_2 of skew fields</u>

For a skew field D there is the "Dieudonné determinant"

$$\det : GL_n(D) \longrightarrow D^*/[D^*, D^*]$$

which has essentially the same properties as the determinant for fields. Its definition specifies to the <u>ordinary det</u> if D is commutative. Its kernel $SL_n(D)$ is generated by the elementary matrices $u_{ij}(x) = 1 + x e_{ij}$, $i \neq j$, $x \in D$.

Again we have for $n \geq 3$ (we omit $n = 2$)

(A) $u_{ij}(x+y) = u_{ij}(x)\, u_{ij}(y)$

(B) $[u_{ij}(x), u_{kl}(y)] = \begin{cases} u_{il}(xy) & j = k,\ i \neq l \\ u_{kj}(-yx) & i = l,\ j \neq k \\ 1 & \text{otherwise} \end{cases}$ if $(j,i) \neq (k,l)$

(C) $h_{ij}(x)\, h_{ij}(y) = h_{ij}(xy)$ $\qquad x, y \in D^*$

with the standard definitions

$$w_{ij}(x) = x_{ij}(x)\, x_{ij}(-x^{-1})\, x_{ij}(x), \quad h_{ij}(x) = w_{ij}(x)\, w_{ij}(-1)$$

By Steinberg (Yale lectures), the group $St_n(D)$ presented by (A), (B) is a universal central ext. of the perfect group $SL_n(D)$, again, its kernel is $K_2(n,D)$. One also has a Bruhat decomposition:

upper tri   monom

$SL_n(D) = U H U$

let $\pi : St_n(D) \longrightarrow SL_n(D)$
be the canonical map, let us denote the generators of $St_n(D)$ by $\tilde{x}_{ij}(x)$, $\tilde{w}_{ij}(x)$, $\tilde{h}_{ij}(x)$.

Now, the elements

$$c_{ij}(x,y) := \tilde{h}_{ij}(xy)\,\tilde{h}_{ij}(x)^{-1}\,\tilde{h}_{ij}(y)^{-1} \in St_n(D)$$

are <u>no longer in kernel π</u>, as they map to

$$\begin{array}{cc} & i \\ & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & xyx^{-1}y^{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} i$$

However, for $x_i, y_i \in D^*$ such that $\prod [x_i, y_i] = 1$
one has, obviously,

$$\prod c(x_i, y_i) \in \operatorname{Ker} \pi \quad \left(\begin{array}{l}\text{and all elements of } \ker \pi \\ \text{are of this type}\end{array}\right)$$

and one has the following <u>replacement</u>
<u>for Matsumoto's theorem</u> (UR, 1977)

Let U_D denote the group generated by $c(x,y)$ $(x, y \in D^*)$, subject to the following relations:

(S°1) $\quad c(xy, z) = c({}^{x}y, {}^{x}z)\, c(x,z)$

(S°2) $\quad c(x, yz) = c(x,y)\, c({}^{y}x, {}^{y}z)$

(S°3) $\quad c(x, 1-x) = 1$

Then the map $U_D \longrightarrow [D^*, D^*]$ $(c(x,y) \longmapsto [x,y])$ defines a central extension of $[D^*, D^*]$, moreover, $U_D \hookrightarrow St_m(D)$ injects via

$$c(x,y) \longmapsto c_{12}(x,y) = \tilde{h}_{12}(xy)\,\tilde{h}_{12}(x)^{-1}\,\tilde{h}_{12}(y)^{-1}.$$

(implicitly $K_2(m,D) = K_2(D)$ for $m \geq 3$ and)

Hence one has an exact sequence

$$0 \to K_2(D) \to U_D \to [D^*, D^*] \to 0.$$

Remark 1:
Obviously, this gives Matsumoto's theorem for D commutative, and in general relates $K_2(D)$ to a central extension of $[D^*, D^*]$.

Remark 2: The relations (S°1), (S°2) together with the relations $c(x,x) = 1$ will give a generating set for all formal commutator relations of a group H, if x, y vary over H. Hochschild-Serre: $H_2(U_D) \to H_2([D^*, D^*]) \to K_2(D) \to H_1(U_D) \to H_1([D^*, D^*]) \to 1$ all with values in $\mathbb{Z}$

Remark 3:
$Sl_m(D)$ in general is not an algebraic group, as the Dieudonné determinant is not a polynomial function.

However, this is true if D is a finite central k-div-Alg:
Then both D and $M_m(D)$ are central simple k-alg, finite over k.

Digression on the reduced norm:
let A be a finite central simple k-alg.
Then $A \otimes_k \bar{k} \simeq M_m(\bar{k})$ for some m, hence $\dim_k A = m^2$.
The ch. pol $\chi_{a\otimes 1}$ of the matrix $a \otimes 1$ for $a \in A$ has coeff in k and is independent of the embedding of A in $M_m(\bar{k})$ (cf. Bourbaki, Algèbre ...):

$$\chi_{a\otimes 1}(X) = X^m - s_1(a)X^{m-1} + s_2(a)X^{m-2} \mp \dots + (-1)^m s_m(a)$$

Clearly, $s_1(a)$, $s_m(a)$ are trace and det of $a \otimes 1$, they are called reduced trace, reduced norm of a:

$$\left.\begin{aligned} rs(a) &= s_1(a) \\ rn(a) &= s_m(a) \end{aligned}\right\} \text{hence} \quad \begin{aligned} rs &: A \to k \ (k\text{-linear}) \\ rn &: A \to k \ (\text{multiplicative}) \end{aligned}$$

For D/k finite, central, the Dieudonné determinant

$$\det GL_n(D) \longrightarrow D^*/[D^*,D^*]$$

is given by the reduced norm for $M_n(D)$,

hence $SL_n(D)$ is an algebraic group.

In order to determine its type, switch (notation) to $SL_{r+1}(D)$,

assume $\dim_k D = d^2$ ($d = \mathrm{index}(D)$).

A minimal parabolic is

$$P = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = \text{group of upper triangular mat. in } SL_{r+1}(D)$$

Levi-decomp:

$$P = L \ltimes R_u(P) = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

Each $(*)$ in the L-part is a copy of $D^* = GL_1(D)$

with a central torus $\simeq \mathbb{G}_m$ (centre of $D^* = k^*$)

$r+1$ copies, but $\det = 1$ hence central torus S in L has

dimension r: $S = \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_r \end{pmatrix} \,\middle|\, \alpha_i \in k^*, \prod \alpha_i = 1 \right\}$ max k split Torus of $G = SL_{r+1}$ (?

semisimple anisotropic kernel (= ss. reductive anisotropic kernel)

$r+1$ copies of $SL_1(D)$

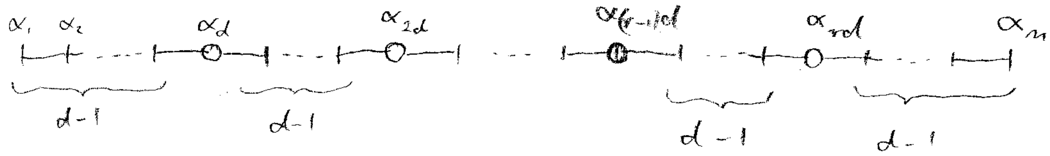
This group is of inner type (the roots are not changed under $Gal(k_s/k)$), hence its type is

$${}^1A^d_{n,r}$$

Since $SL_{r+1}(D) \times_k k_{sep} \simeq SL_{r+1}(M_d(k_{sep})) \simeq SL_{(r+1)d}(k_{sep})$

we have $\underline{n = (r+1)d - 1}$

Tits-Dynkin diagram:



Distinguished vertices: α_d , α_{2d} , $\dots$, α_{rd}

$$n+1 = (r+1)d$$

Consequences for the determination of $K_2(D)$:

Only for special fields k (cf. Stalla – R, 1978)

e.g., if k is a global function field, then

$$K_2(D) = K_2(k) \times \text{finite}$$

and if D in addition is quaternion, then:

$$K_2(D) = K_2(k)$$

But the isomorphism is not induced by the natural embedding $k \hookrightarrow D$

(it is more like $\frac{1}{d}$ of this map, $d = \text{index}(D)$).

There are also partial results ~~for~~ of this type for number fields.

On SK_1 of finite dimensional central div. alg D:

We have $SL_1(D) = \{a \in D^*,\ nr(a) = 1\}$,

$[D^*, D^*] \subseteq SL_1(D)$

Define $SK_1(D) := SL_1(D) / [D^*, D^*]$

Question: $SK_1(D) = ?$

Wang (1950) $SK_1(D) = 1$ if $\text{ind}(D) \overset{d}{=}$ is square free.

Also: $SK_1(D) = 1$ always for D/k, k local, global

But:

Platonov gave an example $\exists\, D/k$ (k two fold valued) : $SK_1(D) \neq 0$

Draxl extended: For every finite abelian group A

$\exists\, D/k$: $SK_1(D) = A$

Lit: Draxl-Kneser: SK_1 of skew fields LNM

Conjecture (Suslin 1990)

$$SK_1(D_{k(SL_1(D))}) = 1 \iff \operatorname{ind} D \text{ square free}$$

One knows (Merkurev 1993):

$\operatorname{char} k \neq 2$: there is a D/k: $\operatorname{ind} D = 4$ and

$$SK_1(D_{k(SL_1(D))}) \neq 1.$$

Compare Knus/Merkurjev/Rost/Tignol:
Book of involutions
AMS Colloq. Publ. Vol 44 (1998)