The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

SMR/1840-10

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Motivic cohomology operations

Charles Weibel
Rutgers University, Piscataway, USA

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

2 Lecture 2: Motivic cohomology operations

Fix a field k of characteristic 0 for simplicity, and consider the category Sm/k of smooth schemes over k. For each coefficient ring R, and each q, there is a chain complex of étale sheaves $R(q) = \mathbb{Z}(q) \otimes R$, constructed in [4, 3.1] and elsewhere; by definition the motivic cohomology groups $H^{p,q}(X, R)$ are the Zariski hypercohomology groups $H^p_{\text{Zar}}(X, R(q))$. We write $H^{p,q}$ for $H^{p,q}(\text{Spec}(k), R)$ when R is understood.

A *cohomology operation* ϕ is a natural transformation from $H^{p',q'}(-, R')$ to $H^{p,q}(-, R)$; it depends upon the integer p', q', p, q and the rings R' and R.

Examples 2.1. (a) If X is connected then $H^{0,0}(X, R) = R$; see [4, 3.1]. Hence any R'-indexed sequence ϕ_j of elements in $H^{p,q}$ with $\phi_0 = 0$, regarded as a function $R' \to H^{pq}$, determines a cohomology operation from $H^{0,0}(X, R')$ to $H^{p,q}(X, R)$. Clearly every such operation has this form.

(b) If t is an indeterminate of bidegree (p', q') then any homogeneous polynomial $f(t) = \Sigma_{i>0} a_i t^i$ of bidegree (p, q), whose coefficients a_i are in $H^{**}(k, R)$, determines a cohomology operation $H^{p',q'}(X, R') \to H^{p,q}(X, R)$ sending λ to $f(\lambda)$.

We will see in 4.6(c) that every cohomology operation $\phi\colon\ H^{2,1}(X, \mathbb{Z}) \to H^{p,q}(X, R)$ is of this form, and that ϕ uniquely determines the polynomial f.

(c) Associated to the exact sequence $0 \to \mathbb{Z}(q) \xrightarrow{\ell} \mathbb{Z}(q) \to \mathbb{Z}/\ell(q) \to 0$ is the *integral Bockstein* $\tilde{\beta}\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+1,q}(X, \mathbb{Z})$ and its reduction modulo ℓ, the *Bockstein* $\beta\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+1,q}(X, \mathbb{Z}/\ell)$.

(d) In [RPO, p. 33], Voevodsky constructed the *reduced power* operations

$$P^i\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+2i(\ell-1),q+i(\ell-1)}(X, \mathbb{Z}/\ell).$$

By [RPO, §9–§10], the P^i satisfy the following axioms when $\ell > 2$:

Axioms for Steenrod Operations 2.2. When $\ell > 2$ the operations P^i satisfy:

1. The operation P^i has bidegree $(2i(\ell-1), i(\ell-1))$, and $P^0x = x$.
2. $P^ix = x^\ell$ if x has bidegree $(2i, i)$.
3. $P^ix = 0$ if x has bidegree (p, q) and $q \leq i$, $p < q + i$.
4. The usual Cartan formula $P^n(xy) = \sum P^i(x)P^j(y)$ holds.
5. The usual Adem relations hold. (These are described in [SE, p.77].)

It follows that the P^i and the Bockstein β generate a bigraded ring, isomorphic to the topological Steenrod Algebra described in [SE]. In particular, every monomial in β and the P^i is a $\mathbb{Z}/\ell$-linear combination of the *admissible* monomials

$$\beta^{\epsilon_0}P^{s_1}\beta^{\epsilon_1}\cdots P^{s_k}\beta^{\epsilon_k}, \quad \epsilon = 0, 1 \text{ and } s_i \geq \ell s_{i+1} + \epsilon_i.$$

The admissible monomials are linearly independent in the (left) $H^{*,*}(k, \mathbb{Z}/\ell)$-module of all cohomology operations by [RPO, 11.5]. We also get operations Q_i in bidegree $(2\ell^i - 1, \ell^i - 1)$ by setting $Q_0 = \beta$ (the Bockstein), $Q_1 = P^1\beta - \beta P^1$ and $Q_{i+1} = [P^{\ell^i}, Q_i]$. The Q_i generate an exterior algebra under composition, and each Q_i is a derivation: $Q_i(xy) = Q_i(x)y + xQ_i(y)$.

2.3. We now turn to a more topological interpretation of motivic cohomology operations. We begin with a quick review of Voevodsky's triangulated category **DM**. Let $\mathbf{Cor} = \mathbf{Cor}(R)$ denote the additive category of finite correspondences between smooth varieties, with coefficients in R; it contains Sm/k and is described in [4, §1]. The category $\mathbf{PST} = \mathbf{PST}(R)$ of *presheaves with transfers* is the category of contravariant R-linear functions from $\mathbf{Cor}(R)$ to R-**mod**; the Yoneda functor $\mathbf{Cor} \to \mathbf{PST}$ is written as $X \mapsto R_{\mathrm{tr}}(X)$; see [4, §2].

A presheaf with transfers which is also a sheaf for the Nisnevich topology on Sm/k is called a *Nisnevich sheaf with transfers.* These form a full subcategory $\mathbf{NST}(R)$ of $\mathbf{PST}(R)$, and sheafification is an exact left adjoint to the inclusion; see [4, §13]. The triangulated category $\mathbf{DM}^{\text{eff}}(k, R)$ of *effective motives* is the localization of $D^-(\mathbf{NST})$ with respect to the class of $\mathbb{A}^1$-weak equivalences, and the motive of X is the class of $R_{\text{tr}}(X)$ in this category. This construction is presented in [4, §14]. In fact, $\mathbf{DM}^{\text{eff}}(k, R)$ is a full subcategory of a larger category $\mathbf{DM}(k, R)$ — but that is irrelevant for us. The key fact for us is that

$$H^{p,q}(X, R) = \text{Hom}_{\mathbf{DM}}(R_{\text{tr}}X, R(q)[p]), \tag{2.3.0}$$

and this is proven in [4, 14.16].

2.3.1. A variant which we shall need is the *motivic homology* $H_{p,q}(X, R)$ of a smooth X, defined as the R-module $\text{Hom}_{\mathbf{DM}}(R(q)[p], R_{\text{tr}}X)$; see [4, 14.7]. When X is smooth projective of dimension d, then by Duality

$$H_{-p,-q}(X, R) \cong \text{Hom}(R, R_{\text{tr}}X(q)[p]) \cong \text{Hom}(R_{\text{tr}}X, R(q+d)[p+2d]) = H^{p+2d,q+d}(X, R).$$

See [4, 16.24]. For $d = 0$, when $X = \text{Spec}(k)$, we have $H_{-p,-q}(k, R) \cong H^{p,q}(k, R)$, and in particular $H_{-1,-1}(k, \mathbb{Z}) = H^{1,1}(k, \mathbb{Z}) = k^\times$.

2.4. We now introduce Voevodsky's "radditive" model for the fundamental adjunction between the Morel-Voevodsky category [7] of pointed spaces and $\mathbf{DM}$. Let $\mathbf{Rad} = \mathbf{Rad}(Sm/k)$ denote the category of contravariant functors $F\colon Sm/k \to \mathbf{Sets}_*$ (pointed sets) sending ϕ to the point, and disjoint unions to products ("radditive functors"). Using the Yoneda embedding, we can identify Sm/k with a full subcategory of $\mathbf{Rad}$.

Now R-$\mathbf{mod} \to \mathbf{Sets}$ induces a forgetful functor $u\colon \mathbf{PST} \to \mathbf{Rad}$. It has a left adjoint R_{tr}, defined by Kan extension from the formula that $Sm/k \to \mathbf{Rad} \to \mathbf{PST}$ is R_{tr}. Thus $R_{\text{tr}}F$ is the quotient of $\oplus_{\alpha \in F(X)} R_{\text{tr}}(X)$ by the relation that for every $f\colon Y \to X$ the copy of $R_{\text{tr}}(X)$ indexed by α is identified with its image in the copy of $R_{\text{tr}}(Y)$ indexed by $f^*(\alpha)$.

A more natural construction involves simplicial objects in **Rad**. There is a canonical resolution $\mathbf{Lres}(F)$ of any simplicial object F in **Rad** by wedges of representable functors and we define $R^L_{\rm tr}(F)$ to be the simplicial object $R_{\rm tr}(\mathbf{Lres}(F))$. This is left adjoint to the simplicial extension of the forgetful functor on the appropriate (global) homotopy category of simplicial presheaves. The details of this machinery are in several of Voevodsky's preprints, including [V07], and there are several variants such as in [8], [6, p.241], and [13].

Definition 2.5. Let U^q denote the simplicial cone of $(\mathbb{A}^q - 0) \to \mathbb{A}^q$, regarded as a simplicial (representable) object in **Rad**. The Lefschetz object $\mathbb{L}^q = \mathbb{L}^q(R)$ is defined to be the simplicial presheaf with transfers $R_{\rm tr}(U^q)$. By [4, 15.3], $\mathbb{L}^q$ is isomorphic to $R(q)[2q]$ in **DM**. As a special case, we have $U^0 = S^0$ and $\mathbb{L}^0 = R$.

Now it is possible to localize the categories $\Delta^{\rm op}\mathbf{Rad}$ and $\Delta^{\rm op}\mathbf{PST}$ with respect to $\mathbb{A}^1$-equivalences. The former yields the Morel-Voevodsky homotopy category of pointed spaces **Ho**, and the latter yields a full subcategory of $\mathbf{DM}^{\rm eff}(k,R)$. Moreover, the pair $(R_{\rm tr}, u)$ form a Quillen adjunction with respect to the $\mathbb{A}^1$-local closed model structures. This is somewhat technical, and the details are in [V07].

Theorem 2.6. *The space $K_a = u\mathbb{L}^a(R)$ represents $H^{2a,a}(-,R)$ in the sense that*

$$H^{2a,a}(X,R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{tr}X, \mathbb{L}^a) \cong \mathrm{Hom}_{\mathbf{Ho}}(X, K_a)$$

Proof. This is immediate from the adjunction. □

Corollary 2.7. *Motivic cohomology operations from $H^{2a,a}(X,R')$ to $H^{p,q}(X,R)$ are in $1-1$ correspondence with elements of $H^{pq}(K_a,R)$, where $K_a = u\mathbb{L}^a(R')$.*

Proof. Since K_a represents $H^{2a,a}(-,R')$, this is the Yoneda theorem. □

Example 2.8. When $a=0$, K_0 is the pointed space $uR' = \bigvee_{R'-0} S^0$, and cohomology operations $H^{0,0}(-,R') \to H^{p,q}(-,R)$ correspond to elements of $H^{p,q}(uR',R) = \prod_{R'-0} H^{pq}(k,R)$.