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Mixed motives and triangulated categories of motives

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Recall from last time:

The category $\mathrm{Cor}_{fin_\sim}(k)_F$: objects $h(X)$ for $X \in \mathbf{SmProj}/k$. Morphisms (for X irreducible) are

$$\mathrm{Hom}_{\mathrm{Cor}_{fin_\sim}(k)_F}(X, Y) := \mathcal{Z}^{d_X}_\sim(X \times Y)_F$$

with composition the composition of correspondences.

The category $\mathrm{Cor}_{fin^*_\sim}(k)_F$: objects are direct sums of $h(X)(r)$, for $X \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$. Morphisms (for X irreducible)

$$\mathrm{Hom}_{\mathrm{Cor}_{fin^*_\sim}(k)_F}(h(X)(r), h(Y)(s)) := \mathcal{Z}^{d_X+s-r}_\sim(X \times Y)_F$$

with composition as correspondences.

The category of *pure effective motives* with F-coefficients is

$$M_{\sim}^{\text{eff}}(k)_F := \text{Cor}_{fin_{\sim}}(k)_F^{\natural}$$

We have $\mathbb{L} := (\mathbb{P}^1, \mathbb{P}^1 \times 0) \in M_{\sim}^{\text{eff}}(k)$.

The category of *pure motives* with F-coefficients is

$$M_{\sim}(k)_F := \text{Cor}_{fin_{\sim}^{*}}(k)_F^{\natural} \cong M_{\sim}^{\text{eff}}(k)_F[\mathbb{L}^{\otimes -1}].$$

These are all tensor categories (over F). $M_{\sim}^{\text{eff}}(k)_F$ and $M_{\sim}(k)_F$ are pseudo-abelian.

Sending $h(X)$ to $h(X)(0)$ extends to a faithful embedding

$$i : M_{\sim}^{\mathrm{eff}}(k)_F \to M_{\sim}(k)_F.$$

Example. $i(\mathbb{L}) = \mathbb{1}(-1) = h(\mathrm{Spec}\, k)(-1)$.

Sending X to $h(X)$ or $h(X)(0)$ gives symmetric monoidal functors

$$\mathfrak{h}_{\sim} : \mathbf{SmProj}/k^{\mathrm{op}} \to M_{\sim}^{\mathrm{eff}}(k)_F;\ \ \mathfrak{h}_{\sim} : \mathbf{SmProj}/k^{\mathrm{op}} \to M_{\sim}(k)_F$$

For a Weil cohomology theory H^*, the functor $H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ extends canonically to tensor functors

$$H^* : M_{\mathrm{hom}}^{\mathrm{eff}}(k)_K \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K;\ \ H^* : M_{\mathrm{hom}}(k)_K \to \mathrm{GrVec}_K$$

In $M_\sim(k)_F$, each object $M = (h(X)(r), \alpha)$ has a dual:

$$(h(X)(r), \alpha)^\vee := (h(X)(d_X - r), {}^t\alpha)$$

The unit $\mathbb{1} \to M^\vee \otimes M$ and trace $M \otimes M^\vee \to \mathbb{1}$ (for $M = \mathfrak{h}(X)$) are given by the diagonal

$$\begin{aligned}[\Delta_X]_\sim \in \mathcal{Z}^{d_X}_\sim(X \times X) &= \mathrm{Hom}_{M_\sim(k)}(\mathbb{1}, \mathfrak{h}(X)(d_X) \otimes \mathfrak{h}(X)) \\ &= \mathrm{Hom}_{M_\sim(k)}(\mathfrak{h}(X) \otimes \mathfrak{h}(X)(d_X), \mathbb{1}).\end{aligned}$$

This makes $M_\sim(k)_F$ a rigid tensor category.

Chow motives and numerical motives

If $\sim \succ \approx$, the surjection $\mathcal{Z}_\sim \to \mathcal{Z}_\approx$ yields functors $\mathsf{Cor}_{fin_\sim}(k) \to \mathsf{Cor}_{fin_\approx}(k)$, $\mathsf{Cor}_{fin^*_\sim}(k) \to \mathsf{Cor}_{fin^*_\approx}(k)$ and thus

$$M^{\mathsf{eff}}_\sim(k) \to M^{\mathsf{eff}}_\approx(k);\ M_\sim(k) \to M_\approx(k).$$

Thus the category of pure motives with the most information is for the finest equivalence relation $\sim = \sim_{\mathsf{rat}}$.

Set $CHM(k)_F := M_{\mathsf{rat}}(k)_F$.

For example $\mathsf{Hom}_{CHM(k)}(\mathbb{1}(-r), \mathfrak{h}(X)) = \mathsf{CH}^r(X)$.

The coarsest equivalence is $\sim_{\mathsf{num}}$, so $M_{\mathsf{num}}(k)$ should be the most simple category of motives.

Set $NM(k)_F := M_{\mathsf{num}}(k)_F$.

Jannsen's semi-simplicity theorem

Theorem (Jannsen) *Fix F a field, char $F = 0$. $NM(k)_F$ is a semi-simple abelian category. If $M_\sim(k)_F$ is semi-simple abelian, then $\sim = \sim_{\text{num}}$.*

Proof. $d := d_X$. We show $\text{End}_{NM(k)_F}(\mathfrak{h}(X)) = \mathcal{Z}^d_{\text{num}}(X^2)_F$ is a finite dimensional semi-simple F-algebra for all $X \in \mathbf{SmProj}/k$. We may extend F, so can assume $F = K$ is the coefficient field for a Weil cohomology on $\mathbf{SmProj}/k$.

Consider the surjection $\pi : \mathcal{Z}^d_{\text{hom}}(X^2)_F \to \mathcal{Z}^d_{\text{num}}(X^2)_F$. $\mathcal{Z}^d_{\text{hom}}(X^2)_F$ is finite dimensional, so $\mathcal{Z}^d_{\text{num}}(X^2)_F$ is finite dimensional.

Also, the radical $\mathcal{N}$ of $\mathcal{Z}^d_{\text{hom}}(X^2)_F$ is nilpotent and it suffices to show that $\pi(\mathcal{N}) = 0$.

Take $f \in \mathcal{N}$. Then $f \circ {}^t g$ is in $\mathcal{N}$ for all $g \in \mathcal{Z}^d_{\mathrm{hom}}(X^2)_F$, and thus $f \circ {}^t g$ is nilpotent. Therefore

$$Tr(H^+(f \circ {}^t g)) = Tr(H^-(f \circ {}^t g)) = 0.$$

By the LTF

$$\deg(f \cdot g) = Tr(H^+(f \circ {}^t g)) - Tr(H^-(f \circ {}^t g)) = 0$$

hence $f \sim_{\mathrm{num}} 0$, i.e., $\pi(f) = 0$.

Chow motives $CHM(k)_F$ has a nice universal property extending the one we have already described:

Theorem *Giving a Weil cohomology theory H^* on $\mathbf{SmProj}/k$ with coefficient field $K \supset F$ is equivalent to giving a tensor functor*

$$H^* : CHM(k)_F \to \mathsf{GrVec}_K$$

with $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2$.

"Weil cohomology" $\rightsquigarrow H^*$ because $\sim_{\text{rat}} \succ \sim_H$.

$H^* \rightsquigarrow$ Weil cohomology: $\mathbb{1}(-1)$ is invertible and $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2 \implies H^2(\mathbb{P}^1) \cong K$.

$\mathfrak{h}(X)^\vee = \mathfrak{h}(X)(d_X) \rightsquigarrow H^*(\mathfrak{h}(X))$ is supported in degrees $[0, 2d_X]$

Rigidity of $CHM(k)_F \rightsquigarrow$ Poincaré duality.

Adequate equivalence relations revisited

Definition Let $\mathcal{C}$ be an additive category. The *Kelly radical* $\mathcal{R}$ is the collection

$$\mathcal{R}(X,Y) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid \forall g \in \mathrm{Hom}_{\mathcal{C}}(Y,X), 1-gf \text{ is invertible}\}$$

$\mathcal{R}$ forms an *ideal* in $\mathcal{C}$ (subgroups $\mathcal{I}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ closed under $- \circ g$, $g \circ -$).

Lemma *$\mathcal{C} \to \mathcal{C}/\mathcal{R}$ is conservative, and $\mathcal{R}$ is the largest such ideal.*

Note. If $\mathcal{I} \subset \mathcal{C}$ is an ideal such that $\mathcal{I}(X,X)$ is a nil-ideal in $\mathrm{End}(X)$ for all X, then $\mathcal{I} \subset \mathcal{R}$.

Definition $(\mathcal{C}, \otimes)$ a tensor category. A ideal $\mathcal{I}$ in $\mathcal{C}$ is a $\otimes$ *ideal* if $f \in \mathcal{I}$, $g \in \mathcal{C} \Rightarrow f \otimes g \in \mathcal{I}$.

$\otimes$ descends to $\mathcal{C}/\mathcal{I}$ iff $\mathcal{I}$ is a tensor ideal. $\mathcal{R}$ is *not* in general a $\otimes$ ideal.

Theorem *There is a 1-1 correspondence between adequate equivalence relations on* $\mathbf{SmProj}/k$ *and proper* $\otimes$ *ideals in* $CHM(k)_F$*:* $M_\sim(k)_F := (CHM(k)_F/\mathcal{I}_\sim)^\natural$.

In particular: Let $\mathcal{N} \subset CHM(k)_\mathbb{Q}$ be the tensor ideal defined by numerical equivalence. Then $\mathcal{N}$ is the largest proper $\otimes$ ideal in $CHM(k)_\mathbb{Q}$.

Mixed motives

and
triangulated categories of motives

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Outline:

- Mixed motives
- Triangulated categories
- Geometric motives

Mixed motives

Why mixed motives?

Pure motives describe the cohomology of smooth projective varieties

Mixed motives should describe the cohomology of *arbitrary* varieties.

Weil cohomology is replaced by *Bloch-Ogus* cohomology: Mayer-Vietoris for open covers and a purity isomorphism for cohomology with supports.

An analog: Hodge structures

The cohomology H^n of a smooth projective variety over $\mathbb{C}$ has a natural pure Hodge structure.

Deligne gave the cohomology of an arbitrary variety over $\mathbb{C}$ a natural *mixed* Hodge structure.

The category of (polarizable) pure Hodge structures is a semi-simple abelian rigid tensor category. The category of (polarizable) mixed Hodge structures is an abelian rigid tensor category, but has non-trivial extensions. The semi-simple objects in MHS are the pure Hodge structures.

MHS has a natural exact finite *weight filtration* on W_*M on each object M, with graded pieces $\mathrm{gr}_n^W M$ pure Hodge structures.

There is a functor

$$RHdg : \mathbf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to D^b(MHS)$$

with $R^nHdg(X) = H^n(X)$ with its MHS, lifting the singular cochain complex functor

$$C^*(-,\mathbb{Z}) : \mathbf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to D^b(\mathbf{Ab}).$$

In addition, all natural maps involving the cohomology of X: pull-back, proper pushforward, boundary maps in local cohomology or Mayer-Vietoris sequences, are maps of MHS.

Beilinson's conjectures

Beilinson conjectured that the semi-simple abelian category of pure motives $M_{\text{hom}}(k)_{\mathbb{Q}}$ should admit a full embedding as the semi-simple objects in an abelian rigid tensor category of *mixed motives* $MM(k)$.

$MM(k)$ should have the following structures and properties:

• a natural finite exact weight filtration W_*M on each M with graded pieces $\text{gr}^W_n M$ pure motives.

• For $\sigma : k \to \mathbb{C}$ a *realization functor* $\Re_\sigma : MM(k) \to MHS$ compatible with all the structures.

• A functor $R\mathfrak{h} : \mathbf{Sch}_k^{\text{op}} \to D^b(MM(k))$ such that $\Re_\sigma(R^n\mathfrak{h}(X))$ is $H^n(X)$ as a MHS.

• A natural isomorphism ($\mathbb{Q}(n)[2n] \cong \mathbb{1}(n)$)

$$\mathrm{Hom}_{D^b(MM(k))}(\mathbb{Q}, R\mathfrak{h}(X)(q)[p]) \cong K^{(q)}_{2q-p}(X),$$

in particular $\mathrm{Ext}^p_{MM(k)}(\mathbb{Q}, \mathbb{Q}(q)) \cong K^{(q)}_{2q-p}(k)$.

• All "universal properties" of the cohomology of algebraic varieties should be reflected by identities in $D^b(MM(k))$ of the objects $R\mathfrak{h}(X)$.

Motivic sheaves

In fact, Beilinson views the above picture as only the story over the generic point $\operatorname{Spec} k$.

He conjectured further that there should be a system of categories of "motivic sheaves"

$$S \mapsto MM(S)$$

together with functors Rf_*, f^*, $f^!$ and $Rf_!$, as well as $\mathcal{H}om$ and $\otimes$, all satisfying the yoga of Grothendieck's six operations for the categories of sheaves for the étale topology.

A partial success

The categories $MM(k)$, $MM(S)$ have not been constructed.

However, there are now a number of (equivalent) constructions of *triangulated tensor categories* that satisfy all the structural properties expected of the derived categories $D^b(MM(k))$ and $D^b(MM(S))$, except those which exhibit these as a derived category of an abelian category (t-structure).

There are at present various attempts to extend this to the triangulated version of Beilinson's vision of motivic sheaves over a base S.

We give a discussion of the construction of various versions of triangulated categories of mixed motives over k due to Voevodsky.

Triangulated categories

Translations and triangles

A *translation* on an additive category $\mathcal{A}$ is an equivalence $T : \mathcal{A} \to \mathcal{A}$. We write $X[1] := T(X)$.

Let $\mathcal{A}$ be an additive category with translation. A *triangle* (X, Y, Z, a, b, c) in $\mathcal{A}$ is the sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & X[1] \\ f\big\downarrow & & g\big\downarrow & & h\big\downarrow & & f[1]\big\downarrow \\ X' & \xrightarrow[a']{} & Y' & \xrightarrow[b']{} & Z' & \xrightarrow[c']{} & X'[1]. \end{array}$$

Verdier has defined a *triangulated category* as an additive category $\mathcal{A}$ with translation, together with a collection $\mathcal{E}$ of triangles, called the *distinguished triangles* of $\mathcal{A}$, which satisfy

TR1

$\mathcal{E}$ is closed under isomorphism of triangles.

$A \xrightarrow{\text{id}} A \to 0 \to A[1]$ is distinguished.

Each $X \xrightarrow{u} Y$ extends to a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to X[1]$$

TR2

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished

$\Leftrightarrow Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished

TR3

Given a commutative diagram with distinguished rows

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & X'[1] \end{array}$$

there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & X'[1] \end{array}$$

TR4

If we have three distinguished triangles $(X, Y, Z', u, i, *)$, $(Y, Z, X', v, *, j)$, and $(X, Z, Y', w, *, *)$, with $w = v \circ u$, then there are morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that

- (id_X, v, f) is a morphism of triangles
- (u, id_Z, g) is a morphism of triangles
- $(Z', Y', X', f, g, i[1] \circ j)$ is a distinguished triangle.

A graded functor $F : \mathcal{A} \to \mathcal{B}$ of triangulated categories is called *exact* if F takes distinguished triangles in $\mathcal{A}$ to distinguished triangles in $\mathcal{B}$.

Remark Suppose $(\mathcal{A}, T, \mathcal{E})$ satisfies (TR1), (TR2) and (TR3). If (X, Y, Z, a, b, c) is in $\mathcal{E}$, and A is an object of $\mathcal{A}$, then the sequences

$$\ldots \xrightarrow{c[-1]_*} \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{A}}(A, Z) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{A}}(A, X[1]) \xrightarrow{a[1]_*} \ldots$$

and

$$\ldots \xrightarrow{a[1]^*} \mathrm{Hom}_{\mathcal{A}}(X[1], A) \xrightarrow{c^*} \mathrm{Hom}_{\mathcal{A}}(Z, A) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{A}}(Y, A) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{c[-1]^*} \ldots$$

are exact.

This yields:

• (five-lemma): If (f, g, h) is a morphism of triangles in $\mathcal{E}$, and if two of f, g, h are isomorphisms, then so is the third.

• If (X, Y, Z, a, b, c) and (X, Y, Z', a, b', c') are two triangles in $\mathcal{E}$, there is an isomorphism $h : Z \to Z'$ such that

$$(\mathrm{id}_X, \mathrm{id}_Y, h) : (X, Y, Z, a, b, c) \to (X, Y, Z', a, b', c')$$

is an isomorphism of triangles.

If (TR4) holds as well, then $\mathcal{E}$ is closed under taking finite direct sums.

The main point

A triangulated category is a machine for generating natural long exact sequences.

An example Let $\mathcal{A}$ be an additive category, $C^?(\mathcal{A})$ the category of cohomological complexes (with boundedness condition $? = \emptyset, +, -, b$), and $K^?(\mathcal{A})$ the homotopy category.

For a complex (A, d_A), let $A[1]$ be the complex

$$A[1]^n := A^{n+1}; \quad d^n_{A[1]} := -d^{n+1}_A.$$

For a map of complexes $f : A \to B$, we have the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1]$$

where $\mathrm{Cone}(f) := A^{n+1} \oplus B^n$ with differential

$$d(a, b) := (-d_A(a), f(a) + d_B(b))$$

i and p are the evident inclusions and projections.

We make $K^?(\mathcal{A})$ a triangulated category by declaring a triangle to be exact if it is isomorphic to the image of a cone sequence.

Tensor structure

Definition Suppose $\mathcal{A}$ is both a triangulated category and a tensor category (with tensor operation $\otimes$) such that $(X \otimes Y)[1] = X[1] \otimes Y$.

Suppose that, for each distinguished triangle (X, Y, Z, a, b, c), and each $W \in \mathcal{A}$, the sequence

$$X \otimes W \xrightarrow{a \otimes \mathrm{id}_W} Y \otimes W \xrightarrow{b \otimes \mathrm{id}_W} Z \otimes W \xrightarrow{c \otimes \mathrm{id}_W} X[1] \otimes W = (X \otimes W)[1]$$

is a distinguished triangle. Then $\mathcal{A}$ is a *triangulated tensor category*.

Example If $\mathcal{A}$ is a tensor category, then $K^{?}(\mathcal{A})$ inherits a tensor structure, by the usual tensor product of complexes, and becomes a triangulated tensor category. (For $? = \emptyset$, $\mathcal{A}$ must admit infinite direct sums).

Thick subcategories

Definition A full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is *thick* if $\mathcal{B}$ is closed under taking direct summands.

If $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, the set of morphisms $s : X \to Y$ in $\mathcal{A}$ which fit into a distinguished triangle $X \xrightarrow{s} Y \to Z \to X[1]$ with Z in $\mathcal{B}$ forms a *saturated multiplicative system* of morphisms.

The intersection of thick subcategories of $\mathcal{A}$ is a thick subcategory of $\mathcal{A}$, So, for each set $\mathcal{T}$ of objects of $\mathcal{A}$, there is a smallest thick subcategory $\mathcal{B}$ containing $\mathcal{T}$, called the thick subcategory *generated* by $\mathcal{T}$.

Remark The original definition (Verdier) of a thick subcategory had the condition:

Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{A}$, with Z in $\mathcal{B}$. If f factors as $X \xrightarrow{f_1} B' \xrightarrow{f_2} Y$ with B' in $\mathcal{B}$, then X and Y are in $\mathcal{B}$.

This is equivalent to the condition given above, that $\mathcal{B}$ is closed under direct summands in $\mathcal{A}$ (*cf.* Rickard).

Localization of triangulated categories

Let $\mathcal{B}$ be a thick subcategory of a triangulated category $\mathcal{A}$. Let $\mathcal{S}$ be the saturated multiplicative system of map $A \xrightarrow{s} B$ with "cone" in $\mathcal{B}$.

Form the category $\mathcal{A}[\mathcal{S}^{-1}] = \mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$, with

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X,Y) = \varinjlim_{s:X'\to X\in\mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X',Y).$$

Composition of diagrams

$$\begin{array}{ccccc} & & Y' & \xrightarrow{g} & Z \\ & & \downarrow{\scriptstyle t} & & \\ X' & \xrightarrow{f} & Y & & \\ {\scriptstyle s}\downarrow & & & & \\ X & & & & \end{array}$$

is defined by filling in the middle

$$\begin{array}{ccccc} X'' & \xrightarrow{f'} & Y' & \xrightarrow{g} & Z \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle t} & & \\ X' & \xrightarrow[f]{} & Y & & \\ {\scriptstyle s}\downarrow & & & & \\ X. & & & & \end{array}$$

One can describe $\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X,Y)$ by a calculus of left fractions as well, i.e.,

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X,Y) = \varinjlim_{s:Y \to Y' \in \mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X,Y').$$

Let $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical functor.

Theorem (Verdier) *(i)* $\mathcal{A}/\mathcal{B}$ *is a triangulated category, where a triangle* T *in* $\mathcal{A}/\mathcal{B}$ *is distinguished if* T *is isomorphic to the image under* $Q_{\mathcal{B}}$ *of a distinguished triangle in* $\mathcal{A}$.

(ii) The functor $Q_{\mathcal{B}}$ *is universal for exact functors* $F : \mathcal{A} \to \mathcal{C}$ *such that* $F(B)$ *is isomorphic to 0 for all* B *in* $\mathcal{B}$.

(iii) $\mathcal{S}$ *is equal to the collection of maps in* $\mathcal{A}$ *which become isomorphisms in* $\mathcal{A}/\mathcal{B}$ *and* $\mathcal{B}$ *is the subcategory of objects of* $\mathcal{A}$ *which becomes isomorphic to zero in* $\mathcal{A}/\mathcal{B}$.

Remark If $\mathcal{A}$ admits some infinite direct sums, it is sometimes better to preserve this property. A subscategory $\mathcal{B}$ of $\mathcal{A}$ is called *localizing* if $\mathcal{B}$ is thick and is closed under direct sums which exist in $\mathcal{A}$.

For instance, if $\mathcal{A}$ admits arbitrary direct sums and $\mathcal{B}$ is a localizing subcategory, then $\mathcal{A}/\mathcal{B}$ also admits arbitrary direct sums.

Localization with respect to localizing subcategories has been studied by Thomason and by Ne'eman.

Localization of triangulated tensor categories If $\mathcal{A}$ is a triangulated tensor category, and $\mathcal{B}$ a thick subcategory, call $\mathcal{B}$ a *thick tensor subcategory* if A in $\mathcal{A}$ and B in $\mathcal{B}$ implies that $A \otimes B$ and $B \otimes A$ are in $\mathcal{B}$.

The quotient $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ of $\mathcal{A}$ by a thick tensor subcategory inherits the tensor structure, and the distinguished triangles are preserved by tensor product with an object.

Example The classical example is the *derived category* $D^?(\mathcal{A})$ of an abelian category $\mathcal{A}$. $\mathcal{D}^?(\mathcal{A})$ is the localization of $K^?(\mathcal{A})$ with respect to the multiplicative system of *quasi-isomorphisms* $f : A \to B$, i.e., f which induce isomorphisms $H^n(f) : H^n(A) \to H^n(B)$ for all n.

If $\mathcal{A}$ is an abelian tensor category, then $D^-(\mathcal{A})$ inherits a tensor structure $\otimes^L$ if each object A of $\mathcal{A}$ admits a surjection $P \to A$ where P is *flat*, i.e. $M \mapsto M \otimes P$ is an exact functor on $\mathcal{A}$. If each A admits a finite flat (right) resolution, then $D^b(\mathcal{A})$ has a tensor structure $\otimes^L$ as well. The tensor structure $\otimes^L$ is given by forming for each $A \in K^?(\mathcal{A})$ a quasi-isomorphism $P \to A$ with P a complex of flat objects in $\mathcal{A}$, and defining

$$A \otimes^L B := \mathrm{Tot}(P \otimes B).$$

Geometric motives

Voevodsky constructs a number of categories: the category of geometric motives $DM_{\mathrm{gm}}(k)$ with its effective subcategory $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, as well as a sheaf-theoretic construction DM^{eff}_{-}, containing $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a full dense subcategory. In contrast to almost all other constructions, these are based on *homology* rather than cohomology as the starting point, in particular, the motives functor from $\mathbf{Sm}/k$ to these categories is covariant.

Finite correspondences To solve the problem of the partially defined composition of correspondences, Voevodsky introduces the notion of *finite* correspondences, for which all compositions are defined.

Definition Let X and Y be in $\mathbf{Sch}_k$. The group $c(X, Y)$ is the subgroup of $z(X \times_k Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that

1. the projection $p_1 : W \to X$ is finite
2. the image $p_1(W) \subset X$ is an irreducible component of X.

The elements of $c(X, Y)$ are called the *finite* correspondences from X to Y.

The following basic lemma is easy to prove:

Lemma *Let X, Y and Z be in $\mathbf{Sch}_k$, $W \in c(X, Y)$, $W' \in c(Y, Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of $|W| \times Z \cap X \times |W'|$ is finite over X and $p_X(C) = X$.*

Thus: for $W \in c(X, Y)$, $W' \in c(Y, Z)$, we have the composition:

$$W' \circ W := p_{XZ*}(p^*_{XY}(W) \cdot p^*_{YZ}(W')),$$

This operation yields an associative bilinear *composition law*

$$\circ : c(Y, Z) \times c(X, Y) \to c(X, Z).$$

The category of finite correspondences

Definition The category $\mathrm{Cor}_{fin}(k)$ is the category with the same objects as $\mathbf{Sm}/k$, with

$$\mathrm{Hom}_{\mathrm{Cor}_{fin}(k)}(X, Y) := c(X, Y),$$

and with the composition as defined above.

Remarks (1) We have the functor $\mathbf{Sm}/k \to \mathrm{Cor}_{fin}(k)$ sending a morphism $f : X \to Y$ in $\mathbf{Sm}/k$ to the graph $\Gamma_f \subset X \times_k Y$.

(2) We write the morphism corresponding to Γ_f as f_*, and the object corresonding to $X \in \mathbf{Sm}/k$ as $[X]$.

(3) The operation $\times_k$ (on smooth k-schemes and on cycles) makes $\mathrm{Cor}_{fin}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor}_{fin}(k))$ is a triangulated tensor category.

The category of effective geometric motives

Definition The category $\widehat{DM}^{\text{eff}}_{\text{gm}}(k)$ is the localization of $K^b(\text{Cor}_{fin}(k))$, as a triangulated tensor category, by

• *Homotopy.* For $X \in \mathbf{Sm}/k$, invert $p_* : [X \times \mathbb{A}^1] \to [X]$

• *Mayer-Vietoris.* Let X be in $\mathbf{Sm}/k$. Write X as a union of Zariski open subschemes U, V: $X = U \cup V$.

We have the canonical map

$$\text{Cone}([U \cap V] \xrightarrow{(j_{U,U\cap V*}, -j_{V,U\cap V*})} [U] \oplus [V]) \xrightarrow{(j_{U*}+j_{V*})} [X]$$

since $(j_{U*} + j_{V*}) \circ (j_{U,U\cap V*}, -j_{V,U\cap V*}) = 0$. Invert this map.

The category $DM^{\text{eff}}_{\text{gm}}(k)$ of *effective geometric motives* is the pseudo-abelian hull of $\widehat{DM}^{\text{eff}}_{\text{gm}}(k)$ (Balmer-Schlichting).

The motive of a smooth variety

Let $M_{\mathsf{gm}}(X)$ be the image of $[X]$ in $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$. Sending $f : X \to Y$ to $M_{\mathsf{gm}}(f) := [\Gamma_f] = f_*$ defines

$$M^{\mathsf{eff}}_{\mathsf{gm}} : \mathbf{Sm}/k \to DM^{\mathsf{eff}}_{\mathsf{gm}}(k).$$

Note. $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$ is modeled on *homology*, so $M^{\mathsf{eff}}_{\mathsf{gm}}$ is a covariant functor. In fact, $M^{\mathsf{eff}}_{\mathsf{gm}}$ is a symmetric monoidal functor.

The category of geometric motives

To define the category of geometric motives we invert the Lefschetz motive.

For $X \in \mathbf{Sm}_k$, the *reduced motive* (in $C^b(\mathrm{Cor}_{fin}(k))$) is

$$\widetilde{[X]} := \mathrm{Cone}(p_* : [X] \to [\mathrm{Spec}\, k])[-1].$$

If X has a k-point $0 \in X(k)$, then $p_* i_{0*} = \mathrm{id}_{[\mathrm{Spec}\, k]}$ so

$$[X] = \widetilde{[X]} \oplus [\mathrm{Spec}\, k]$$

and

$$\widetilde{[X]} \cong \mathrm{Cone}(i_{0*} : [\mathrm{Spec}\, k] \to [X])$$

in $K^b(\mathrm{Cor}_{fin}(k))$.

Write $\widetilde{M_{\text{gm}}^{\text{eff}}(X)}$ for the image of $[\widetilde{X}]$ in $M_{\text{gm}}^{\text{eff}}(k)$.

Set $\mathbb{Z}(1) := \widetilde{M_{\text{gm}}^{\text{eff}}(\mathbb{P}^1)}[-2]$, and set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Thus $M_{\text{gm}}^{\text{eff}}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, $\mathbb{Z} := M_{\text{gm}}(\text{Spec}\, k) = \mathbb{Z}(0)$. So $\mathbb{Z}(1)[2]$ is like the Lefschetz motive.

Definition The category of *geometric motives*, $DM_{\text{gm}}(k)$, is defined by inverting the functor $\otimes\mathbb{Z}(1)$ on $DM_{\text{gm}}^{\text{eff}}(k)$, i.e., for $r, s \in \mathbb{Z}$, $M, N \in DM_{\text{gm}}^{\text{eff}}(k)$,

$$\begin{aligned} &\text{Hom}_{DM_{\text{gm}}(k)}(M(r), N(s)) \\ &\qquad := \varinjlim_{n} \text{Hom}_{DM_{\text{gm}}^{\text{eff}}(k)}(M \otimes \mathbb{Z}(n+r), N \otimes \mathbb{Z}(n+s)). \end{aligned}$$

Remark In order that $DM_{\text{gm}}(k)$ be again a triangulated category, it suffices that the commutativity involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ be the identity, which is in fact the case.

Of course, there arises the question of the behavior of the evident functor $DM_{\text{gm}}^{\text{eff}}(k) \to DM_{\text{gm}}(k)$. Here we have

Theorem (Cancellation) *The functor $i : DM_{\text{gm}}^{\text{eff}}(k) \to DM_{\text{gm}}(k)$ is a fully faithful embedding.*

Let $M_{\text{gm}} : \mathbf{Sm}/k \to DM_{\text{gm}}(k)$ be $i \circ M_{\text{gm}}^{\text{eff}}$. $M_{\text{gm}}(X)$ is the *motive of X*.