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Reduction to axioms

Charles Weibel
Rutgers University, Piscataway, USA

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

3 Lecture 3: Rost Motives

The material in this lecture is based on [W-ax], which is based on [MC/l].

We begin by giving the definition of a Rost variety. By a ν_{n-1}-*variety* over k we mean a smooth projective variety X of dimension $d = \ell^{n-1} - 1$, with the degree of $s_d(X)$ being $\not\equiv 0 \pmod{\ell^2}$; see [9, 1.20]. Here $s_d(X)$ is the characteristic class of the tangent bundle of X corresponding to the symmetric polynomial $\Sigma\, t_j^d$ in the Chern roots t_j, see [RPO, 14.3].

Definition 3.1. A *Rost variety* for a sequence $\underline{a} = (a_1, \ldots, a_n)$ of units of k is a ν_{n-1}-variety X satisfying:

(a) X splits $\underline{a}$, i.e., $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$;

(b) For each $i < n$ there is a ν_i-variety mapping to X;

(c) The following motivic homology sequence is exact (for $R = \mathbb{Z}$):

$$H_{-1,-1}(X \times X) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \longrightarrow H_{-1,-1}(k) = k^\times$$

is exact. See 2.3.1 for the definition of $H_{-1,-1}$ and the calculation that $H_{-1,-1}(k) = k^\times$.

As mentioned in Lecture 1, Rost varieties exist for all n, ℓ and $\underline{a}$. When $n = 1$, Spec(L) is a Rost variety for a when $L = k\left(\sqrt[\ell]{a}\,\right)$. When $n = 2$, the Severi-Brauer variety corresponding to the degree ℓ division algebra with symbol $\underline{a}$ is a Rost variety for $\underline{a}$.

In this lecture we write $\mathfrak{X}$ for the simplicial Čech scheme $\check{C}(X)$ of Lecture 1, and $\Sigma\mathfrak{X}$ for its suspension i.e., the cone of $\mathfrak{X} \to \mathrm{Spec}(k)$. Note that $H^{p,q}(\mathfrak{X}, \mathbb{Z}) \cong H^{p+1,q}(\Sigma\mathfrak{X}, \mathbb{Z})$ when $p > q$, as $H^{pq}(k, \mathbb{Z}) = 0$ in this range. A transfer argument [W-ax, 2.3] shows that the motivic cohomology groups $H^{**}(\Sigma\mathfrak{X}, \mathbb{Z})$ have exponent ℓ. Hence we have exact sequences

$$0 \to H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}) \to H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\tilde{\beta}} H^{p+1,q}(\Sigma\mathfrak{X}, \mathbb{Z}) \to 0.$$

By 3.1(b), Theorem 3.2 of [MC/2] translates to:

Theorem 3.2. *If $i < n$, the sequences* $\xrightarrow{Q_i} H^{**}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{Q_i}$ *are exact.*

Proposition 3.3. (Voevodsky) *The cohomology operations $Q_0 = \beta$, $Q = Q_{n-2} \cdots Q_0$ and $Q_{n-1}Q$ are injections on $H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$, and restrict to injections*

$$H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\beta} H^{n+1,n-1}(\mathfrak{X}, \mathbb{Z}) \xrightarrow{Q_{n-2}\cdots Q_1} H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}) \xrightarrow{Q_{n-1}} H^{2b\ell+2,b\ell+1}(\mathfrak{X}, \mathbb{Z}),$$

where $b = b/\ell - 1 = 1 + \ell + \cdots + \ell^{n-2}$. In particular, $\mu = Q(\delta) \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$ is nonzero.

Proof. It suffices to show that each $Q_i \ldots Q_0$ is injective on the group $H^{n+1,n-1}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \cong H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$. This follows from Theorem 3.2 and the observation in [MC/2, 6.9] that

$$H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) = 0 \text{ when } (p, q) \text{ is in the region } q < n,\ p \le 1 + q.$$

Indeed, each Q_i is injective on the group $H^{**}(\Sigma\mathfrak{X})$ containing $Q_{i-1} \ldots Q_1 H^{n+2,n}(\Sigma\mathfrak{X})$, because the preceeding term in Theorem 3.2 is zero. Figure 1 shows the case $i = 2$. ☐

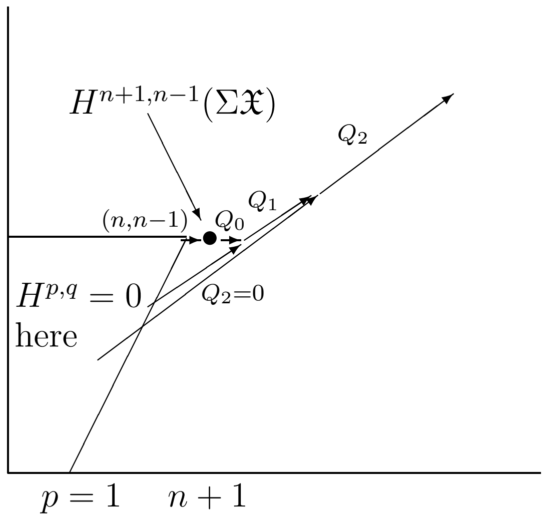

Figure 1: The composition $Q_2Q_1Q_0$ is an injection on $H^{n+1,n}(\Sigma\mathfrak{X})$

The goal of Lectures 4-6 is to use $\mu \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$ to construct a summand M of the motive of the Rost variety X satisfying the following axioms.

Rost motives 3.4. A *Rost motive* for $\underline{a}$ is a motive M with coefficients $\mathbb{Z}_{(\ell)}$ satisfying:

(a) M is a direct summand of a Rost variety X, i.e., a Chow motive (X,e).

(b) The transpose $M' = (X, e^t)$ is isomorphic to M via $M' \to X \xrightarrow{p} M$, p being the projection. Here $M' \to X$ is defined as $M' = M^* \otimes \mathbb{L}^d \xrightarrow{p^* \otimes \mathbb{L}^d} X^* \otimes \mathbb{L}^d \cong X$.

(c) There is a motive D related to the evident map $y\colon\ M \to X \to \mathfrak{X}$ by two distinguished triangles

$$D \otimes \mathbb{L}^b \to M \xrightarrow{y} \mathfrak{X} \to\ , \tag{3.4.1}$$

$$\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{Dy} M \to D \to\ . \tag{3.4.2}$$

Here Dy is the dual map $\mathfrak{X} \otimes \mathbb{L}^d \to \mathbb{L}^d \xrightarrow{y^*} M^* \otimes \mathbb{L}^d \cong M$, where the final isomorphism comes from axiom (b).

In the rest of this lecture, we will assume that a Rost motive M exists for $\underline{a}$, and use it to verify the Bloch-Kato Conjecture. We will use without comment the standard fact that if $p > q$ then $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, R_{\mathrm{tr}}(Y)(q)[p]) = 0$ for every smooth Y.

Lemma 3.5. *The structure map $H_{-1,-1}(\mathfrak{X}) \to H_{-1,1}(k) = k^{\times}$ is injective.*

Proof. (Voevodsky) For all p and $n \geq 2$, $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, X^p(1)[n]) = 0$ (as $n > 1$). Therefore every row in the fourth quadrant spectral sequence

$$E^1_{pq} = \mathrm{Hom}(\mathbb{Z}[q], X^{p+1}(1)) \Longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathfrak{X}(1)[p-q]) = H_{q-p,-1}(\mathfrak{X})$$

is zero except for $q = 0, -1$. The homology at $(p,q) = (0,-1)$ yields the exact sequence

$$0 \longleftarrow H_{-1,-1}(\mathfrak{X}) \longleftarrow H_{-1,-1}(X) \longleftarrow H_{-1,-1}(X \times X).$$

The result follows from the homology exact sequence 3.1(c) of a Rost variety X. $\square$

Corollary 3.6. *The structure map $H_{-1,-1}(M) \xrightarrow{y} H_{-1,-1}(k) = k^*$ is injective.*

Proof. By axiom 3.4(c), it suffices to observe that $\mathrm{Hom}(\mathbb{Z}, D(b+1)[2b+1]) = 0$. This follows from the vanishing of both $\mathrm{Hom}(\mathbb{Z}, M(b+1)[2b+1])$ and $\mathrm{Hom}(\mathbb{Z}, \mathfrak{X}\otimes\mathbb{L}^{b+d+1})$. $\square$

Lemma 3.7. $H^{2d+1,d+1}(D,\mathbb{Z}) = 0$.

Proof. Set $(p,q) = (2d+1, d+1)$. From (3.4.2) we get an exact sequence

$$H^{0,1}(\mathfrak{X}) \to H^{p,q}(D) \to H^{p,q}(M) \xrightarrow{Dy} H^{p,q}(\mathfrak{X}\otimes\mathbb{L}^d).$$

The first group is 0 because it equals $H^{0,1}(k,\mathbb{Z}) = H^{-1}_{\mathrm{Zar}}(k, \mathcal{O}_X^\times) = 0$. Hence $H^{p,q}(D)$ is the kernel of Dy. But by axiom 3.4(b), for any u in $H_{-1,-1}(M) = \mathrm{Hom}(\mathbb{Z}, M(1)[1])$ the dual of $\mathbb{Z}(-1)[-1] \xrightarrow{u} M \xrightarrow{y} \mathbb{Z}$ is an element of $H^{p,q}(\mathfrak{X}\otimes\mathbb{L}^d)$ represented by:

$$\mathfrak{X}\otimes\mathbb{L}^d \longrightarrow \mathbb{L}^d \xrightarrow{y^*} M^*\otimes\mathbb{L}^d \xrightarrow{u^*} \mathbb{Z}(1)[1]\otimes\mathbb{L}^d = \mathbb{Z}(d+1)[2d+1].$$

Hence $Dy\colon H^{p,q}(M) \to H^{p,q}(\mathfrak{X}\otimes\mathbb{L}^d)$ may be identified with the given structural map $H_{-1,-1}(M) \xrightarrow{y} H_{-1,-1}(k)$, and is an injection by 3.6. Hence $H^{p,q}(D) = \ker(Dy) = 0$. $\square$

Proposition 3.8. $H^{n+1,n}(\mathfrak{X},\mathbb{Z}) = 0$.

Proof. In the cohomology sequence arising from (3.4.1),

$$H^{2b\ell+1,b\ell+1}(D\otimes\mathbb{L}^b) \to H^{2b\ell+2,b\ell+1}(\mathfrak{X}) \to H^{2b\ell+2,b\ell+1}(M),$$

the first term is $H^{2d+1,d+1}(D)$ because $b\ell = d+b$, and it vanishes by 3.7. The third term is a summand of $H^{2b\ell+2,b\ell+1}(X)$, which is zero because $H^{p,q}(X) = 0$ whenever $p > d+q$ by the Vanishing Theorem [4, 3.6]. Hence $H^{2b\ell+2,b\ell+1}(\mathfrak{X},\mathbb{Z}) = 0$. By Proposition 3.3, ths implies that $H^{n+1,n}(\mathfrak{X},\mathbb{Z}) = 0$. $\square$

Theorem 3.9. $H^{n+1}_{\acute{e}t}(k,\mathbb{Z}(n)) \to H^{n+1}_{\acute{e}t}(k(X),\mathbb{Z}(n))$ *is an injection.*

As pointed out in Lecture 1 (Prop. 1.3), this establishes the Bloch-Kato conjecture for n.

Proof. Set $E = \operatorname{Spec} k(X)$, and let $L(n)$ denote the truncation $\tau^{\leq n+1}\mathbb{Z}^{\text{ét}}(n)$ of the complex in **DM** representing étale motivic cohomology. Then $H^{n+1}(E, L(n)) \cong H^{n+1}_{\text{ét}}(E, \mathbb{Z}(n))$ and

$$H^{n+1}(\mathfrak{X}, L(n)) \cong H^{n+1}_{\text{ét}}(\mathfrak{X}, \mathbb{Z}(n)) \cong H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)).$$

Write $K(n)$ for the cofiber of $\mathbb{Z}(n) \to L(n)$. Then $\operatorname{Spec}(E) \to \mathfrak{X}$ induces a commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \overset{3.8}{=} H^{n+1,n}(\mathfrak{X}, \mathbb{Z}) & \longrightarrow & H^{n+1}(\mathfrak{X}, L(n)) & \longrightarrow & H^{n+1}(\mathfrak{X}, K(n)) \\
\downarrow & & \downarrow & & \downarrow \\
0 = H^{n+1,n}(E, \mathbb{Z}) & \longrightarrow & H^{n+1}(E, L(n)) & \longrightarrow & H^{n+1}(E, K(n)).
\end{array}$$

We have to prove that that the middle vertical is an injection. This will follow once we show that the right vertical is an injection. Now $H^{n+1}(X, K(n)) \to H^{n+1}(E, K(n))$ is an injection by [4, 11.1, 13.8, 13.10], and $H^{n+1}(M, K(n))$ is a summand of the former group. It suffices to show that $y\colon M \to \mathfrak{X}$ induces an injection on $H^{n+1}(-, K(n))$. By (3.4.1) it suffices to show that $H^n(D \otimes \mathbb{L}^b, K(n))$ vanishes. By (3.4.2) we have an exact sequence

$$(3.9.1) \qquad H^{n-1}(\mathfrak{X} \otimes \mathbb{L}^{b+d}, K(n)) \to H^n(D \otimes \mathbb{L}^b, K(n)) \to H^n(M \otimes \mathbb{L}^b, K(n)).$$

We are deduced to showing the outside terms vanish in (3.9.1). For this we invoke the fact (proven in [MC/2, 6.12]) that $H^*(Y(1), K(n)) = 0$ for every smooth Y. Since M is a summand of X by 3.4(a), this fact implies that $H^n(M \otimes \mathbb{L}^b, K(n)) = 0$ in (3.9.1). It also implies that (for any $m > 1$) the spectral sequence $E^1_{pq} = H^q(X^p \otimes \mathbb{L}^m, K(n)) \Rightarrow H^{p+1}(\mathfrak{X} \otimes \mathbb{L}^m, K(n))$ collapses, yielding $H^{n-1}(\mathfrak{X} \otimes \mathbb{L}^m, K(n)) = 0$. Thus the left term in (3.9.1) also vanishes. □

Remark 3.10. Markus Rost has proposed a construction of M in [R-BC]. He shows that the element μ of Proposition 3.3 determines an equivalence class of summands (X, e) of the Chow motive of X. By construction $M = (X, e)$ satisfies axioms 3.4(a,b). I do not know if it satisfies axiom 3.4(c).