The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

SMR/1840-25

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Noncommutative locally affine "spaces" and schemes

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Lecture 1. Noncommutative locally affine 'spaces' and schemes.

1. Noncommutative 'spaces' represented by categories and morphisms between them. Continuous, affine and locally affine morphisms.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$, denotes the bicategory of categories which belong to a fixed universum $\mathfrak{U}$. We call objects of Cat^{op} 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the corresponding object of Cat^{op} (the underlying 'space') defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of the morphism* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p^*_\Sigma} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_f the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p^*_f = p^*_{\Sigma_f} : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $\Sigma^{-1}C_Y \xrightarrow{f^*_{\mathfrak{c}}} C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms only, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f^*_{\mathfrak{c}}$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a unique decomposition of a morphism f into a localization and a conservative morphism.

1.3. Continuous, flat, and affine morphisms. A morphism is called continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called 'affine' if its direct image functor is conservative (i.e. reflects isomorphisms) and has a right adjoint.

1.4. Categoric spectrum of a unital ring. For an associative unital ring R, we define the *categoric spectrum* of R as the object $\mathbf{Sp}(R)$ of $|Cat|^o$ such that $C_{\mathbf{Sp}(R)} = R-mod$.

Let $R \xrightarrow{\phi} S$ be a unital ring morphism and $R-mod \xrightarrow{\bar{\phi}^*} S-mod$ the functor $S\otimes_R -$. The canonical right adjoint to $\bar{\phi}^*$ is the pull-back functor by the ring morphism ϕ. A right adjoint to ϕ_* is given by

$$\phi^! : S - mod \longrightarrow R - mod, \quad L \longmapsto Hom_R(\phi_*(S), L).$$

The map

$$\Big(R \xrightarrow{\phi} S\Big) \longmapsto \Big(\mathbf{Sp}(S) \xrightarrow{\bar{\phi}} \mathbf{Sp}(R)\Big)$$

is a functor

$$\mathbf{Sp} : Rings^{op} \longrightarrow |Cat|^o$$

which takes values in the subcategory formed by affine morphisms.

The image $\mathbf{Sp}(R) \xrightarrow{\bar{\phi}} \mathbf{Sp}(T)$ of a ring morphism $T \xrightarrow{\phi} R$ is flat (resp. faithful) iff ϕ turns R into a flat (resp. faithful) right T-module.

1.4.1. Continuous, flat, and affine morphisms from $\mathbf{Sp}(S)$ to $\mathbf{Sp}(R)$. Let R and S be associative unital rings. A morphism $f : \mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ with an inverse image functor f^* is continuous iff

$$f^* \simeq \mathcal{M}\otimes_R \ : L \longmapsto \mathcal{M} \otimes_R L \tag{1}$$

for an (S,R)-bimodule $\mathcal{M}$ defined uniquely up to isomorphism. The functor

$$f_* = Hom_S(\mathcal{M}, -) : N \longmapsto Hom_S(\mathcal{M}, N) \tag{2}$$

is a direct image of f.

The morphism f with an inverse image functor (1) is conservative iff $\mathcal{M}$ is *faithful* as a right R-module, i.e. the functor $\mathcal{M} \otimes_R -$ is faithful.

The direct image functor (2) is conservative iff $\mathcal{M}$ is a cogenerator in the category of left S-modules, i.e. for any nonzero S-module N, there exists a nonzero S-module morphism $\mathcal{M} \longrightarrow N$.

The morphism f is flat iff $\mathcal{M}$ is flat as a right R-module.

The functor (2) has a right adjoint, $f^!$, iff f_* is isomorphic to the tensoring (over S) by a bimodule. This happens iff $\mathcal{M}$ is a projective S-module of finite type. The latter is equivalent to the condition: the natural functor morphism $\mathcal{M}^*_S \otimes_S - \longrightarrow Hom_S(\mathcal{M}, -)$ is an isomorphism. Here $\mathcal{M}^*_S = Hom_S(M, S)$. In this case, $f^! \simeq Hom_R(\mathcal{M}^*_S, -)$.

1.5. Example. Let $\mathcal{G}$ be a monoid and R a $\mathcal{G}$-graded unital ring. We define the 'space' $\mathbf{Sp}_{\mathcal{G}}(R)$ by taking as $C_{\mathbf{Sp}_{\mathcal{G}}(R)}$ the category $gr_{\mathcal{G}}R-mod$ of left $\mathcal{G}$-graded R-modules. There is a natural functor $gr_{\mathcal{G}}R - mod \xrightarrow{\phi_*} R_0 - mod$ which assigns to each graded R-module its zero component ('zero' is the unit element of the monoid $\mathcal{G}$). The functor ϕ_* has a left adjoint, ϕ^*, which maps every R_0-module M to the graded R-module $R \otimes_{R_0} M$. The adjunction arrow $Id_{R_0-mod} \longrightarrow \phi_*\phi^*$ is an isomorphism. This means that the functor ϕ^* is fully faithful, or, equivalently, the functor ϕ_* is a localization.

The functors ϕ_* and ϕ^* are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism ϕ is affine iff ϕ is an isomorphism (i.e. ϕ^* is an equivalence of categories).

In fact, if ϕ is affine, the functor ϕ_* should be conservative. Since ϕ_* is a localization, this means, precisely, that ϕ_* is an equivalence of categories.

1.6. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $\mathcal{T}_{R_+}$ denote the full subcategory of the category $R - mod$ whose objects are all R-modules annihilated by R_+. Let $\mathcal{T}_{R_+}^-$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - mod$ spanned by $\mathcal{T}_{R_+}$.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}_{R_+}^-$. The localization functor $R - mod \xrightarrow{u^*} R - mod/\mathcal{T}_{R_+}^-$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see C3.2.1). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

1.7. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}_{R_+}^-.$$

Here $gr_{\mathcal{G}}\mathcal{T}_{R_+}^-$ is the Serre subcategory of the category $gr_{\mathcal{G}}R - mod$spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}_{R_+}^- = gr_{\mathcal{G}}R - mod \cap \mathcal{T}_{R_+}^-$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

1.7.1. Example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \oplus_{n\ge 0} R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \oplus_{n\ge 1} R_n$ its 'irrelevant' ideal. Thus, we have the *cone* of R_+, $\mathbf{Cone}(R_+)$, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

2. Noncommutative schemes and locally affine 'spaces'. Descent.

2.1. Locally affine morphisms of 'spaces'. We call a morphism $X \xrightarrow{f} S$ of 'spaces' *locally affine* if there exists a family $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms such that

– all inverse image functors u_i^* are *exact* (i.e. the functors u_i^* preserve finite limits and colimits),

– the family $\{u_i^* \mid i \in J\}$ is *conservative* (i.e. if $u_i^*(s)$ is an isomorphism for all $i \in J$, then s is an isomorphism),

– all the compositions $f \circ u_i$ are affine.

2.2. Weak locally affine schemes over S. These are locally affine morphisms which have a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ formed by *localizatios*. The latter means that each inverse image functor u_i^* is the composition of a localization functor (at $\Sigma_{u_i^*} \stackrel{\mathrm{def}}{=} \{s \in HomC_X \mid u_i^*(s) \text{ is invertible }\}$) and an equivalence of categories.

2.3. Descent.

2.3.1. The Beck's Theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism in with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_*f^* \quad \text{and} \quad f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{F}_f$ denote the monad (F_f, μ_f) on Y, where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$. There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\widetilde{f_*}} & (\mathcal{F}_f/Y) - mod \\ & {\scriptstyle f_*} \searrow \quad \swarrow {\scriptstyle \mathfrak{f}_*} & \\ & C_X & \end{array} \tag{3}$$

Here $\widetilde{f_*}$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

and $\mathfrak{f}^*$ is the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

The following assertion is one of the versions of Beck's theorem.

2.3.1.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ∎

2.3.1.2. Corollary. *Let $X \xrightarrow{f} Y$ be an affine morphism (cf. 1.5). If the category C_Y has cokernels of reflexive pairs of arrows (e.g. C_Y is an abelian category), then the canonical morphism $X \xrightarrow{\mathfrak{f}} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.*

2.3.1.3. Monadic morphisms. A continuous morphism $X \xrightarrow{f} Y$ such that the functor

$$C_X \xrightarrow{\widetilde{f_*}} \mathcal{F}_f - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

is an equivalence of categories.

2.3.2. Continuous monads and affine morphisms. A functor F is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on Y (i.e. on the category C_Y) is called *continuous* if the functor F is continuous.

2.3.2.1. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous iff the canonical morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ is affine.*

Proof. A proof in the case of a continuous monad can be found in [KR2, 6.2], or in [R3, 4.4.1] (see also [R4, 2.2]). ∎

2.3.2.2. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ is affine iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous monad $\mathcal{F}_f$ on Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fullfiled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from 2.3.2.1. ∎

2.4. The category of affine schemes over a 'space' and the category of monads on this 'space'.

2.4.1. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & f \searrow \quad \swarrow g & \\ & S & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose C_Z has colimits of reflexive pairs of arrows. If f and g are affine, then h is affine.

Let Aff_S denote the full subcategory of the category $|Cat|^o/S$ of 'spaces' over S whose objects are pairs $(X, X \xrightarrow{f} S)$, where f is an affine morphism. On the other hand, we have the category $\mathfrak{Mon}_{\mathfrak{c}}(S)$ of continuous monads on the 'space' S (i.e. on the category C_S) and the functor

$$\mathfrak{Mon}_{\mathfrak{c}}(S)^{op} \longrightarrow Aff_S \tag{1}$$

which assigns to every continuous monad $\mathcal{F}$ the object $(\mathbf{Sp}(\mathcal{F}/S, \mathfrak{f})$, where $\mathbf{Sp}(\mathcal{F}/S)$ is the 'space' represented by the category $\mathcal{F} - mod$ and the morphism $\mathfrak{f}$ has the forgetful functor $\mathcal{F} - mod \longrightarrow C_S$ as a direct image functor. It follows from 2.4.1 and 2.3.2.2 that this functor is essentially full (that is its image is equivalent to the category Aff_S).

For every endofunctor $C_S \xrightarrow{G} C_S$, let $|G|$ denote the set $Hom(Id_{C_S}, G)$ of *elements* of G. If $\mathcal{F} = (F, \mu)$ is a monad, then the set of elements of F has a natural monoid structure; we denote this monoid by $|\mathcal{F}|$. And we denote by $|\mathcal{F}|^*$ the group of the invertible elements of the monoid $|\mathcal{F}|$. We say that two monad morphisms $\mathcal{F} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{G}$ are *conjugate* to each other of $\phi = t \cdot \psi \cdot t^{-1}$ for some $t \in |\mathcal{G}|^*$.

Let $\mathfrak{Mon}_{\mathfrak{c}}^{\mathfrak{r}}(S)$ denote the category whose objects are continuous monads on C_S and morphisms are *conjugacy classes* of morphisms of monads.

2.4.2. Proposition *The functor (1) induces an equivalence between the category $\mathfrak{Mon}_{\mathfrak{c}}^{\mathfrak{r}}(S)$ and the category Aff_S of affine schemes over S.*

2.4.3. Example. Let $S = \mathbf{Sp}(R)$ for an associative ring R. Then the category $\mathfrak{Mon}_{\mathfrak{c}}(S)$ of monads on $C_S = R - mod$ is naturally equivalent to the category $R\backslash Rings$ of associative rings over R. The conjugacy classes of monad morphisms correspond to conjugacy classes of ring morphisms. Let $\mathfrak{Ass}$ denote the category whose objects are associative rings and morphisms the conjugacy classes of ring morphisms.

One deduces from 2.4.2 the following assertion:

2.4.3.1. Proposition. *The category Aff_S of affine schemes over $S = \mathbf{Sp}(R)$ is naturally equivalent to the category $(R\backslash\mathfrak{Ass})^{op}$.*

2.5. Descent: "covers", comonads, and glueing.

2.5.1. Comonads associated with "covers". Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a family of continuous morphisms and $\mathfrak{u}$ the corresponding morphism $\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{\mathfrak{u}} X$ with the inverse image functor

$$C_X \xrightarrow{\mathfrak{u}^*} \prod_{i \in J} C_{U_i} = C_{\mathcal{U}}, \quad M \longmapsto (u_i^*(M)|i \in J).$$

It follows that the family of inverse image functors $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is conservative iff the functor $\mathfrak{u}^*$ is conservative.

Suppose that the category C_X has products of $|J|$ objects. Then the morphism $\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{\mathfrak{u}} X$ is continuous: its direct image functor assigns to every object $(L_i|i \in J)$ of the category $C_{\mathcal{U}} = \prod_{i \in J} C_{U_i}$ the product $\prod_{i \in J} u_{i*}(L_i)$.

The adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$ assigns to each object M of C_X the morphism $M \longrightarrow \prod_{i\in J} u_{i*}u_i^*(M)$ determined by adjunction arrows $Id_{C_X} \xrightarrow{\eta_{u_i}} u_{i*}u_i^*$.

The adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$ assigns to each object $\mathcal{L} = (L_i|i \in J)$ of $C_{\mathcal{U}}$ the morphism $(\epsilon_{\mathfrak{u},i}(\mathcal{L})|i \in J)$, where

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{\epsilon_{\mathfrak{u},i}(\mathcal{L})} L_i$$

is the composition of the image

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{u_i^*(p_i)} u_i^*u_{i*}(L_i)$$

of the image of the projection p_i and the adjunction arrow $u_i^*u_{i*}(L_i) \xrightarrow{\epsilon_{u_i}(L_i)} L_i$.

2.5.2. Beck's theorem and glueing. Suppose that for each $i \in J$, the category C_{U_i} has kernels of coreflexive pairs of arrows and the functor u_i^* preserves them. Then the inverse and direct image functors of the morphism $\mathfrak{u}$ satisfy the conditions of Beck's theorem, hence the category C_X is equivalent to the category of comodules over the comonad $\mathcal{G}_{\mathfrak{u}} = (G_{\mathfrak{u}}, \delta_{\mathfrak{u}}) = (\mathfrak{u}^*\mathfrak{u}_*, \mathfrak{u}^*\eta_{\mathfrak{u}}\mathfrak{u}_*)$ associated with the choice of inverse and direct image functors of $\mathfrak{u}$ together with an adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$.

Recall that $\mathcal{G}_{\mathfrak{u}}$-comodule is a pair $(\mathcal{L}, \zeta)$, where $\mathcal{L}$ is an object of $C_{\mathcal{U}}$ and ζ a morphism $\mathcal{L} \longrightarrow G_{\mathfrak{u}}(\mathcal{L})$ such that $\epsilon_{\mathfrak{u}}(\mathcal{L}) \circ \zeta = id_{\mathcal{L}}$ and $G_{\mathfrak{u}}(\zeta) \circ \zeta = \delta_{\mathfrak{u}}(\mathcal{L}) \circ \zeta$. Beck's theorem says that if the category $C_{\mathcal{U}}$ has kernels of coreflexive pairs of arrows and the functor $\mathfrak{u}^*$ preserves and reflects them, then the functor $C_X \xrightarrow{\widetilde{\mathfrak{u}^*}} (\mathcal{U}\backslash\mathcal{G}_{\mathfrak{u}}) - comod$ which assigns to each object M of C_X the $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathfrak{u}^*(M), \delta_{\mathfrak{u}}(M))$ is an equivalence of categories.

In terms of our local data – the "cover" $\{U_i \xrightarrow{u_i} X \mid i \in J\}$, a $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathcal{L}, \zeta)$ is the data $(L_i, \zeta_i|i \in J)$, where $(L_i|i \in J) = \mathcal{L}$ and ζ_i is a morphism

$$L_i \longrightarrow u_i^*\mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_{j\in J} u_{j*}(L_j))$$

which equalizes the pair of arrows

$$u_i^*\mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_j u_{j*}(L_j)) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\rightrightarrows}} u_i^*(\prod_m u_{m*}u_m^*(\prod_j u_{j*}(L_j))) = u_i^*\mathfrak{u}_*\mathfrak{u}^*\mathfrak{u}_*(\mathcal{L})$$

and such that $\epsilon_{\mathfrak{u},i}(\mathcal{L}) \circ \zeta_i = id_{L_i}$, $i \in J$.

The exactness of the diagram

$$\mathcal{L} \xrightarrow{\zeta} G_{\mathfrak{u}}(\mathcal{L}) \underset{G_{\mathfrak{u}}(\zeta)}{\overset{\delta_{\mathfrak{u}}(\mathcal{L})}{\rightrightarrows}} G_{\mathfrak{u}}^2(\mathcal{L})$$

is equivalent to the exactness of the diagram

$$L_i \xrightarrow{\zeta_i} u_i^*(\prod_{j\in J} u_{j*}(L_j)) \overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\underset{u_i^*(u_{j*}\zeta_j)}{\rightrightarrows}} u_i^*(\prod_{m\in J} u_{m*}u_m^*(\prod_{j\in J} u_{j*}(L_j))) \qquad (1)$$

for every $i \in J$. If the functors u_k^* preserve products of J objects (or just the products involved into (1)), then the diagram (1) is isomorphic to the diagram

$$L_i \xrightarrow{\zeta_i} \prod_{j\in J} u_i^* u_{j*}(L_j) \overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\underset{u_i^*(u_{j*}\zeta_j)}{\rightrightarrows}} \prod_{j,m\in J} u_i^* u_{m*}u_m^* u_{j*}(L_j) \qquad (2)$$

2.5.3. Remark. The exactness of the diagram (1) might be viewed as a sort of sheaf property. This interpretation looks more plausible (or less streched) when the diagram (1) is isomorphic to the diagram (2), because $u_i^* u_{j*}(L_j)$ can be regarded as the section of L_j over the 'intersection' of U_i and U_j and $u_i^* u_{m*}u_m^* u_{j*}(L_j)$ as the section of L_j over the intersection of the elements U_j, U_m, and U_i of the "cover".

2.5.4. The condition of the continuity of the comonad associated with a "cover". Suppose that each direct image functor $C_{U_i} \xrightarrow{u_{i*}} C_X$, $i \in J$, has a right adjoint, $u_i^!$; and let $\mathfrak{u}^!$ denote the functor $C_X \longrightarrow C_{\mathcal{U}} = \prod_{i\in J} C_{U_i}$ which maps every object M to $(u_i^!(M)|i \in J)$. If the category C_X has coproducts of $|J|$ objects, then the functor $\mathfrak{u}^!$ has a left adjoint which maps every object $(L_i|i \in J)$ of $C_{\mathcal{U}}$ to the coproduct $\coprod_{i\in J} u_{i*}(L_i)$.

Therefore, if the canonical morphism $\coprod_{i\in J} u_{i*}(L_i) \longrightarrow \prod_{i\in J} u_{i*}(L_i)$ is an isomorphism for every object $(L_i|i \in J)$ of the category $C_{\mathcal{U}}$, then (and only then) the functor $\mathfrak{u}^!$ is a right adjoint to the functor $\mathfrak{u}_*$.

In particular, $\mathfrak{u}^!$ is a right adjoint to $\mathfrak{u}_*$, if the category C_X is additive and J is finite.

2.5.5. Note. If, in addition, the functors u_{i*} are conservative for all $i \in J$, then the functor $\mathfrak{u}_*$ is conservative, and the category $C_{\mathcal{U}}$ is equivalent to the category of modules over the continuous monad $\mathcal{F}_{\mathfrak{u}} = (F_{\mathfrak{u}}, \mu_{\mathfrak{h}})$, where $F_{\mathfrak{u}} = \mathfrak{u}_*\mathfrak{u}^*$ and $\mu_{\mathfrak{u}} = \mathfrak{u}_*\epsilon_{\mathfrak{u}}\mathfrak{u}^*$ for an adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$.

3. Some motivating examples.

3.1. Noncommutative schemes related with quantized enveloping algebras: quantum flag variety and associated quantum D-scheme.

3.1.1. The base affine 'space' and the flag variety of a reductive Lie algebra from the point of view of noncommutative algebraic geometry. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda\in\mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

3.1.2. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of $\mathfrak{g}$.

3.1.2.1. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W,$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ is naturally equivalent to $(S_w^{-1}R)_0 - mod$. By 1.5, it suffices to verify that the canonical functor $gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow S_w^{-1}R)_0 - mod$ which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if z is a nonzero element of λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e^\lambda_{w\lambda})^{-1} z$ is a nonzero element of the zero component of this module.

3.2. Noncommutative Grassmannians. Fix an associative unital k-algebra R. Let $R\backslash Alg_k$ be the category of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism). We call them for convenience R-rings. We denote by R^e the k-algebra $R \otimes_k R^o$. Here R^o is the algebra opposite to R.

3.2.1. The functor $Gr_{M,V}$. Let M, V be left R-modules. Consider the functor, $Gr_{M,V} : R\backslash Alg_k \longrightarrow \mathbf{Sets}$, which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of isomorphism classes of epimorphisms $s^*(M) \longrightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any R-ring morphism $(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$ the map $Gr_{M,V}(S, s) \longrightarrow Gr_{M,V}(T, t)$ induced by the inverse image functor $S - mod \xrightarrow{\phi^*} T - mod$, $\mathcal{N} \longmapsto T \otimes_S \mathcal{N}$.

3.2.2. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $R\backslash Alg_k \longrightarrow$ **Sets** which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of pairs of morphisms $s^*(V) \xrightarrow{v} s^*(M) \xrightarrow{u} s^*(V)$ such that $u \circ v = id_{s^*(V)}$ and acts naturally on morphisms. Since V is a projective module, the map

$$\pi = \pi_{M,V} : G_{M,V} \longrightarrow Gr_{M,V}, \ (v, u) \longmapsto [u], \tag{1}$$

is a (strict) functor epimorphism.

3.2.3. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \times_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$ which assigns to each R-ring, $(S, R \xrightarrow{s} S)$, the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms u_1, u_2 are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus, $\mathfrak{R}_{M,V}(S, s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S, s) \times G_{M,V}(S, s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S, s) \times G_{M,V}(S, s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2 \tag{3}$$

Denote by $p_1, \ p_2$ the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \ \underset{p_2}{\overset{p_1}{\rightrightarrows}} \ G_{M,V} \ \xrightarrow{\pi} \ Gr_{M,V} \tag{4}$$

is exact.

3.2.4. Proposition. *If both M and V are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. See [KR2, 10.4.3]. ■

3.2.5. Quasi-coherent presheaves on $Gr_{M,V}$. Suppose that M and V are projective modules of a finite type, hence the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable by R-rings resp. $(\mathfrak{G}_{M,V}, R \to \mathfrak{G}_{M,V})$ and $(\mathcal{R}_{M,V}, R \to \mathcal{R}_{M,V})$. Then the category $Qcoh(G_{M,V})$ (resp. $Qcoh(\mathfrak{R}_{M,V})$) is equivalent to $\mathfrak{G}_{M,V} - mod$ (resp. $\mathcal{R}_{M,V} - mod$), and the category $Qcoh(Gr_{M,V})$ of quasi-coherent presheaves on $Gr_{M,V}$ is equivalent to the kernel of the diagram

$$Qcoh(G_{M,V}) \ \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \ Qcoh(\mathfrak{R}_{M,V}) \tag{5}$$

This means that, after identifying categories of quasi-coherent presheaves in (5) with corresponding categories of modules, quasi-coherent presheaves on $Gr_{M,V}$ can be realized as pairs (L,ϕ), where L is a $\mathfrak{G}_{M,V}$-module and ϕ is an isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. Morphisms $(L,\phi) \longrightarrow (N,\psi)$ are given by morphisms $L \xrightarrow{g} N$ such that the diagram

$$\begin{array}{ccc}
p_1^*(L) & \xrightarrow{p_1^*(g)} & p_1^*(N) \\
\phi \downarrow \wr & & \wr \downarrow \psi \\
p_2^*(L) & \xrightarrow{p_2^*(g)} & p_2^*(N)
\end{array}$$

commutes. The functor

$$Qcoh(Gr_{M,V}) \xrightarrow{\pi^*} Qcoh(G_{M,V}), \quad (L,\phi) \longmapsto L,$$

is an inverse image functor of the projection $G_{M,V} \xrightarrow{\pi} Gr_{M,V}$ (see 3.2.3(4)).

3.2.6. Quasi-coherent presheaves on presheaves and sheaves of sets. Consider the category $\mathbf{Aff}_k$ of *affine k-schemes* which we identify with the category of representable functors on the category Alg_k of k-algebras, and the fibered category with the base $\mathbf{Aff}_k$ whose fibers are categories of left modules over corresponding algebras. Let X be a presheaf of sets on $\mathbf{Aff}_k$. Then we have a fibered category $\widetilde{\mathbf{Aff}_k}/X$ with the base $\mathbf{Aff}_k/X$ induced by the forgetful functor $\mathbf{Aff}_k/X \longrightarrow \mathbf{Aff}_k$. The category $Qcoh(X)$ of *quasi-coherent presheaves* on X is the opposite to the category of *cartesian sections* of $\widetilde{\mathbf{Aff}_k}/X$. Given a *(pre)topology* τ on $\mathbf{Aff}_k/X$, we define the subcategory $Qcoh(X,\tau)$ of *quasi-coherent sheaves* on (X,τ) [KR4].

3.2.7. Theorem ([KR4]). *(a) A topology τ on $\mathbf{Aff}_k$ is subcanonical (i.e. all representable presheaves are sheaves) iff $Qcoh(X) = Qcoh(X,\tau)$ for every presheaf of sets X on $\mathbf{Aff}_k$ (in other words, 'descent' topologies on $\mathbf{Aff}_k$ are precisely subcanonical topologies). In this case, $Qcoh(X) = Qcoh(X,\tau) \hookrightarrow Qcoh(X^\tau) = Qcoh(X^\tau,\tau)$, where X^τ is the sheaf associated to X and $\hookrightarrow$ is a natural full embedding.*

(b) If τ is a topology of effective descent [KR4] (e.g. the **fpqc** *or smooth topology [KR2]), then the categories $Qcoh(X,\tau)$ and $Qcoh(X^\tau)$ are naturally equivalent.*

This theorem says, roughly speaking, that the category $Qcoh(X)$ of quasi-coherent presheaves knows which topologies to choose. A topology that seems to be the most plausible for Grassmannians, in particular, for $N\mathbb{P}_k^n$, is the *smooth* topology introduced in [KR2]. It is of effective descent, and the category of quasi-coherent sheaves on $N\mathbb{P}_k^n$ defined in [KR1] is naturally equivalent to the category of quasi-coherent sheaves of the projective space defined via smooth topology on $\mathbf{Aff}_k$.

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