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An Introduction to K-theory

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AN INTRODUCTION TO K-THEORY

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1. $K_0(-)$, $K_1(-)$, AND $K_2(-)$

Perhaps the first new concept that arises in the study of K-theory, and one which recurs frequently, is that of the group completion of an abelian monoid.

The basic example to keep in mind is that the abelian group of integers $\mathbb{Z}$ is the group completion of the monoid $\mathbb{N}$ of natural numbers. Recall that an abelian monoid M is a set together with a binary, associative, commutative operation $+ : M \times M \to M$ and a distinguished element $0 \in M$ which serves as an identify (i.e., $0 + m = m$ for all $m \in M$). Then we define the group completion $\gamma : M \to M^+$ by setting M^+ equal to the quotient of the free abelian group with generators $[m], m \in M$ modulo the subgroup generated by elements of the form $[m] + [n] - [m + n]$ and define $\gamma : M \to M^+$ by sending $m \in M$ to $[m]$. We frequently refer to M^+ as the *Grothendieck group* of M.

The group completion map $\gamma : M \to M^+$ satisfies the following *universal property.* For any homomorphism $\phi : M \to A$ from M to a group A, there exists a unique homomorphism $\phi^+ : M^+ \to A$ such that $\phi^+ \circ \gamma = \phi : M \to A$.

1.1. **Algebraic K_0 of rings.** This leads almost immediately to K-theory. Let R be a ring (always assumed associative with unit, but not necessarily commutative). Recall that an (always assumed left) R-module P is said to be projective if there exists another R-module Q such that $P \oplus Q$ is a free R-module.

Definition 1.1. Let $\mathcal{P}(R)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of finitely generated projective R-modules. Then we define $K_0(R)$ to be $\mathcal{P}(R)^+$.

Warning: The group completion map $\gamma : \mathcal{P}(R) \to K_0(R)$ is frequently not injective.

Exercise 1.2. *Verify that if $j : R \to S$ is a ring homomorphism and if P is a finitely generated projective R-module, then $S \otimes_R P$ is a finitely generated projective S-module. Using the universal property of the Grothendieck group, you should also check that this construction determines $j_* : K_0(R) \to K_0(S)$.*

Indeed, we see that $K_0(-)$ is a (covariant) functor from rings to abelian groups.

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Example 1.3. If $R = F$ is a field, then a finitely generated F-module is just a finite dimensional F-vector space. Two such vector spaces are isomorphic if and only if they have the same dimension. Thus, $\mathcal{P}(F) \simeq \mathbb{N}$ and $K_0(F) = \mathbb{Z}$.

Example 1.4. Let $K/\mathbb{Q}$ be a finite field extension of the rational numbers (K is said to be a *number field*) and let $\mathcal{O}_K \subset K$ be the ring of algebraic integers in K. Thus, $\mathcal{O}$ is the subring of those elements $\alpha \in K$ which satisfy a monic polynomial $p_\alpha(x) \in \mathbb{Z}[x]$. Recall that $\mathcal{O}_K$ is a Dedekind domain. The theory of Dedekind domains permits us to conclude that

$$K_0(\mathcal{O}_K) \ = \ \mathbb{Z} \oplus Cl(K)$$

where $Cl(K)$ is the ideal class group of K.

A well known theorem of Minkowski asserts that $Cl(K)$ is finite for any number field K (cf. [Rosenberg]). Computing class groups is devilishly difficult. We do know that there only finitely many cyclotomic fields (i.e., of the form $\mathbb{Q}(\zeta_n)$ obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$) with class group $\{1\}$. The smallest n with non-trivial class group is $n = 23$ for which $Cl(Q(\zeta_{23})) = \mathbb{Z}/3$. A check of tables shows, for example, that $Cl(\mathbb{Q}(\zeta_{100})) = \mathbb{Z}/65$.

The reader is referred to the book by [3] for an accessible introduction to this important topic.

The K-theory of integral group rings has several important applications in topology. For a group π, the integral group ring $\mathbb{Z}[\pi]$ is defined to be the ring whose underlying abelian group is the free group on the set $[g], g \in \pi$ and whose ring structure is defined by setting $[g] \cdot [h] = [g \cdot h]$. Thus, if π is not abelian, then $\mathbb{Z}[\pi]$ is not a commutative ring.

Application 1.5. *Let X be a path-connected space with the homotopy type of a C.W. complex and with fundamental group π. Suppose that X is a retract of a finite C.W. complex. Then the Wall finiteness obstruction is an element of $K_0(\mathbb{Z}[\pi])$ which vanishes if and only if X is homotopy equivalent to a finite C.W. complex.*

1.2. **Topological K_0.** We now consider topological K-theory for a topological space X. This is also constructed as a Grothendieck group and is typically easier to compute than algebraic K-theory of a ring R. Moreover, results first proved for topological K-theory have both motivated and helped to prove important theorems in algebraic K-theory.

Definition 1.6. Let $\mathbb{F}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. An $\mathbb{F}$-vector bundle on a topological space X is a continuous open surjective map $p : E \to X$ satisfying

- (a) For all $x \in X$, $p^{-1}(x)$ is a finite dimensional $\mathbb{F}$-vector space.
- (b) There are continuous maps $E \times E \to E, \mathbb{F} \times E \to E$ which provide the vector space structure on $p^{-1}(x)$, all $x \in X$.

- (c) For all $x \in X$, there exists an open neighborhood $U_x \subset X$, an $\mathbb{F}$-vector space V, and a homeomorphism $\psi_x : V \times U_x \to p^{-1}(U_x)$ over U_x (i.e., $pr_2 = p \circ \psi_x : V \times U_x \to U_x$) compatible with the structure in (b.).

Example 1.7. Let $X = S^1$, the circle. The projection of the Möbius band M to its equator $p : M \to S^1$ is a rank 1, real vector bundle over S^1.

Let $X = S^2$, the 2-sphere. Then the projection $p : T_{S^2} \to S^2$ of the tangent bundle is a non-trivial vector bundle.

Let $X = S^2$, but now view X as the complex projective line, so that points of X can be viewed as complex lines through the origin in $\mathbb{C}^2$ (i.e., complex subspaces of $\mathbb{C}^2$ of dimension 1). Then there is a natural rank 1, complex line bundle $E \to X$ whose fibre above $x \in X$ is the complex line parametrized by x; if $E - o(X) \to X$ denotes the result of removing the origin of each fibre, then we can identify $E - o(X)$ with $\mathbb{C}^2 - \{0\}$.

Definition 1.8. Let $Vect_{\mathbb{F}}(X)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of $\mathbb{F}$-vector bundles of X. We define

$$K^0_{top}(X) = Vect_{\mathbb{C}}(X)^+, \quad KO^0_{top}(X) = Vect_{\mathbb{R}}(X)^+.$$

(This definition will agree with our more sophisticated definition of topological K-theory given in a later lecture provided that the X has the homotopy type of a finite dimensional C.W. complex.)

The reason we use a superscript 0 rather than a subscript 0 for topological K-theory is that it determines a contravariant functor. Namely, if $f : X \to Y$ is a continuous map of topological spaces and if $p : E \to Y$ is an $\mathbb{F}$-vector bundle on Y, then $pr_2 : E \times_Y X \to X$ is an $\mathbb{F}$-vector bundle on X. This determines

$$f^* : K^0_{top}(Y) \to K^0_{top}(X).$$

Example 1.9. Let n_{S^2} denote the "trivial" rank n, real vector bundle over S^2 (i.e., $pr_2 : \mathbb{R}^n \times S^2 \to S^2$) and let T_{S^2} denote the tangent bundle of S^2. Then $T_{S^2} \oplus 1_{S^2} \simeq 3_{S^2}$. We conclude that $Vect_{\mathbb{R}}(S^2) \to K\mathcal{O}^0_{top}(S^2)$ is *not injective* in this case.

Here is one of the early theorems of K-theory, a theorem proved by Richard Swan. You can find a full proof, for example, in [4].

Theorem 1.10. *(Swan) Let $\mathbb{F} = \mathbb{R}$ (respectively, $= \mathbb{C}$), let X be a compact Hausdorff space, and let $\mathcal{C}(X, \mathbb{F})$ denote the ring of continuous functions $X \to \mathbb{F}$. For any $E \in Vect_{\mathbb{F}}(X)$, define the $\mathbb{F}$-vector space of global sections $\Gamma(X, E)$ to be*

$$\Gamma(X, E) = \{s : X \to E \text{ continuous}; p \circ s = id_X\}.$$

Then sending E to $\Gamma(X, E)$ determines isomorphisms

$$KO^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbb{R})), \quad K^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbb{C})).$$

1.3. **Quasi-projective Varieties.** We briefly recall a few basic notions of classical algebraic geometry. Let us assume our ground field k is algebraically closed, so that we need only consider k-rational points. For more general fields k, we could have to consider "points with values in some finite field extension L/k."

Recollection 1.11. Recall projective space $\mathbb{P}^N$, whose k-rational points are equivalence classes of $N+1$-tuple, $\langle a_0, \ldots, a_N \rangle$, some entry of which is non-zero. Two $N+1$-tuples $(a_0, \ldots, a_N), (b_0, \ldots, b_N)$ are equivalent if there exists some $0 \neq c \in k$ such that $(a_0, \ldots, a_N) = (cb_0, \ldots, cb_N)$.

If $F(X_0, \ldots, X_N)$ is a homogeneous polynomial, then the zero locus $Z(F) \subset \mathbb{P}^N$ is well defined.

Recall that $\mathbb{P}^N$ is covered by standard affine opens $U_i = \mathbb{P}^N \backslash Z(X_i)$.

Recall the Zariski topology on $\mathbb{P}^N$, a base of open sets for which are the subsets of the form $U_G = \mathbb{P}^N \backslash Z(G)$.

Recollection 1.12. Recall the notion of a presheaf on a topological space T: a contravariant functor from the category whose objects are open subsets of T and whose morphisms are inclusions.

Recall that a sheaf is a presheaf satisfying the sheaf axiom: for T compact, this axiom can be simply expressed as requiring for each pair of open subsets U, V that

$$F(U \cup V) = F(U) \times_{F(U \cap V)} F(V).$$

Recall the structure sheaf of "regular functions" $\mathcal{O}_{\mathbb{P}^N}$ on $\mathbb{P}^N$, sections of $\mathcal{O}_{\mathbb{P}^N}(U)$ on any open U are given by quotients $\frac{P(X_0,\ldots,X_N)}{Q(X_0,\ldots,X_N)}$ of homogeneous polynomilas of the same degree satisfying the condition that Q has no zeros in U. In particular,

$$\mathcal{O}_{\mathbb{P}^N}(U_G) = \{F(X)/G^j, j \geq 0; F \text{ homgeneous of deg} = j \cdot deg(G)\}.$$

Definition 1.13. A projective variety X is a space with a sheaf of commutative rings $\mathcal{O}_X$ which admits a closed embedding into some $\mathbb{P}^N$, $i : X \subset \mathbb{P}^N$, so that $\mathcal{O}_X$ is the quotient of the sheaf $\mathcal{O}_{\mathbb{P}^N}$ by the ideal sheaf of those regular functions which vanish on X.

A quasi-projective variety U is once again a space with a sheaf of commutative rings $\mathcal{O}_U$ which admits a locally a closed embedding into some $\mathbb{P}^N$, $j : U \subset \mathbb{P}^N$, so that the closure $\overline{U} \subset \mathbb{P}^N$ of U admits the structure of a projective variety and so that $\mathcal{O}_U$ equals the restriction of $\mathcal{O}_{\overline{U}}$ to $U \subset \overline{U}$.

A quasi-projective variety U is said to be affine if U admits a closed embedding into some $\mathbb{A}^N = \mathbb{P}^N \backslash Z(X_0)$ so that $\mathcal{O}_U$ is the quotient of $\mathcal{O}_{\mathbb{A}^N}$ by the sheaf of ideals which vanish on U.

Remark 1.14. Any quasi-projective variety U has a base of (Zariski) open subsets which are affine.

Most quasi-projective varieties are neither projective nor affine.

There is a bijective correspondence between affine varieties and finitely generated commutative k-algebras. If U is an affine variety, then $\Gamma(\mathcal{O}_U)$ is the corresponding finitely generated k-algebra. Conversely, if A is written as a quotient $k[x_1, \dots, x_N] \to A$, then $SpecA \to Speck[x_1, \dots, x_N] = \mathbb{A}^N$ is the corresponding closed embedding of the affine variety $SpecA$.

Example 1.15. Let F be a polynomial in variables $X_0, \dots, X_N$ homogeneous of degree d (i.e., $F(ca_0, \dots, ca_N) = c^d F(a_0, \dots, a_N)$. Then the zero locus $Z(F) \subset \mathbb{P}^N$ is called a hypersurface of degree d. For example if $N = 2$, then $Z(F)$ is 1-dimensional (i.e., a curve). If $k = \mathbb{C}$ and if the Jacobian of F does not vanish anywhere on $C = Z(F)$ (i.e., if C is *smooth*), then C is a projective, smooth, algebraic curve of genus $\frac{(d-1)(d-2)}{2}$.

1.4. **Algebraic vector bundles.**

Definition 1.16. Let X be a quasi-projective variety. A *quasi-coherent sheaf* $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules (i.e., an abelian sheaf equipped with a pairing $\mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ of sheaves satisfying the condition that for each open $U \subset X$ this pairing gives $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module) with the property that there exists an open covering $\{U_i \subset X; i \in I\}$ by affine open subsets so that $\mathcal{F}_{|U_i}$ is the sheaf associated to an $O_X(U_i)$-module M_i for each i.

If each of the M_i can be chosen to be finitely generated as an $O_X(U_i)$-module, then such a quasi-coherent sheaf is called *coherent*.

Definition 1.17. Let X be a quasi-projective variety. A coherent sheaf $\mathcal{E}$ on X is said to be an algebraic vector bundle if $\mathcal{E}$ is locally free. In other words, if there exists a (Zariski) open covering $\{U_i; i \in I\}$ of X such that $\mathcal{E}_{|U_i} \simeq \mathcal{O}_{X|U_i}^{e_i}$ for each i.

Remark 1.18. If quasi-projective variety is affine, then an algebraic vector bundle on X is equivalent to a projective $\Gamma(\mathcal{O}_X)$-module.

Construction 1. If M is a free A-module of rank r, then the symmetric algebra $Sym_A^\bullet(M)$ is a polynomial algebra of r generators over A and the structure map $\pi : \operatorname{Spec} Sym_A^\bullet(M) \to \operatorname{Spec} A$ is just the projection $\mathbb{A}^r \times \operatorname{Spec} A \to \operatorname{Spec} A$. This construction readily globalizes: if $\mathcal{E}$ is an algebraic vector bundle over X, then

$$\pi_{\mathcal{E}} : \mathbb{V}(\mathcal{E}) \equiv \operatorname{Spec} Sym_{\mathcal{O}_X}^\bullet(\mathcal{E})^* \ \to \ X$$

is locally in the Zariski topology a product projection: if $\{U_i \subset X; i \in\}$ is an open covering restricted to which $\mathcal{E}$ is trivial, then the restriction of $\pi_{\mathcal{E}}$ above each U_i is isomorphic to the product projection $\mathbb{A}^r \times U_i \to U_i$. In the above definition of $\pi_{\mathcal{E}}$ we consider the symmetric algebra on the dual $\mathcal{E}^* = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, so that the association $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E}^*)$ is covariantly functorial.

Thus, we may alternatively think of an algebraic vector bundle on X as a map of varieties

$$\pi_{\mathcal{E}} : \mathbb{V}(\mathcal{E}^*) \ \to \ X$$

satisfying properties which arethe algebraic analogues of the properties of the structure map of a topological vector bundle over a topological space.

Remark 1.19. We should be looking at the maximal ideal spectrum of a variety over a field k, rather than simply the k rational points, whenever k is not algebraically closed. We suppress this point, for we will soon switch to prime ideal spectra (i.e., work with schemes of finite type over k). However, we do point out that the reason it suffices to consider the maximal ideal spectrum rather the spectrum of all prime ideals is the validity of the Hilbert Nullstellensatz. One form of this important theorem is that the subset of maximal ideals constitute a dense subset of the space of prime ideals (with the Zariski topology) of a finitely generated commutative k-algebra.

1.5. **Examples of Algebraic Vector Bundles.**

Example 1.20. Rank 1 vector bundles $O_{\mathbb{P}^N}(k), k \in \mathbb{Z}$ on $\mathbb{P}^N$. The sections of $O_{\mathbb{P}^N}(j)$ on the basic open subset $U_G = \mathbb{P}^N\ Z(G)$ are given by the formula

$$O_{\mathbb{P}^N}(k)(U_G) = k[X_0, \dots, X_N, 1/G]_{(j)}$$

(i.e., ratios of homogeneous polynomials of total degree j).

In terms of the trivialization on the open covering $U_i, 0 \leq i \leq N$, the patching functions are given by $X_i^j / X_{i'}^j$.

$\Gamma(O_{\mathbb{P}^N}(j))$ has dimension $\binom{N+j}{j}$ if $j > 0$, dimension 1 if $j = 0$, and 0 otherwise. Thus, using the fact that $O_{\mathbb{P}^N}(j) \otimes_{\mathcal{O}_X} O_{\mathbb{P}^N}(j') = O_{\mathbb{P}^N}(j + j')$, we conclude that $\Gamma(O_{\mathbb{P}^N}(j))$ is not isomorphic to $\Gamma(O_{\mathbb{P}^N}(j'))$ provided that $j' \neq j$.

Proposition 1.21. *(Grothendieck) Each vector bundle on $\mathbb{P}^1$ has a unique decomposition as a finite direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(k), k \in \mathbb{Z}$.*

Example 1.22. Serre's conjecture (proved by Quillen and Suslin) asserts that every algebraic vector bundle on $\mathbb{A}^N$ (or any affine open subset of $\mathbb{A}^N$) is trivial. In more algebraic terms, every finitely generated projective $k[x_1, \dots, x_n]$-module is free.

Example 1.23. Let $X = Grass_{n,N}$, the Grassmann variety of $n-1$-planes in P^N (i.e., n-dimensional subspaces of k^{N+1}). We can embed $Grass_{n,N}$ as a Zariski closed subset of $\mathbb{P}^{M-1}$, where $M = \binom{N+1}{n}$, by sending the subspace $V \subset k^{N+1}$ to its n-th exterior power $\Lambda^n V \subset \Lambda^n(K^{N+1})$. There is a natural rank n algebraic vector bundle $\mathcal{E}$ on X provided with an embedding in the trivial rank $N+1$ dimensional vector bundle $\mathcal{O}_X^{N+1}$ (in the special case $n = 1$, this is $\mathcal{O}_{\mathbb{P}^N}(-1) \subset \mathcal{O}_{\mathbb{P}^N}^{N+1}$) whose fibre above a point in X is the corresponding subspace. Of equal importance is the natural rank $N-n$-dimensional quotient bundle $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^N}^{N+1}/\mathcal{E}$.

This example readily generalizes to flag varieties.

Example 1.24. Let A be a commutative k-algebra and recall the module $\Omega_{A/k}$ of Kähler differentals. These globalize to a quasi-coherent sheaf Ω_X on a quasi-projective variety X over k. If X is *smooth* of dimension r, then Ω_X is an algebraic vector bundle over X of rank r.

1.6. **Picard Group** $Pic(X)$**.**

Definition 1.25. Let X be a quasi-projective variety. We define $Pic(X)$ to be the abelian group whose elments are isomorphism classes of rank 1 algebraic vector bundles on X (also called "invertible sheaves"). The group structure on $Pic(X)$ is given by tensor product.

So defined, $Pic(X)$ is a generalization of the construction of the Class Group (of fractional ideals modulo principal ideal) for $X = \operatorname{Spec} A$ with A a Dedekind domain.

Example 1.26. By examing patching data, we readily verify that

$$H^1(X, \mathcal{O}_X^*) \;=\; Pic(X)$$

where $\mathcal{O}_X^*$ is the sheaf of abelian groups on X with sections $\Gamma(U, \mathcal{O}_X^*)$ defined to be the invertible elements of $\Gamma(U, \mathcal{O}_X)$ (with group structure given by multiplication).

If $k = \mathbb{C}$, then we have a short exact sequence of *analytic sheaves* of abelian sheaves on the analytic space $X(\mathbb{C})^{an}$,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \stackrel{exp}{\to} \mathcal{O}_X^* \to 0.$$

We use identification due to Serre of analytic and algebraic vector bundles on a projective variety. If $X = C$ is a smooth curve, this identification enables us to conclude the short exact sequence

$$0 \to \mathbb{C}^g/\mathbb{Z}^{2g} \to Pic(C) \to H^2(C, \mathbb{Z})$$

since $H^1(C, \mathcal{O}_C) \simeq H^0(C, \Omega_C) = \mathbb{C}^g$ (where g is the genus of C). In particular, we conclude that for a curve of positive genus, $Pic(C)$ is very large, having a "continuous part" (which is an abelian variety).

Example 1.27. A $K3$ surface S over the complex numbers is characterized by the conditions that $H^0(S, \Lambda^2(\Omega_S)) = 0 = H^1_{sing}(S, \mathbb{Q})$. Even though the homotopy type of a smooth $K3$ surface does not depend upon the choice of such a surface S, the rank of $Pic(S)$ can vary from 1 to 20. [The dimension of $H^2_{sing}(S, \mathbb{Q})$ is 22.]

1.7. K_0 **of Quasi-projective Varieties.**

Definition 1.28. Let X be a quasi-projective variety. We define $K_0(X)$ to be the quotient of the free abelian group generated by isomorphism classes $[\mathcal{E}]$ of (algebraic) vector bundles $\mathcal{E}$ on X modulo the equivalence relation generated pairs $([\mathcal{E}], [\mathcal{E}_1] + [\mathcal{E}_2])$ for each short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of vector bundles.

Remark 1.29. Let A be a finitely generated k-algebra. Observe that every short exact sequence of projective A-modules splits. Thus, the equivalence relation defining $K_0(A)$ is generated by pairs $([\mathcal{E}_1 \oplus \mathcal{E}_2], [\mathcal{E}_1] + [\mathcal{E}_2])$. Every element of $K_0(A)$ can be written as $[P] - [m]$ for some non-negative integer m; moreover, projective modules P, Q determine the same class in $K_0(A)$ if and only if there exists some non-negative integer m such that $P \oplus A^m \simeq Q \oplus A^m$.

Proposition 1.30. *$K_0(\mathbb{P}^N)$ is a free abelian group of rank $N+1$. Moreover, for any $k \in \mathbb{Z}$, the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(k), \ldots, \mathcal{O}_{\mathbb{P}^N}(k+N)$ generate $K_0(\mathbb{P}^N)$.*

Proof. One obtains a relation among $N+2$ invertible sheaves from the Koszul complex on $N+1$ dimensional vector space V:

$$0 \to \Lambda^{N+1}V \otimes S^{*-N-1}(V) \to \cdots \to V \otimes S^{*-1}(V) \to S^*(V) \to k \to 0.$$

One shows that the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(j), j \in \mathbb{Z}$ generate $K_0(\mathbb{P}^N)$ using Serre's theorem that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ there exist integers $m, n > 0$ and a surjective map of $\mathcal{O}_{\mathbb{P}^N}$-modules $\mathcal{O}_{\mathbb{P}^N}(m)^n \to \mathcal{F}$.

One way to show that the rank of $K_0(\mathbb{P}^N)$ equals $N+1$ is to use Chern classes. □

1.8. K_1 **of rings.** So far, we have only considered degree 0 algebraic and topological K-theory. Before we consider $K_n(R), n \in \mathbb{N}, K^n_{top}(X), n \in \mathbb{Z}$, we look explicitly at $K_1(R)$.

Definition 1.31. Let R be a ring (assumed associative, as always and with unit). We define $K_1(R)$ by the formula

$$K_1(R) \ \equiv \ GL(R)/[GL(R), GL(R)],$$

where $GL(R) = \varinjlim_n GL(n, R)$ and where $[GL(r), GL(R)]$ is the commutator subgroup of the group $GL(R)$. Thus, $K_1(R)$ is the maximal abelian quotient of $GL(R)$,

$$K_1(R) \ = \ H_1(GL(R), \mathbb{Z}).$$

The commutator subgroup $[GL(R), GL(R)]$ equals the subgroup $E(R) \subset GL(R)$ defined as the subgroup generated by elementary matrices $E_{i,j,}(r), r \in R, i \neq j$ (i.e., matrices which differ by the identity matrix by having r in the (i, j) position). This group is readily seen to be *perfect* (i.e., $E(R) = [E(r), E(R)]$); indeed, it is an elementary exercise to verify that $E(n, R) = E(R) \cap GL(n, R)$ is perfect for $n \geq 3$.

Proposition 1.32. *If R is a commutative ring, then the determinant map*

$$det : K_1(R) \to R^\times$$

from $K_1(R)$ to the mulitplicative group of units of R provides a natural splitting of $R^\times = GL(1, R) \to GL(R) \to K_1(R)$. Thus, we can write

$$K_1(R) \ = \ R^\times \times SL(R)$$

where $SL(R) = ker\{det\}$.

If R is a field or more generally a local ring, then $SK_1(R) = 0$.

The following theorem is not at all easy, but it does tell us that nothing surprising happens for rings of integers in number fields.

Theorem 1.33. *(Bass-Milnor-Serre) If $\mathcal{O}_K$ is the ring of integers in a number field K, then $SK_1(\mathcal{O}_K) = 0$.*

Application 1.34. *The work of Bass-Milnor-Serre was dedicated to solving the following question: is every subgroup $H \subset SL(\mathcal{O}_K)$ of finite index a "congruent subgroup" (i.e., of the form $ker\{SL(\mathcal{O}_K) \to SL(\mathcal{O}_K/p^n)\}$ for some prime ideal $p \subset \mathcal{O}_K$. The answer is yes if the number field F admits a real embedding, and no otherwise.*

The Bass-Milnor-Serre theorem is complemented by the following classical result due to Dirichlet (cf. [4]).

Theorem 1.35. *(Dirichlet's Theorem) Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then*

$$O_K^* = \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1}$$

where $\mu(K) \subset K$ denotes the finite subgroup of roots of unity and where r_1 (respectively, r_2) denotes the number of embeddings of K into $\mathbb{R}$ (resp., number of conjugate pairs of embeddings of K into $\mathbb{C}$).

We conclude this brief commentary on K_1 with the following early application to topology.

Application 1.36. *Let π be a finitely generated group and consider the Whitehead group*

$$Wh(\pi) = K_1(R)/\{\pm g; g \in \pi\}.$$

A homotopy equivalence of finite complexes with fundamental group π has an invariant (its "Whitehead torsion") in $Wh(\pi)$ which determines whether or not this is a simple homotopy equivalence (given by a chain of "elementary expansions" and "elementary collapses").

1.9. K_2 **of rings.** One can think of $K_0(R)$ as the "stable group" of projective modules "modulo trivial projective modules" and of $K_1(R)$ of the stabilized group of automorphisms of the trivial projective module modulo "trivial automorphisms" (i.e., the elementary matrices up to isomorphism. This philosophy can be extended to the definition of K_2, but has not been extended to $K_i, i > 2$. Namely, $K_2(R)$ can be viewed as the relations among the trivial automorphisms (i.e., elementary matrices) modulo those relations which hold universally.

Definition 1.37. Let $St(R)$, the Steinberg group of R, denote the group generated by elements $X_{i,j}(r), i \neq j, r \in R$ subject to the following commutator relations:

$$[X_{i,j(r)}, X_{k,\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq \ell \\ X_{i,\ell}(rs) & \text{if } j = k, i \neq \ell \\ X_{k,j}(-rs) & \text{if } j \neq k, i = \ell \end{cases}$$

We define $K_2(R)$ to be the kernel of the map $St(R) \to E(R)$, given by sending $X_{i,j}(r)$ to the elementary matrix $E_{i,j}(r)$, so that we have a short exact sequence

$$1 \to K_2(R) \to St(R) \to E(R) \to 1.$$

Proposition 1.38. *The short exact sequence*

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

is the universal central extension of the perfect group $E(R)$. Thus, $K_2(R) = H_2(E(R), \mathbb{Z})$, the Schur multiplier of $E(r)$.

Proof. Once can show that a universal central extension of a group E exists if and only E is perfect. In this case, a group S mapping onto E is the universal central extension if and only if S is also perfect and $H_2(S, \mathbb{Z}) = 0$. □

Example 1.39. If R is a field, then $K_1(F) = F^\times$, the non-zero elements of the field viewed as an abelian group under multiplication. By a theorem of Matsumoto, $K_2(F)$ is characterized as the target of the "universal Steinberg symbol". Namely, $K_2(F)$ is isomorphic to the free abelian group with generators "Steinberg symbols" $\{a, b\},\ a, b \in F^\times$ and relations

- i. {ac,b} = {a,b} {c,b},
- ii. {a,bd} = {a,b} {a,d},
- iii. $\{a, 1-a\} = 1, \quad a \neq 1 \neq 1-a.$ (*Steinberg relation*)

Observe that for $a \in F^\times,\ -a = \frac{1-a}{1-a^{-1}}$, so that

$$\{a, -a\} = \{a, 1-a\}\{a, 1-a^{-1}\}^{-1} = \{a, 1-a^{-1}\}^{-1} = \{a^{-1}, 1-a^{-1}\} = 1.$$

Then we conclude the skew symmetry of these symbols:

$$\{a, b\}\{b, a\} = \{a, -a\}\{a, b\}\{b, a\}\{b, -b\} = \{a, -ab\}\{b, -ab\} = \{ab, -ab\} = 1.$$

Milnor used this of $K_2(F)$ as the starting point of his definition of the *Milnor K-theory* K_*^{Milnor} of a field F, discussed briefly in Lecture 5.

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