Lecture 3. First notions and facts of non-abelian homological algebra.

The preliminaries are dedicated to kernels of arrows of arbitrary categories with initial objects. They are complemented by Appendix. In the treatment of non-abelian homological algebra, we adopt here an intermediate level of generality – right exact (instead of fibred and cofibred) categories, which turns the main body of this text into an exercise on satellites along the lines of [Gr], in which abelian categories are replaced by right (or left) exact categories with initial (resp. final) objects. Finally, an analysis of obtained facts, which leads to the notions of stable category of a left exact category and to the notions of quasi-suspended and quasi-triangulated categories, is influenced by the philosophy of [KeV] and [Ke1].

1. Preliminaries: kernels and cokernels of morphisms. Let C_X be a category with an initial object, x. For a morphism $M \xrightarrow{f} N$ we define the *kernel of* f as the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & N \end{array}$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & Cok(f) \\ {\scriptstyle f}\uparrow & cocart & \uparrow{\scriptstyle f'} \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X.

If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the diagram $Ker(f) \xrightarrow{\mathfrak{k}(f)} M \overset{f}{\underset{0}{\rightrightarrows}} N$ is exact.

Dually, the cokernel of f makes the diagram $M \overset{f}{\underset{0}{\rightrightarrows}} N \xrightarrow{\mathfrak{c}(f)} Cok(f)$ exact.

1.1. Lemma. *Let C_X be a category with an initial object x.*

(a) Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.

(b) If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{i_M} M$ is the kernel of f.

Proof. The pull-backs of monomorphisms are monomorphisms. ■

1.2. Corollary. *Let C_X be a category with an initial object x. The following conditions are equivalent:*

(a) If $M \xrightarrow{f} N$ has a kernel, then the canonical arrow $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism.

(b) The unique arrow $x \xrightarrow{i_M} M$ is a monomorphism for any $M \in ObC_X$.

Proof. $(a) \Rightarrow (b)$, because, by 1.1(b), the unique morphism $x \xrightarrow{i_M} M$ is the kernel of the identical morphism $M \longrightarrow M$. The implication $(b) \Rightarrow (a)$ follows from 1.1(a). ∎

1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

1.4. Examples.

1.4.1. Kernels of morphisms of unital k-algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k. For any k-algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathfrak{k}(\varphi)} \uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k-algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \longrightarrow A$ and the k-algebra structure $k \longrightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \gamma} \uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k-algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k-algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k-algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k-algebra structure $k \longrightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k, because this category does not have final objects.

1.4.2. Kernels and cokernels of maps of sets. Since the only initial object of the category *Sets* is the empty set $\emptyset$ and there are no morphisms from a non-empty set to $\emptyset$, the kernel of any map $X \longrightarrow Y$ is $\emptyset \longrightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

1.4.3. Presheaves of sets. Let C_X be a svelte category and $C_X^\wedge$ the category of non-trivial presheaves of sets on C_X (that is we exclude the *trivial* presheaf which assigns to every object of C_X the empty set). The category $C_X^\wedge$ has a final object which is the constant presheaf with values in a one-element set. If C_X has a final object, y, then $\widehat{y} = C_X(-, y)$ is a final object of the category $C_X^\wedge$. Since $C_X^\wedge$ has small colimits, it has cokernels of arbitrary morphisms which are computed object-wise, that is using 1.4.2.

If the category C_X has an initial object, x, then the presheaf $\widehat{x} = C_X(-, x)$ is an initial object of the category $C_X^\wedge$. In this case, the category $C_X^\wedge$ has kernels of all its morphisms (because $C_X^\wedge$ has limits) and the Yoneda functor $C_X \xrightarrow{h} C_X^\wedge$ preserves kernels.

Notice that the initial object of $C_X^\wedge$ is not isomorphic to its final object unless the category C_X is pointed, i.e. initial objects of C_X are its final objects.

1.5. Some properties of kernels. See Appendix.

2. Right exact categories and (right) 'exact' functors. We define a *right exact* category as a pair $(C_X, \mathfrak{E}_X)$, where C_X is a category and $\mathfrak{E}_X$ is a pretopology on C_X whose covers are *strict epimorphisms*; that is for any element $M \longrightarrow L$ of $\mathfrak{E}$ (– a cover), the diagram $M \times_L M \rightrightarrows M \longrightarrow L$ is exact. This requirement means precisely that the pretopology $\mathfrak{E}_X$ is *subcanonical*; i.e. every representable presheaf of sets on C_X is a sheaf. We call the elements of $\mathfrak{E}_X$ *deflations* and assume that all isomorphisms are deflations.

2.1. The coarsest and the finest right exact structures. The coarsest right exact structure on a category C_X is the discrete pretopology: the class of deflations coincides with the class $Iso(C_X)$ of all isomorphisms of the category C_X.

Let $\mathfrak{E}_X^{\mathfrak{s}}$ denote the class of all *universally strict* epimorphisms of C_X; i.e. elements of $\mathfrak{E}_X^{\mathfrak{s}}$ are strict epimorphisms $M \xrightarrow{\mathfrak{e}} N$ such that for any morphism $\widetilde{N} \xrightarrow{f} N$, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & M \\ \widetilde{\mathfrak{e}} \downarrow & cart & \downarrow \mathfrak{e} \\ \widetilde{N} & \xrightarrow{f} & N \end{array}$$

whose left vertical arrow is a strict epimorphism. It follows that $\mathfrak{E}_X^{\mathfrak{s}}$ is the finest right exact structure on the category C_X. We call this structure *standard*.

If C_X is an abelian category or a topos, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all epimorphisms.

If C_X is a quasi-abelian category, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all strict epimorphisms.

2.2. Right 'exact' and 'exact' functors. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories. A functor $C_X \xrightarrow{F} C_Y$ will be called *right 'exact'* (resp. *'exact'*) if it maps deflations to deflations and for any deflation $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ and any morphism $\widetilde{N} \xrightarrow{f} N$, the canonical arrow $F(\widetilde{N} \times_N M) \longrightarrow F(\widetilde{N}) \times_{F(N)} F(M)$ is a deflation (resp. an isomorphism).

In other words, the functor F is 'exact' if it maps deflations to deflations and preserves pull-backs of deflations.

2.3. Weakly right 'exact' and weakly 'exact' functors. A functor $C_X \xrightarrow{F} C_Y$ is called *weakly right 'exact'* (resp. *weakly 'exact'*) if it maps deflations to deflations and for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, the canonical morphism $F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$ is a deflation (resp. an isomorphism). In particular, weakly 'exact' functors are weakly right 'exact'.

2.4. Note. Of cause, 'exact' (resp. right 'exact') functors are weakly 'exact' (resp. weakly right 'exact'). In the additive (actually a more general) case, weakly 'exact' functors are 'exact' (see 2.7 and 2.7.2).

2.5. Interpretation: 'spaces' represented by right exact categories. Weakly right 'exact' functors will be interpreted as inverse image functors of morphisms between 'spaces' represented by right exact categories. We consider the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$ whose objects are pairs $(X, \mathfrak{E}_X)$, where $(C_X, \mathfrak{E}_X)$ is a svelte right exact category. A morphism from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ is a morphism of 'spaces' $X \xrightarrow{\varphi} Y$ whose inverse image functor $C_Y \xrightarrow{\varphi^*} C_X$ is a weakly right 'exact' functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$. The map which assigns to every 'space' X the pair $(X, Iso(C_X))$ is a full embedding of the category $|Cat|^o$ of 'spaces' into the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$. This full embedding is a right adjoint functor to the forgetful functor

$$\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}} \longrightarrow |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto X.$$

2.5.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive right exact categories and $C_X \xrightarrow{F} C_Y$ an additive functor. Then*

(a) The functor F is weakly right 'exact' iff it maps deflations to deflations and the sequence

$$F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is exact for any deflation $M \xrightarrow{\mathfrak{e}} N$.

(b) The functor F is weakly 'exact' iff it maps deflations to deflations and the sequence

$$0 \longrightarrow F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is 'exact' for any deflation $M \xrightarrow{\mathfrak{e}} N$.

Proof. See A.2(b). ∎

2.6. Conflations and fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x. We denote by $\mathcal{E}_X$ the class of all sequences of the form $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $K \xrightarrow{\mathfrak{k}} M$ is a kernel of $\mathfrak{e}$. Expanding the terminology of exact additive categories, we call such sequences *conflations.*

2.6.1. Fully exact subcategories of a right exact category. We call a full subcategory $\mathcal{B}$ of C_X a *fully exact* subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, if $\mathcal{B}$

contains the initial object x and is *closed under extensions*; i.e. if objects K and N in a conflation $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$ belong to $\mathcal{B}$, then M is an object of $\mathcal{B}$.

In particular, fully exact subcategories of $(C_X, \mathfrak{E}_X)$ are strictly full subcategories.

2.6.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its fully exact subcategory. Then the class $\mathfrak{E}_{X,\mathcal{B}}$ of all deflations $M \xrightarrow{\mathfrak{e}} N$ such that M, N, and $Ker(\mathfrak{e})$ are objects of $\mathcal{B}$ is a structure of a right exact category on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}}) \longrightarrow (C_X, \mathfrak{E}_X)$.*

Proof. The argument is an application of facts of Appendix. ∎

2.6.3. Remark. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its strictly full subcategory containing x. Let $\mathfrak{E}$ be a right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \xrightarrow{\mathfrak{J}} C_X$ maps deflations to deflations and preserves kernels of deflations. Then $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$. In particular, $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$ if the inclusion functor is an 'exact' functor from $(\mathcal{B}, \mathfrak{E})$ to $(C_X, \mathfrak{E}_X)$. This shows that if $\mathcal{B}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, then $\mathfrak{E}_{X,\mathcal{B}}$ is the finest right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an exact functor from $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ to $(C_X, \mathfrak{E}_X)$.

2.7. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and F a functor $C_X \longrightarrow C_Y$ which maps conflations to conflations. Suppose that the category C_Y is additive. Then the functor F is 'exact'.*

2.7.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive k-linear right exact categories and F an additive functor $C_X \longrightarrow C_Y$. Then the functor F is weakly 'exact' iff it is 'exact'.*

Proof. By 2.5.1, a k-linear functor $C_X \xrightarrow{F} C_Y$ is a weakly 'exact' iff it maps conflations to conflations. The assertion follows now from 2.7. ∎

2.7.2. The property (†). In Proposition 2.7, the assumption that the category C_Y is additive is used only at the end of the proof (part (b)). Moreover, additivity appears there only because it garantees the following property:

(†) if the rows of a commutative diagram

$$\begin{array}{ccccc}
\widetilde{L} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{N} \\
\downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & M & \longrightarrow & N
\end{array}$$

are conflations and its right and left vertical arrows are isomorphisms, then the middle arrow is an isomorphism.

So that the additivity of C_Y in 2.7 can be replaced by the property (†) for $(C_Y, \mathfrak{E}_Y)$.

2.7.3. An observation. The following obvious observation helps to establish the property (†) for many non-additive right exact categories:

If $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ are right exact categories and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations, then the property (†) holds in $(C_X, \mathfrak{E}_X)$ provided it holds in $(C_Y, \mathfrak{E}_Y)$.

2.7.3.1. Example. Let $(C_Y, \mathfrak{E}_Y)$ are right exact k-linear category, $(C_X, \mathfrak{E}_X)$ a right exact category, and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations. Then the property (†) holds in $(C_X, \mathfrak{E}_X)$.

For instance, the property (†) holds for the right exact category $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ of associative unital k-algebras with strict epimorphisms as deflations, because the forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k-mod$ is conservative, maps deflations to deflations (that is to epimorphisms) and is left exact. Therefore, it maps conflations to conflations.

2.8. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. The Yoneda embedding induces an 'exact' fully faithful functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} (C_{X_\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}})$, where $C_{X_\mathfrak{E}}$ is the category of sheaves of sets on the presite $(C_X, \mathfrak{E}_X)$ and $\mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}}$ the family of all universally strict epimorphisms of $C_{X_\mathfrak{E}}$ (– the standard structure of a right exact category).*

(b) Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ a weakly right 'exact' functor. There exists a functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ \mathfrak{j}_X^* \big\downarrow & & \big\downarrow \mathfrak{j}_Y^* \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array} \qquad (1)$$

quasi commutes, i.e. $\widetilde{\varphi}^*\mathfrak{j}_X^* \simeq \mathfrak{j}_Y^*\varphi^*$. The functor $\widetilde{\varphi}^*$ is defined uniquely up to isomorphism and has a right adjoint, $\widetilde{\varphi}_*$.

Proof. (a) Since the right exact structure $\mathfrak{E}_X$ of C_X is a subcanonical pretopology, the Yoneda embedding takes values in the category $C_{X_\mathfrak{E}}$ of sheaves on $(C_X, \mathfrak{E}_X)$, hence it induces a full embedding of C_X into $C_{X_\mathfrak{E}}$ which preserves all small limits and maps deflations to deflations. In particular it is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_{X_\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}})$.

(b) Every weakly right exact functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ determines a continuous (i.e. having a right adjoint) functor between the categories of presheaves of sets, which is compatible with the sheafification functor, hence determines uniquely a continuous functor between the corresponding categories of sheaves making commute the diagram (1). ■

2.9. Application: right exact additive categories and exact categories.

2.9.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be an additive k-linear right exact category. Then there exists an exact category $(C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ and a fully faithful k-linear 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ which is universal; that is any 'exact' k-linear functor from $(C_X, \mathcal{E}_X)$ to an exact k-linear category factorizes uniquely through γ_X^*.*

Proof. We take as $C_{X_\mathfrak{e}}$ the smallest fully exact subcategory of the category $C_{X_\mathfrak{E}}$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$ containing all representable sheaves. Objects of the category $C_{X_\mathfrak{e}}$ are sheaves $\mathcal{F}$ such that there exists a finite filtration

$$0 = \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \ \dots \ \longrightarrow \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_m/\mathcal{F}_{m-1}$ is representable for $1 \le m \le n$. The subcategory $C_{X_\mathfrak{e}}$, being a fully exact subcategory of an abelian category, is exact. The remaining details are left as an exercise. ■

3. Satellites in right exact categories.

3.1. Preliminaries: trivial morphisms, pointed objects, and complexes. Let C_X be a category with initial objects. We call a morphism of C_X *trivial* if it factors through an initial object. It follows that an object M is initial iff $id_{_M}$ is a trivial morphism. If C_X is a pointed category, then the trivial morphisms are usually called *zero morphisms*.

3.1.1. Trivial compositions and pointed objects. If the composition of arrows $L \xrightarrow{f} M \xrightarrow{g} N$ is trivial, i.e. there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \xi \big\downarrow & & \big\downarrow g \\ x & \xrightarrow{i_N} & N \end{array}$$

where x is an initial object, and the morphism g has a kernel, then f is the composition of the canonical arrow $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a morphism $L \xrightarrow{f_g} Ker(g)$ uniquely determined by f and ξ. If the arrow $x \xrightarrow{i_N} N$ is a monomorphism, then the morphism ξ is uniquely determined by f and g ; therefore in this case, the arrow f_g does not depend on ξ.

3.1.1.1. Pointed objects. In particular, f_g does not depend on ξ, if N is a *pointed* object. The latter means that therre exists an arrow $N \longrightarrow x$.

3.1.2. Complexes. A sequence of arrows

$$\ldots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \qquad (1)$$

is called a *complex* if each its arrow has a kernel and the next arrow factors *uniquely* through this kernel.

3.1.3. Lemma. *Let each arrow in the sequence*

$$\ldots \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0 \qquad (2)$$

of arrows have a kernel and the composition of any two consecutive arrows is trivial. Then

$$\ldots \xrightarrow{f_4} M_4 \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \qquad (3)$$

is a complex. If M_0 is a pointed object, then (2) is a complex.

Proof. The objects M_i are pointed for $i \ge 2$, which implies that $(Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ are monomorphisms for all $i \ge 2$, hence) (3) is a complex (see 3.1.1). ■

3.1.4. Corollary. *A sequence of morphisms*

$$\dots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots$$

unbounded on the right is a complex iff the composition of any pair of its consecutive arrows is trivial and for every i, there exists a kernel of the morphism f_i.

3.1.5. Example. Let C_X be the category Alg_k of unital associative k-algebras. The algebra k is its initial object, and every morphism of k-algebras has a kernel. Pointed objects of C_X which have a morphism to initial object are precisely augmented k-algebras. If the composition of pairs of consecutive arrows in the sequence

$$\dots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$$

is trivial, then it follows from the argument of 3.1.2 that A_i is an augmented k-algebra for all $i \geq 2$. And any unbounded on the right sequence of algebras with trivial compositions of pairs of consecutive arrows is formed by augmented algebras.

3.1.6. 'Exact' complexes. Let $(C_X, \mathcal{E}_X)$ be a right exact category with an initial object. We call a sequence of two arrows $L \xrightarrow{f} M \xrightarrow{g} N$ in C_X 'exact' if the arrow g has a kernel, and f is the composition of $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a deflation $L \xrightarrow{f_g} Ker(g)$. A complex is called 'exact' if any pair of its consecutive arrows forms an 'exact' sequence.

3.2. ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object. A ∂^**-functor* from $(C_X, \mathcal{E}_X)$ to C_Y is a system of functors $C_X \xrightarrow{T_i} C_Y$, $i \geq 0$, together with a functorial assignment to every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ and every $i \geq 0$ a morphism $T_{i+1}(L) \xrightarrow{\mathfrak{d}_i(E)} T_i(N)$ which depends functorially on the conflation E and such that the sequence of arrows

$$\dots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is a complex. Taking the trivial conflation $x \longrightarrow x \longrightarrow x$, we obtain that $T_i(x) \xrightarrow{id_{T_i(x)}} T_i(x)$ is a trivial morphism, or, equivalently, $T_i(x)$ is an initial object, for every $i \geq 1$.

Let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ and $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ be a pair of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y. A morphism from T to T' is a family $f = (T_i \xrightarrow{f_i} T'_i \mid i \geq 0)$ of functor morphisms such that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the exact category C_X and every $i \geq 0$, the diagram

$$\begin{array}{ccc} T_{i+1}(L) & \xrightarrow{\mathfrak{d}_i(E)} & T_i(N) \\ {\scriptstyle f_{i+1}(L)}\downarrow & & \downarrow{\scriptstyle f_i(N)} \\ T'_{i+1}(L) & \xrightarrow{\mathfrak{d}'_i(E)} & T'_i(N) \end{array}$$

commutes. The composition of morphisms is naturally defined. Thus, we have the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y.

3.2.1. Trivial ∂^*-functors. We call a ∂^*-functor $T = (T_i, \mathfrak{d}_i| \ i \geq 0)$ *trivial* if all T_i are functors with values in initial objects. One can see that trivial ∂^*-functors are precisely initial objects of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Once an initial object y of the category C_Y is fixed, we have a canonical trivial functor whose components equal to the constant functor with value in y – it maps all arrows of C_X to id_y.

3.2.2. Some natural functorialities. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and C_Y a category with initial object. If C_Z is another category with an initial object and $C_Y \xrightarrow{F} C_Z$ a functor which maps initial objects to initial objects, then for any ∂^*-functor $T = (T_i, \mathfrak{d}_i| \ i \geq 0)$, the composition $F \circ T = (F \circ T_i, F\mathfrak{d}_i| \ i \geq 0)$ of T with F is a ∂^*-functor. The map $(F, T) \longmapsto F \circ T$ is functorial in both variables; i.e. it extends to a functor

$$Cat_*(C_Y, C_Z) \times \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \longrightarrow \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Z). \tag{1}$$

Here Cat_* denotes the subcategory of Cat whose objects are categories with initial objects and morphisms are functors which map initial objects to initial objects.

On the other hand, let $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ be another right exact category with an initial object and Φ a functor $C_{\mathfrak{X}} \longrightarrow C_X$ which maps conflations to conflations. In particular, it maps initial objects to initial objects (because if x is an initial object of $C_{\mathfrak{X}}$, then $x \longrightarrow M \xrightarrow{id_M} M$ is a conflation; and $\Phi(x \longrightarrow M \xrightarrow{id_M} M)$ being a conflation implies that $\Phi(x)$ is an initial object). For any ∂^*-functor $T = (T_i, \mathfrak{d}_i| \ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y, the composition $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i \Phi| \ i \geq 0)$ is a ∂^*-functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to C_Y. The map $(T, \Phi) \longmapsto T \circ \Phi$ extends to a functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \times \mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X)) \longrightarrow \mathcal{H}om^*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), C_Y), \tag{2}$$

where $\mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X))$ denotes the full subcategory of $\mathcal{H}om(C_{\mathfrak{X}}, C_X)$ whose objects are preserving conflations functors $C_{\mathfrak{X}} \longrightarrow C_X$.

3.3. Universal ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object y.

A ∂^*-functor $T = (T_i, \mathfrak{d}_i| \ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is called *universal* if for every ∂^*-functor $T' = (T'_i, \mathfrak{d}'_i| \ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y and every functor morphism $T'_0 \xrightarrow{g} T_0$, there exists a unique morphism $f = (T'_i \xrightarrow{f_i} T_i \mid i \geq 0)$ from T' to T such that $f_0 = g$.

3.3.1. Interpretation. Consider the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{3}$$

which assigns to every ∂^*-functor (resp. every morphism of ∂^*-functors) its zero component. For any functor $C_X \xrightarrow{F} C_Y$, we have a presheaf of sets $\mathcal{H}om(\Psi^*(-), F)$ on the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Suppose that this presheaf is representable by an object (i.e. a ∂^*-functor) $\Psi_*(F)$. Then $\Psi_*(F)$ is a universal ∂^*-functor.

Conversely, if $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ is a universal ∂^*-functor, then $T \simeq \Psi_*(T_0)$.

3.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered systems. Then, for any functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ such that $T_0 = F$.*

In other words, the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \qquad (3)$$

which assigns to each morphism of ∂^*-functors its zero component has a right adjoint, Ψ_*.

Proof. For an arbitrary functor $C_X \xrightarrow{F} C_Y$, we set

$$S_-(F)(L) = \lim Ker(F(\mathfrak{k}(\mathfrak{e}))),$$

where the limit is taken by the (filtered) system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since deflations form a pretopology, the map $L \longmapsto S_-(F)(L)$ extends naturally to a functor $C_X \xrightarrow{S_-(F)} C_Y$. By the definition of $S_-(F)$, for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, there exists a unique morphism $S_-(F)(L) \xrightarrow{\widetilde{\partial}^0_F(E)} Ker(F(\mathfrak{j}))$. We denote by $\partial^0_F(E)$ the composition of $\widetilde{\partial}^0_F(E)$ and the canonical morphism $Ker(F(j)) \longrightarrow F(N)$.

Notice that the correspondence $F \longmapsto S_-(F)$ is functorial. Applying the iterations of the functor S_- to F, we obtain a ∂^*-functor $S^\bullet_-(F) = (S^i_-(F)|i \geq 0)$. This ∂^*-functor is universal. ∎

3.3.3. Remark. Let the assumptions of 3.3.2 hold. Then we have a pair of adjoint functors

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$$

By 3.3.2, the adjunction morphism $\Psi^*\Psi_* \longrightarrow Id$ is an isomorphism which means that Ψ_* is a fully faithful functor and Ψ^* is a localization functor at a left multiplicative system.

3.3.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y. Let C_Z be another category with an initial object and F a functor from C_Y to C_Z which preserves initial objects, kernels of morphisms and limits of filtered systems. Then*

(a) If T is a universal ∂^*-functor, then $F \circ T = (F \circ T_i, F\mathfrak{d}_i|\ i \geq 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F \circ T$ is universal iff T is universal.

Proof. (a) Since the functor F preserves kernels of morphisms and filtered limits (that is all types of limits which appear in the construction of $S_-(G)(L)$), the natural morphism

$$F \circ S_-(G)(L) \longrightarrow S_-(F \circ G)(L)$$

is an isomorphism for any functor $C_X \xrightarrow{G} C_Y$ such that $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ exists. Therefore, the natural morphism $F \circ S_-^i(T_0)(L) \longrightarrow S_-^i(F \circ T_0)(L)$ is an isomorphism for all $i \geq 0$ and all $L \in ObC_X$.

(b) By (a), we have a functor isomorphism $F \circ T_{i+1} \xrightarrow{\sim} F \circ S_-(T_i)$ for all $i \geq 0$. Since the functor F is fully faithful, this isomorphism is the image of a uniquely determined isomorphism $T_{i+1} \xrightarrow{\sim} S_-(T_i)$. The assertion follows now from (the argument of) 3.3.2. Details are left as an exercise. ∎

3.3.5. An application. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category. We assume that both categories, C_X and C_Y have initial objects. Consider the Yoneda embedding

$$C_Y \xrightarrow{h_Y} C_Y^{\wedge}, \quad M \longmapsto \widehat{M} = C_Y(-, M).$$

of the category C_Y into the category $C_Y^{\wedge}$ of presheaves of sets on C_Y. The functor h_Y is fully faithful and preserves all limits. In particular, it satisfies the conditions of 3.3.4(b). Therefore, a ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is universal iff the ∂^*-functor $\widehat{T} \stackrel{\text{def}}{=} h_Y \circ T = (\widehat{T}_i, \widehat{\mathfrak{d}}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to $C_Y^{\wedge}$ is universal.

Since the category $C_Y^{\wedge}$ has all limits (and colimits), it follows from 3.3.2 that, for any functor $C_X \xrightarrow{G} C_Y^{\wedge}$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i | \ i \geq 0) = \Psi_*(G)$ whose zero component coincides with G. In particular, for every functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ such that $T_0 = h_Y \circ F = \widetilde{F}$. It follows from 3.3.4(b) that a universal ∂^*-functor whose zero component coincides with F exists if and only if for all $L \in ObC_X$ and all $i \geq 1$, the presheaves of sets $T_i(L)$ are representable.

3.3.6. Remark. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with an initial object x and C_Y a category with an initial object y and limits. Then, by the argument of 3.3.2, we have an endofunctor S_- of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y, together with a cone $S_- \xrightarrow{\lambda} \mathfrak{y}$, where $\mathfrak{y}$ is the constant functor with the values in the initial object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X, \mathfrak{E}_X)$ and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc}
S_-F(L) & \xrightarrow{\lambda(L)} & y & & \\
{\scriptstyle \mathfrak{d}_0(E)} \downarrow & & \downarrow & & \\
F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L)
\end{array}$$

3.4. The dual picture: ∂-functors and universal ∂-functors. Let $(C_X, \mathfrak{I}_X)$ be a left exact category, which means by definition that $(C_X^{op}, \mathfrak{I}_X^{op})$ is a right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is the data which becomes a ∂^*-functor in the dual right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is *universal* if its dualization is a universal ∂^*-functor. We leave to the reader the reformulation in the context of ∂-functors of all notions and facts about ∂^*-functors.

3.5. Universal ∂^*-functors and 'exactness'.

3.5.1. The properties $(CE5)$ **and** $(CE5^*)$**.** Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We say that it satisfies $(CE5^*)$ (resp. $(CE5)$) if the limit of a filtered system (resp. the colimit of a cofiltered system) of conflations in $(C_Y, \mathfrak{E}_Y)$ exists and is a conflation.

In particular, if $(C_X, \mathfrak{E}_X)$ satisfies $(CE5^*)$ (resp. $(CE5)$), then the limit of any filtered system (resp. the colimit of any cofiltered system) of deflations is a deflation.

The properties $(CE5)$ and $(CE5^*)$ make sense for left exact categories as well. Notice that a right exact category satisfies $(CE5^*)$ (resp. $(CE5)$) iff the dual left exact category satisfies $(CE5)$ (resp. $(CE5^*)$).

3.5.2. Note. If $(C_X, \mathfrak{E}_X)$ is an abelian category with the standard exact structure, then the property $(CE5)$ for $(C_X, \mathfrak{E}_X)$ is equivalent to the Grothendieck's property (AB5) and, therefore, the property $(CE5^*)$ is equivalent to $(AB5^*)$ (see [Gr, 1.5]).

The property $(CE5)$ holds for Grothendieck toposes.

In what follows, we use $(CE5^*)$ for right exact categories and the dual property $(CE5)$ for left exact categories.

3.5.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and $(C_Y, \mathfrak{E}_Y)$ satisfy $(CE5^*)$. Let F be a weakly right 'exact' functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ such that $S_-(F)$ exists. Then for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the sequence*

$$S_-(F)(N) \xrightarrow{S_-(F)(\mathfrak{j})} S_-(F)(M) \xrightarrow{S_-(F)(\mathfrak{e})} S_-(F)(L) \xrightarrow{\mathfrak{d}_0(E)} F(N) \xrightarrow{F(\mathfrak{j})} F(M) \qquad (1)$$

is 'exact'. The functor $S_-(F)$ is a weakly right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

3.5.4. 'Exact' ∂^*-functors and universal ∂^*-functors. Fix right exact categories $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$, both with initial objects. A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *'exact'* if for every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the complex

$$\ldots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_i(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'.

3.5.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. Suppose that $(C_Y, \mathfrak{E}_Y)$ satisfies $(CE5^*)$. Let $T = (T_i |\ i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. If the functor T_0 is right 'exact', then the universal ∂^*-functor T is 'exact'.*

Proof. If T_0 is right 'exact', then, by 3.5.3, the functor $T_1 \simeq S_-(T_0)$ is right 'exact' and for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the sequence

$$T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'. Since $T_{n+1} = S_-(T_n)$, the assertion follows from 3.5.3 by induction. ∎

3.5.4.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category. For each object L of C_X, the ∂-functor $Ext^\bullet_X(L) = (Ext^i_X(L) \mid i \geq 0)$ is 'exact'.*

Suppose that the category C_X is k-linear. Then for each $L \in ObC_X$, the ∂-functor $\mathcal{E}xt^\bullet_X(L) = (\mathcal{E}xt^i_X(L) \mid i \geq 0)$ is 'exact'.

Proof. In fact, the ∂-functor $Ext^\bullet_X(L)$ is universal by definition (see 3.4.1), and the functor $Ext^0_X(L) = C_X(-, L)$ is left exact. In particular, it is left 'exact'.

If C_X is a k-linear category, then the universal ∂-functors $\mathcal{E}xt^\bullet_X(L)$, $L \in ObC_X$, with the values in the category of k-modules (defined in 3.4.2) are 'exact' by a similar reason. ∎

4. Coeffaceable functors, universal ∂^*-functors, and pointed projectives.

4.1. Projectives and projective deflations. Fix a right exact category $(C_X, \mathfrak{E}_X)$. We call an object P of C_X *projective* if every deflation $M \longrightarrow P$ splits. Equivalently, any morphism $P \xrightarrow{f} N$ factors through any deflation $M \xrightarrow{\mathfrak{e}} N$.

We denote by $\mathcal{P}_{\mathfrak{E}_X}$ the full subcategory of C_X generated by projective objects.

4.1.1. Example. Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations split. Then every object of C_X is a projective object of $(C_X, \mathfrak{E}_X)$.

A deflation $M \longrightarrow L$ is called *projective* if it factors through any deflation of L.

Any deflation $P \longrightarrow L$ with P projective is a projective deflation. On the other hand, an object P is projective iff the identical morphism $P \longrightarrow P$ is a projective deflation.

4.2. Coeffaceble functors and projectives. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category with an initial object. We call a functor $C_X \xrightarrow{F} C_Y$ *coeffaceable*, or $\mathfrak{E}_X$*-coeffaceable*, if for any object L of C_X, there exists a deflation $M \xrightarrow{\mathfrak{t}} L$ such that $F(\mathfrak{t})$ is a trivial morphism.

It follows that if a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable, then it maps all projectives to initial objects and all projective deflations to trivial arrows.

So that if the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective deflations (resp. enough projectives), then a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable iff $F(\mathfrak{e})$ is trivial for any projective deflation $\mathfrak{e}$ (resp. $F(M)$ is an initial object for every projective object M).

4.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Then $T_i(P)$ is an initial object for any pointed projective object P and for all $i \geq 1$.*

4.3.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Suppose that $(C_X, \mathfrak{E}_X)$ has enough projectives and projectives of $(C_X, \mathfrak{E}_X)$ are pointed objects. Then the functors T_i are coeffaceable for all $i \geq 1$.*

Proof. The assertion follows from 4.3 and 4.2. ∎

4.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects; and let $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ be an 'exact' ∂^*-functor from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.*

If the functors T_i are $\mathfrak{E}_X$-coeffaceable for $i \geq 1$, then T is a universal ∂^*-functor.

Proof. The argument is similar to the proof in [Gr] of the corresponding assertion for abelian categories. ∎

4.5. Proposition. *Let* $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$, *and* $(C_Z, \mathfrak{E}_Z)$ *be right exact categories. Suppose that* $(C_X, \mathfrak{E}_X)$ *has enough projectives and* C_Y *has kernels of all morphisms. If* $T = (T_i|\ i \geq 0)$ *is a universal, 'exact'* ∂^**-functor from* $(C_X, \mathfrak{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$ *and* F *a functor from* $(C_Y, \mathfrak{E}_Y)$ *to* $(C_Z, \mathfrak{E}_Z)$ *which respects conflations, then the composition* $F \circ T = (F \circ T_i|\ i \geq 0)$ *is a universal 'exact'* ∂^**-functor.*

Proof. Under the conditions of the proposition, the composition $F \circ T$ is an 'exact' functor such that the functors $F \circ T_i$, $i \geq 1$, map pointed projectives of $(C_X, \mathfrak{E}_X)$ to trivial objects (because T_i map pointed projectives to trivial objects by 4.3 and F maps trivial objects to trivial objects). Since there are enough pointed projectives (hence all projectives are pointed), this implies that the functors $F \circ T_i$ are coeffaceable for $i \geq 1$. Therefore, by 4.4, $F \circ T$ is a universal ∂^*-functor. ∎

4.6. Sufficient conditions for having enough pointed projectives.

4.6.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *and* $(C_Z, \mathfrak{E}_Z)$ *be right exact categories and* $C_Z \xrightarrow{f^*} C_X$ *a functor having a right adjoint* f_*. *Suppose that* f^* *maps deflations of the form* $N \longrightarrow f_*(M)$ *to deflations and the adjunction arrow* $f^*f_*(M) \xrightarrow{\epsilon(M)} M$ *is a deflation for all* M *(which is the case if any morphism* $\mathfrak{t}$ *of* C_X *such that* $f_*(\mathfrak{t})$ *is a split epimorphism belongs to* $\mathfrak{E}_X$*). Let* $(C_Z, \mathfrak{E}_Z)$ *have enough projectives, and all projectives are pointed objects. Then each projective of* $(C_X, \mathfrak{E}_X)$ *is a pointed object.*

If, in addition, f_* *maps deflations to deflations, then* $(C_X, \mathfrak{E}_X)$ *has enough projectives.*

4.6.2. Note. The conditions of 4.6.1 can be replaced by the requirement that if $N \longrightarrow f_*(M)$ is a deflation, then the corresponding morphism $f^*(N) \longrightarrow M$ is a deflation. This requirement follows from the conditions of 4.6.1, because the morphism $f^*(N) \longrightarrow M$ corresponding to $N \xrightarrow{\mathfrak{t}} f_*(M)$ is the composition of $f^*(\mathfrak{t})$ and the adjunction arrow $f^*f_*(M) \xrightarrow{\epsilon(M)} M$.

4.6.3. Example. Let $(C_X, \mathfrak{E}_X)$ be the category Alg_k of associative k-algebras endowed with the standard (that is the finest) right exact structure. This means that class $\mathfrak{E}_X$ of deflations coincides with the class of all are strict epimorphisms of k-algebras. Let $(C_Y, \mathfrak{E}_Y)$ be the category of k-modules with the standard exact structure, and f_* the forgetful functor $Alg_k \longrightarrow k - mod$. Its left adjoint, f^* preserves strict epimorphisms, and the functor f_* preserves and reflects deflations; i.e. a k-algebra morphism t is a strict epimorphism iff $f_*(t)$ is an epimorphism. In particular, the adjunction arrow $f^*f_*(A) \longrightarrow A$ is a strict epimorphism for all A. By 4.6.1, $(C_X, \mathfrak{E}_X)$ has enough projectives and each projective has a morphism to the initial object k; that is each projective has a structure of an augmented k-algebra.

4.7. Acyclic objects and the universality of ∂^*-functors. Given a ∂^*-functor $T = (T_i|\ i \geq 0)$ from a right exact category $(C_X, \mathfrak{E}_X)$ to a category C_Y, we call an object M of C_X *T-acyclic* if $T_i(M)$ is a trivial object for all $i \geq 1$.

4.7.1. Proposition. *Let $(C_{\mathfrak{A}}, \mathfrak{E}_{\mathfrak{A}})$ and $(C_X, \mathfrak{E}_X)$ be right exact categories with initial objects and $C_{\mathfrak{A}} \xrightarrow{G} C_X$ a functor which preserves conflations. Let $T = (T_i|\ i \geq 0)$ be an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to a category $C_{\mathcal{Z}}$ with initial objects. If there are enough objects M of C_X such that $G(M)$ is a T-acyclic object, then $T \circ G$ is a universal ∂^*-functor.*

Proof. Since the functor G maps conflation to conflations, and the ∂^*-functor T is 'exact', its composition $T \circ G$ is an 'exact' ∂^*-functor. Since there are *enough* objects in $C_{\mathfrak{A}}$ which the functor G maps to acyclic objects (i.e. for each object L of $C_{\mathfrak{A}}$, there is a deflation $M \longrightarrow L$ such that $G(M)$ is T-acyclic), the functor $T_i \circ G$ is effaceable for all $i \geq 1$. Therefore, by 4.6, the composition $T \circ G$ is a universal ∂^*-functor. ∎

5. Universal problems for universal ∂^*- and ∂-functors.

5.1. The categories of universal ∂^*- and ∂-functors. Fix a right exact svelte category $(C_X, \mathfrak{E}_X)$ with an initial object. Let $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to categories C_Y (with initial objects). Let T be a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

Dually, for a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ with a final object, we denote by $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to categories with final object. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

5.2. Universal problems for universal ∂-functors with values in complete categories and ∂^*-functors with values in cocomplete categories.

Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. We denote by $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* (i.e. having limits of small diagrams) categories C_Y and morphisms between these universal ∂^*-functors are pairs (F, ϕ), where the functor F preserves limits.

For a svelte left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves small colimits.

5.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ and $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. It suffices to prove the assertion about $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, because the assertion about ∂^*-functors is obtained via dualization.

Consider the Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C_X^{\wedge}, \quad M \longmapsto C_{\mathfrak{X}}(-, M).$$

We denote by $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ the universal ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^{\wedge}_{\mathfrak{X}}$ such that $Ext^{0}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = h_{\mathfrak{X}}$. The claim is that $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

In fact, let C_Y be a cocomplete category. By [GZ, II.1.3], the composition with the Yoneda embedding $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$ is an equivalence between the category $\mathcal{H}om_{\mathfrak{c}}(C^{\wedge}_{\mathfrak{X}}, C_Y)$ of *continuous* (that is having a right adjoint, or, equivalently, preserving colimits) functors $C^{\wedge}_{\mathfrak{X}} \longrightarrow C_Y$ and the category $\mathcal{H}om(C_{\mathfrak{X}}, C_Y)$ of functors from $C_{\mathfrak{X}}$ to C_Y. Let $C_{\mathfrak{X}} \xrightarrow{F} C_Y$ be an arbitrary functor and $C^{\wedge}_{\mathfrak{X}} \xrightarrow{F_{\mathfrak{c}}} C_Y$ the corresponding continuous functor. By definition, $S_+F(L) = \mathrm{colim}(Cok(F(M \longrightarrow Cok(\mathfrak{j}))$, where $L \xrightarrow{\mathfrak{j}} M$ runs through inflations of L. Since $F_{\mathfrak{c}}$ preserves colimits, it follows from (the dual version of) 3.3.4(a) that $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor whose zero component is $F_{\mathfrak{c}} \circ Ext^{0}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = F_{\mathfrak{c}} \circ h_{\mathfrak{X}} = F$. Therefore, by (the dual version of the argument of) 3.3.2, the universal ∂-functor $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is isomorphic to the right satellite $S^{\bullet}_{+}F$ of the functor F. This shows that $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$. ∎

5.3. The universal problem for arbitrary universal ∂- and ∂^*-functors. Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte left exact category with final objects. Let $C_{\mathfrak{X}_{\mathfrak{s}}}$ denote the smallest strictly full subcategory of the category $C^{\wedge}_{\mathfrak{X}}$ containing all presheaves $Ext^{n}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$. Let $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ denote the corestriction of the ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$. Thus, $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is the composition of the ∂-functor $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the inclusion functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{\mathfrak{I}_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$. It follows that $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor.

5.3.1. Proposition. *Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. For any universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a category C_Y (with final objects), there exists a unique (up to isomorphism) functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{T^{\mathfrak{s}}} C_Y$ such that $T = T^{\mathfrak{s}} \circ \mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the diagram*

$$\begin{array}{ccc}
C^{\wedge}_{\mathfrak{X}} & \xrightarrow{T^*_0} & C^{\vee^{op}}_{Y} \\
{\scriptstyle \mathfrak{I}_{\mathfrak{X}}}\uparrow & & \uparrow{\scriptstyle h^o_Y} \\
C_{\mathfrak{X}_{\mathfrak{s}}} & \xrightarrow{T^{\mathfrak{s}}} & C_Y
\end{array}$$

commutes. Here $C^{\vee^{op}}_{Y}$ denote the category of presheaves of sets on C^{op}_{Y} (i.e. functors $C_Y \longrightarrow Sets$) and h^o_Y the (dual) Yoneda functor $C_Y \longrightarrow C^{\vee^{op}}_{Y}$, $L \longmapsto C_Y(L, -)$.; and T^*_0 is a unique continuous (i.e. having a right adjoint) functor such that $T^*_0 \circ h_{\mathfrak{X}} = h^o_Y \circ T_0$.

Proof. The category $C^{\vee^{op}}_{Y}$ is cocomplete (and complete) and the functor h^o_Y preserves colimits. Therefore, by 3.3.4, the composition $h^o_Y \circ T$ is a universal ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^{\vee^{op}}_{Y}$. By 5.2.1, the ∂-functor $h^o_Y \circ T$ is the composition of the universal ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^{\wedge}_{\mathfrak{X}}$ and the unique continuous functor $C^{\wedge}_{\mathfrak{X}} \xrightarrow{T^*_0} C^{\vee^{op}}_{Y}$ such that $T^*_0 \circ h_{\mathfrak{X}} = h^o_Y \circ T_0$. Since the functor h^o_Y is fully faithful, this implies that the universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is isomorphic to the composition of the corestriction of $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$ and a unique functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{T^{\mathfrak{s}}} C_Y$ such that the composition $h^o_Y \circ T^{\mathfrak{s}}$ coincides with the restriction of the functor T^*_0 to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$. ∎

5.3.2. Note. The formulation of the dual assertion about the universal ∂^*-functors is left to the reader.

5.4. The k-linear version. Fix a right exact svelte k-linear additive category $(C_X, \mathfrak{E}_X)$. Let $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to k-linear additive categories C_Y. Let T be a universal k-linear ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a k-linear functor from C_Y to C_Z and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial^*_k\mathfrak{Un}^{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are k-linear ∂^*-functors with values in complete categories and morphisms are pairs (F, ϕ) such that the functor F preserves small limits.

Dually, for a left exact svelte k-linear additive category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal k-linear ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to additive k-linear categories. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a k-linear functor from C_Y to C_Z and ψ a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are k-linear ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves small colimits.

5.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ (resp. $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$) be a svelte right (resp. left) exact additive k-linear category. The categories $\partial^*_k\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ and $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. By duality, it suffices to prove that the category $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ has an initial object. The argument is similar to the argument of 5.2.1, except for the category $C^{\wedge}_{\mathfrak{X}}$ of presheaves of sets on the category $C_{\mathfrak{X}}$ is replaced by the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $C_{\mathfrak{X}}$. The initial object of the category $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the universal k-linear ∂-functor $\mathcal{E}xt^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the category $\mathcal{M}_k(\mathfrak{X})$ whose zero component is the (k-linear) Yoneda embedding $C_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X}), \quad L \longmapsto C_{\mathfrak{X}}(-, L)$. ■

Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte k-linear additive left exact category. Let $\mathcal{M}^{\mathfrak{s}}_k(\mathfrak{X}$ denote the smallest additive strictly full subcategory of the category $\mathcal{M}_k(\mathfrak{X})$ containing all presheaves $\mathcal{E}xt^n_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$. Let $\mathcal{E}\mathfrak{xt}^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ denote the corestriction of the ∂-functor $\mathcal{E}xt^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ to the subcategory $\mathcal{M}^{\mathfrak{s}}_k(\mathfrak{X}$. Thus, $\mathcal{E}xt^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ is the composition of the k-linear ∂-functor $\mathcal{E}\mathfrak{xt}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the inclusion functor

$$\mathcal{M}^{\mathfrak{s}}_k(\mathfrak{X}) \xrightarrow{\mathfrak{I}_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X}).$$

It follows that $\mathfrak{E}\mathfrak{xt}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor.

5.4.2. Proposition. *Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte left exact category with final objects. For any universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a category C_Y (with* *final objects), there exists a unique (up to isomorphism) functor* $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{T^{\mathfrak{s}}} C_Y$ *such that* $T = T^{\mathfrak{s}} \circ \mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ *and the diagram*

$$\begin{array}{ccc} C^{\wedge}_{\mathfrak{X}} & \xrightarrow{T_0^*} & C_Y^{\vee^{op}} \\ {\scriptstyle \mathfrak{I}_{\mathfrak{X}}}\uparrow & & \uparrow {\scriptstyle h_Y^o} \\ C_{\mathfrak{X}_{\mathfrak{s}}} & \xrightarrow{T^{\mathfrak{s}}} & C_Y \end{array}$$

commutes. Here $C_Y^{\vee^{op}}$ *denote the category of presheaves of sets on* C_Y^{op} *(i.e. functors* $C_Y \longrightarrow Sets$*) and* h_Y^o *the (dual) Yoneda functor* $C_Y \longrightarrow C_Y^{\vee^{op}}$, $L \longmapsto C_Y(L,-)$.*; and* T_0^* *is a unique continuous (i.e. having a right adjoint) functor such that* $T_0^* \circ h_{\mathfrak{X}} = h_Y^o \circ T_0$.

Proof. The argument is similar to that of 5.3.1. ■

6. The stable category of a left exact category.

6.1. Reformulations. Fix a svelte left exact category $(C_X, \mathfrak{I}_X)$. Let $\widehat{\Theta}_X^*$ denote the continuous (that is having a right adjoint) functor $C_X^{\wedge} \longrightarrow C_X^{\wedge}$ determined (uniquely up to isomorphism) by the equality $Ext_X^1 = \widehat{\Theta}_X^* \circ h_X$. To any conflation $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$, corresponds the diagram

$$\begin{array}{ccccc} \widehat{N} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{L} \\ & & \downarrow & & \downarrow {\scriptstyle \mathfrak{d}_0(E)} \\ & & \widehat{x} & \xrightarrow{\lambda(\widehat{N})} & \widehat{\Theta}_X^*(\widehat{N}) \end{array} \qquad (1)$$

where $\widehat{L} = h_X(L)$ and $\widehat{x}$ is the final object of the category $C_X^{\wedge}$ of presheaves on C_X.

Due to the universality of $Ext_X^{\bullet}$, all the information about universal ∂-functors from the left exact category $(C_X, \mathfrak{I}_X)$, is encoded in the diagrams (1), where $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ runs through the class of conflations of $(C_X, \mathfrak{I}_X)$.

In fact, the universal ∂-functor $Ext_X^{\bullet}$ is of the form $(\widehat{\Theta}_X^{*n} \circ h_X, \widehat{\Theta}_X^{*n}(\mathfrak{d}_0) | n \geq 0)$; and for any functor F from C_X to a category C_Y with colimits and final objects, the universal ∂-functor $(T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{I}_X)$ to C_Y with $T_0 = F$ is isomorphic to

$$F^* \circ Ext_X^{\bullet} = (F^* \widehat{\Theta}_X^{*n} \circ h_X, F^* \widehat{\Theta}_X^{*n}(\mathfrak{d}_0) \mid n \geq 0). \qquad (2)$$

6.2. Note. If C_X is a pointed category, then the presheaf $\widehat{x} = C_X(-, x)$ is both a final and an initial object of the category $C_X^{\wedge}$. In particular, the morphism $\widehat{x} \xrightarrow{\lambda(\widehat{N})} \widehat{\Theta}_X^*(\widehat{N})$ in (1) is uniquely defined, hence is not a part of the data. In this case, the data consists of diagrams

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}_X^*(\widehat{N}),$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ runs through conflations of $(C_X, \mathfrak{I}_X)$.

6.3. Stable category of $(C_X, \mathfrak{I}_X)$. Consider the full subcategory $C_{X_\mathfrak{s}}$ of the category $C_X^\wedge$ whose objects are $\widehat{\Theta}_X^{*n}(\mathcal{M})$, where $\mathcal{M}$ runs through representable presheaves and n through nonnegative integers. We denote by $\theta_{X_\mathfrak{s}}$ the endofunctor $C_{X_\mathfrak{s}} \longrightarrow C_{X_\mathfrak{s}}$ induced by $\widehat{\Theta}_X^*$. It follows that $C_{X_\mathfrak{s}}$ is the smallest $\widehat{\Theta}_X^*$-stable strictly full subcategory of the category $C_X^\wedge$ containing all presheaves $\widehat{M} = C_X(-, M),\ M \in ObC_X$.

6.3.1. Triangles. We call the diagram

$$\widehat{N} \xrightarrow{\ \widehat{\mathfrak{j}}\ } \widehat{M} \xrightarrow{\ \widehat{\mathfrak{e}}\ } \widehat{L} \xrightarrow{\ \mathfrak{d}_0(E)\ } \widehat{\Theta}_X^*(\widehat{N}), \qquad (1)$$

quasi-suspended where $E = (N \xrightarrow{\ \mathfrak{j}\ } M \xrightarrow{\ \mathfrak{e}\ } L)$ is a conflation in $(C_X, \mathfrak{I}_X)$, a *standard triangle.*

A *triangle* is any diagram in $C_{X_\mathfrak{s}}$ of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{X_\mathfrak{s}}(\mathcal{N}), \qquad (2)$$

which is isomorphic to a standard triangle. It follows that for every triangle, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \mathfrak{e}\ } & \mathcal{L} \\ \big\downarrow & & \big\downarrow \mathfrak{d} \\ \widehat{x} & \xrightarrow{\ \lambda(\mathcal{N})\ } & \widehat{\Theta}_X^*(\mathcal{N}) \end{array}$$

commutes. Triangles form a category $\mathfrak{Tr}_{X_\mathfrak{s}}$: morphisms from

$$\mathcal{N} \xrightarrow{\ \mathfrak{j}\ } \mathcal{M} \xrightarrow{\ \mathfrak{e}\ } \mathcal{L} \xrightarrow{\ \mathfrak{d}\ } \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\ \mathfrak{j}'\ } \mathcal{M}' \xrightarrow{\ \mathfrak{e}'\ } \mathcal{L}' \xrightarrow{\ \mathfrak{d}'\ } \theta_{\mathfrak{X}}(\mathcal{N}')$$

are given by commutative diagrams

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\ \mathfrak{j}\ } & \mathcal{M} & \xrightarrow{\ \mathfrak{e}\ } & \mathcal{L} & \xrightarrow{\ \mathfrak{d}\ } & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \big\downarrow & & \big\downarrow g & & \big\downarrow h & & \big\downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\ \mathfrak{j}'\ } & \mathcal{M}' & \xrightarrow{\ \mathfrak{e}'\ } & \mathcal{L}' & \xrightarrow{\ \mathfrak{d}'\ } & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

The composition is obvious.

6.3.2. The prestable category of a left exact category. Thus, we have obtained a data $(C_{X_\mathfrak{s}}, (\theta_{X_\mathfrak{s}}, \lambda), \mathfrak{Tr}_{X_\mathfrak{s}})$. We call this data the *prestable* category of the left exact category $(C_X, \mathfrak{I}_X)$.

6.3.3. The stable category of a left exact category with final objects. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_\mathfrak{s}}, \theta_{X_\mathfrak{s}}, \lambda; \mathfrak{Tr}_{X_\mathfrak{s}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. Let $\Sigma = \Sigma_{\theta_{X_\mathfrak{s}}}$ be the class of all arrows $\mathfrak{t}$ of $C_{X_\mathfrak{s}}$ such that $\theta_{X_\mathfrak{s}}(\mathfrak{t})$ is an isomorphism.

We call the quotient category $C_{X_{\mathfrak{s}}} = \Sigma^{-1}C_{X_{\mathfrak{s}}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The endofunctor $\theta_{X_{\mathfrak{s}}}$ determines a conservative endofunctor $\theta_{X_{\mathfrak{s}}}$ of the stable category $C_{X_{\mathfrak{s}}}$. The localization functor $C_{X_{\mathfrak{s}}} \xrightarrow{\mathfrak{q}^*_\Sigma} C_{X_{\mathfrak{s}}}$ maps final objects to final objects. Let $\lambda_{\mathfrak{s}}$ denote the image $\widetilde{x} = \mathfrak{q}^*_\Sigma(\widehat{x}) \longrightarrow \theta_{X_{\mathfrak{s}}}$ of the cone $\widehat{x} \xrightarrow{\lambda} \theta_{X_{\mathfrak{s}}}$.

Finally, we denote by $\mathfrak{Tr}_{X_{\mathfrak{s}}}$ the strictly full subcategory of the category of diagrams of the form $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{X_{\mathfrak{s}}}(\mathcal{N})$ generated by $\mathfrak{q}^*_\Sigma(\mathfrak{Tr}_{X_{\mathfrak{s}}})$.

The data $(C_{X_{\mathfrak{s}}}, \theta_{X_{\mathfrak{s}}}, \lambda_{\mathfrak{s}}; \mathfrak{Tr}_{X_{\mathfrak{s}}})$ will be called the *stable* category of $(C_X, \mathfrak{I}_X)$.

6.4. Dual notions. If $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ is a right exact category with an initial object, one obtains, dualizing the definitions of 6.3.2 and 6.3.3, the notions of the *precostable* and *costable* category of $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$.

6.5. The k-linear version. Let $(C_X, \mathfrak{I}_X)$ be a k-linear additive svelte left exact category. Replacing the category of $C_X^\wedge$ of presheaves of sets by the category $\mathcal{M}_k(X)$ of presheaves of k-modules on C_X and the functor $Ext^1_{(X,\mathfrak{I}_X)}$ by its k-linear version, $\mathcal{E}xt^1_{(X,\mathfrak{I}_X)}$, we obtain the k-linear versions of prestable and stable categories of the left exact category $(C_X, \mathfrak{I}_X)$.

6.5.1. Note. If $(C_X, \mathfrak{I}_X)$ is a k-linear exact category (that is $\mathfrak{I}_X$ happen to be the class of inflations of a k-linear exact category) with enough injectives, than its stable category defined above is equivalent to the conventional stable category of $(C_X, \mathfrak{I}_X)$. Recall that the latter has the same objects as C_X and its morphisms are *homotopy* classes of morphisms of C_X: two morphisms $M \overset{f}{\underset{g}{\rightrightarrows}} N$ are *homotopy equivalent* to each other if their difference $f - g$ factors through an injective object.

Notice that our construction of stable category of $(C, \mathfrak{I}_X)$ does not require any additional hypothesis. In particular, it extends the notion of the stable category to arbitrary exact categories.

7. Complement: presuspended and quasi-suspended categories. It is tempting to follow Keller's example [Ke1], [KeV] and turn essential properties of prestable and stable categories of a left exact category into axioms. We call the corresponding notions respectively *presuspended* and *quasi-suspended* categories.

7.1. Presuspended categories. Fix a category $C_{\mathfrak{X}}$ with a final object x and a functor $C_{\mathfrak{X}} \xrightarrow{\widetilde{\theta}_{\mathfrak{X}}} x\backslash C_{\mathfrak{X}}$, or, what is the same, a pair $(\theta_{\mathfrak{X}}, \lambda)$, where $\theta_{\mathfrak{X}}$ is an endofunctor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$ and λ is a cone $x \longrightarrow \theta_{\mathfrak{X}}$. We denote by $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ the category whose objects are all diagrams of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

such that the square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ x & \xrightarrow{\lambda(\mathcal{N})} & \theta_{\mathfrak{X}}(\mathcal{N}) \end{array}$$

commutes. Morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

are triples $(\mathcal{N} \xrightarrow{f} \mathcal{N}', \mathcal{M} \xrightarrow{g} \mathcal{M}', \mathcal{L} \xrightarrow{h} \mathcal{L}')$ such that the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes. The composition of morphisms is natural.

7.1.1. Definition. A *presuspended* category is a triple $(C_{\mathfrak{X}}, \widetilde{\theta}_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ and $\widetilde{\theta}_{\mathfrak{X}} = (\theta_{\mathfrak{X}}, \lambda)$ are as above and $\mathfrak{Tr}_{\mathfrak{X}}$ is a strictly full subcategory of the category $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ whose objects are called *triangles*, which satisfies the following conditions:

(PS1) Let $C_{\mathfrak{X}_0}$ denote the full subcategory of $C_{\mathfrak{X}}$ generated by objects $\mathcal{N}$ such that there exists a triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$. For every $\mathcal{N} \in ObC_{\mathfrak{X}_0}$, the diagram

$$\mathcal{N} \xrightarrow{id_{\mathcal{N}}} \mathcal{N} \longrightarrow x \xrightarrow{\lambda(\mathcal{N})} \theta_{\mathfrak{X}}(\mathcal{N})$$

is a triangle.

(PS2) For any triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$ and any morphism $\mathcal{N} \xrightarrow{f} \mathcal{N}'$ with $\mathcal{N}' \in ObC_{\mathfrak{X}_0}$, there is a triangle $\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$ such that f extends to a morphism of triangles

$$(\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})) \xrightarrow{(f,g,h)} (\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')).$$

(PS2') For any pair of triangles

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

and any commutative square

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} \\ f \downarrow & & \downarrow g \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' \end{array}$$

there exists a morphism $\mathcal{L} \xrightarrow{h} \mathcal{L}'$ such that (f, g, h) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes.

(PS3) For any pair of triangles

$$\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{M} \xrightarrow{x} \mathcal{M}' \xrightarrow{s} \widetilde{\mathcal{M}} \xrightarrow{r} \theta_{\mathfrak{X}}(M),$$

there exists a commutative diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{u} & \mathcal{M} & \xrightarrow{v} & \mathcal{L} & \xrightarrow{w} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ id \downarrow & & x \downarrow & & \downarrow y & & \downarrow id \\ \mathcal{N} & \xrightarrow{u'} & \mathcal{M}' & \xrightarrow{v'} & \mathcal{L}' & \xrightarrow{w'} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ & & s \downarrow & & \downarrow t & & \downarrow \theta_{\mathfrak{X}}(u) \\ & & \widetilde{\mathcal{M}} & \xrightarrow{id} & \widetilde{\mathcal{M}} & \xrightarrow{r} & \theta_{\mathfrak{X}}(\mathcal{M}) \\ & & r \downarrow & & \downarrow r' & & \\ & & \theta_{\mathfrak{X}}(\mathcal{M}) & \xrightarrow{\theta_{\mathfrak{X}}(v)} & \theta_{\mathfrak{X}}(\mathcal{L}) & & \end{array} \tag{2}$$

whose two upper rows and two central columns are triangles.

(PS4) For every triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$, the sequence

$$\dots \longrightarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(\mathcal{N}), -) \longrightarrow C_{\mathfrak{X}}(\mathcal{L}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{M}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{N}, -)$$

is exact.

7.2. The category of presuspended categories. Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; Tr_{\mathfrak{X}})$ and $\mathfrak{T}_+C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \lambda_{\mathfrak{Y}}; Tr_{\mathfrak{Y}})$ be presuspended categories. A *triangle* functor from $\mathfrak{T}_+C_{\mathfrak{X}}$ to $\mathfrak{T}_+C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ which maps initial object to an initial object and ϕ is a functor isomorphism $F \circ \theta_{\mathfrak{X}} \longrightarrow \theta_{\mathfrak{Y}} \circ F$ such that for every triangle $\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N})$ of $\mathfrak{T}_+C_{\mathfrak{X}}$, the sequence

$$F(\mathcal{N}) \xrightarrow{F(u)} F(\mathcal{M}) \xrightarrow{F(v)} F(\mathcal{L}) \xrightarrow{\phi(\mathcal{N})F(w)} \theta_{\mathfrak{Y}}(F(\mathcal{N}))$$

is a triangle of $\mathfrak{T}_+C_{\mathfrak{Y}}$. The composition of triangle functors is defined naturally:

$$(G, \psi) \circ (F, \phi) = (G \circ F, \psi F \circ G\phi).$$

Let (F, ϕ) and (F', ϕ') be triangle functors from $\mathfrak{T}_-C_{\mathfrak{X}}$ to $\mathfrak{T}_-C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc}
\theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\
{\scriptstyle \theta_{\mathfrak{Y}}\lambda}\downarrow & & \downarrow{\scriptstyle \lambda\theta_{\mathfrak{X}}} \\
\theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}}
\end{array}$$

commutes. The composition is the compsition of the functor morphisms.

Altogether gives the definition of a bicategory $\mathfrak{P}\mathfrak{Cat}$ formed by svelte presuspended categories, triangle functors as 1-morphisms and morphisms between them as 2-morphisms.

As usual, the term "category $\mathfrak{P}\mathfrak{Cat}$" means that we forget about 2-morphisms.

Dualizing (i.e. inverting all arrows in the constructions above), we obtain the bicategory $\mathfrak{P}^o\mathfrak{Cat}$ formed by *precosuspended* svelte categories as objects, triangular functors as 1-morphisms, and morphisms between them as 2-morphisms.

7.3. Quasi-suspended categories. We call a presuspended category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ *quasi-suspended* if the functor $\theta_{\mathfrak{X}}$ is conservative. We denote by $\mathfrak{S}\mathfrak{Cat}$ the full subcategory of the category $\mathfrak{P}\mathfrak{Cat}$ of presuspended categories whose objects are conservative presuspended svelte categories.

7.3.1. Example. The main example is, of course, the stable category of a left exact category. In the case when the left exact category is an exact (additive) category, the stable category is suspended in the sense of [KeV]. So 'quasi-' is added to avoid extra confusion in mathematics terminology.

7.3.2. From presuspended categories to quasi-suspended categories. Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category and $\Sigma = \Sigma_{\theta_{\mathfrak{X}}}$ the class of all arrows s of the category $C_{\mathfrak{X}}$ such that $\theta_{\mathfrak{X}}(s)$ is an isomorphism. Let $\Theta_{\mathfrak{X}}$ denote the endofunctor of the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ uniquely determined by the equality $\Theta_{\mathfrak{X}} \circ \mathfrak{q}_{\Sigma}^* = \mathfrak{q}_{\Sigma}^* \circ \theta_{\mathfrak{X}}$, where $\mathfrak{q}_{\Sigma}^*$ denotes the localization functor $C_{\mathfrak{X}} \longrightarrow \Sigma^{-1}C_{\mathfrak{X}}$. Notice that the functor $\mathfrak{q}_{\Sigma}^*$ maps final objects to final objects. Let $\widetilde{\lambda}$ denote the morphism $\mathfrak{q}_{\Sigma}^*(x) \longrightarrow \Theta_{\mathfrak{X}}$ induced by $x \xrightarrow{\lambda} \theta_{\mathfrak{X}}$ (that is by $\mathfrak{q}_{\Sigma}^*(x) \xrightarrow{\mathfrak{q}_{\Sigma}^*(\lambda)} \mathfrak{q}_{\Sigma}^* \circ \theta_{\mathfrak{X}} = \Theta_{\mathfrak{X}} \circ \mathfrak{q}_{\Sigma}^*$) and $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ the essential image of $\mathfrak{Tr}_{\mathfrak{X}}$. Then the data $(\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ is a quasi-suspended category.

The constructed above map $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \longmapsto (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ extends to a functor $\mathfrak{P}\mathfrak{Cat} \xrightarrow{\mathfrak{J}^*} \mathfrak{S}\mathfrak{Cat}$ which is a left adjoint to the inclusion functor $\mathfrak{S}\mathfrak{Cat} \xrightarrow{\mathfrak{J}_*} \mathfrak{P}\mathfrak{Cat}$. The natural triangle (localization) functors $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}_{\Sigma}^*} (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ form an adjunction arrow $Id_{\mathfrak{P}\mathfrak{Cat}} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$. The other adjunction arrow is identical.

7.5. Quasi-triangulated categories. Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category. We call it *quasi-triangulated*, if the endofunctor $\theta_{\mathfrak{X}}$ is an auto-equivalence.

In particular, every quasi-triangulated category is quasi-suspended. Let $\mathfrak{Q}\mathfrak{Tr}$ denote the full subcategory of $\mathfrak{P}\mathfrak{Cat}$ (or $\mathfrak{S}\mathfrak{Cat}$) whose objects are quasi-triangulated subcategories. We call a quasi-triangulated category *strict* if θ_X is an isomorphism of categories.

7.5.1. Proposition. *The inclusion functor* $\mathfrak{QTr} \longrightarrow \mathfrak{PCat}$ *has a left adjoint. More precisely, for each prestable category,* $\mathfrak{TC}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$*, there is a triangle functor from* $\mathfrak{TC}_{\mathfrak{X}}$ *to a strict quasi-triangulated category such that any triangle functor to a quasi-triangulated category factors uniquely through this functor.*

Proof. The argument is a standard procedure of inverting a functor, which was originated, probably, in Grothendieck's work on derivators. One can mimik the argument of the similar theorem (from suspended to strict triangulated categories) from [KeV]. ■

7.6. The k**-linear version.** It is obtained by restricting to k-linear additive categories and k-linear functors. Otherwise all axioms and facts look similarly. Details are left to the reader.

7.7. Remark. Notice that the notion of a *quasi-suspended* k-linear category presented here differs from the notion of *suspended* category proposed by Keller and Vossieck [KeV1]. In particular, the notion of a quasi-triangulated k-linear category is different from the notion of a triangulated k-linear category.

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Appendix: some properties of kernels.

A.1. Proposition. *Let* $M \xrightarrow{f} N$ *be a morphism of* C_X *which has a kernel pair,* $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$. *Then the morphism* f *has a kernel iff* p_1 *has a kernel, and these two kernels are naturally isomorphic to each other.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array} \tag{1}$$

Then we have the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array} \tag{2}$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{i_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $i_N = f \circ i_M$ imply that the left square of the diagram (2) is cartesian, i.e. $Ker(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1.

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle p_1'}\downarrow & cart & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle p_1'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ■

A.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) A.1 is symmetric, i.e. there is an isomorphism $Ker(p_1) \xrightarrow{\tau_f'} Ker(p_2)$ which is an arrow in the commutative diagram

$$\begin{array}{ccccc}
Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_1} & M \\
{\scriptstyle \tau'_f}\big\downarrow\wr & & {\scriptstyle \tau_f}\big\downarrow\wr & & \big\downarrow {\scriptstyle id_M} \\
Ker(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M \times_N M & \xrightarrow{p_2} & M
\end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M$, and a kernel. Then, by A.1, there exists a kernel of p_1, so that we have a morphism $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the left square of the commutative diagram

$$\begin{array}{ccccc}
x & \longrightarrow & Ker(p_1) & \xrightarrow{p'_1} & x \\
\big\downarrow & cart & \mathfrak{c}(p_1)\big\downarrow & & \big\downarrow \\
M & \xrightarrow{\Delta_M} & M \times_N M & \xrightarrow{p_1} & M
\end{array}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $Ker(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $Ker(p_1) \coprod M$, then the pair of morphisms $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$Ker(p_1) \coprod M \longrightarrow M \times_N M.$$

If the category C_X is additive, then this morphism is an isomorphism, or, what is the same, $Ker(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 1.4.

A.3. Proposition. *Let*

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\
{\scriptstyle \widetilde{g}}\big\downarrow & cart & \big\downarrow {\scriptstyle g} \\
M & \xrightarrow{f} & N
\end{array} \qquad (3)$$

be a cartesian square. Then $Ker(f)$ exists iff $Ker(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

A.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x.

A.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of A.3

shows that we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \mathfrak{k}(gf)\Big\downarrow & cart & \Big\downarrow\mathfrak{k}(g) & cart & \Big\downarrow i_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array} \qquad (1)$$

whose both squares are cartesian and all morphisms are uniquely determined by f, g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x \\ t\Big\downarrow & cart & \Big\downarrow\mathfrak{k}(g) & & \Big\downarrow i_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g\circ f} N$.

A.4.2. Remarks. (a) It follows from A.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \wr\Big\downarrow & & \mathfrak{k}(gf)\Big\downarrow & cart & \Big\downarrow\mathfrak{k}(g) & cart & \Big\downarrow i_N \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left vertical arrow is an isomorphism.

(b) Suppose that $(C_X, \mathfrak{E}_X)$ is a right exact category (with an initial object x). If the morphism f above is a deflation, then it follows from this observation that the canonical morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a deflation too. In this case, $Ker(f)$ exists, and we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) \\ \wr\Big\downarrow & & \mathfrak{k}(gf)\Big\downarrow & cart & \Big\downarrow\mathfrak{k}(g) \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \end{array}$$

whose rows are conflations.

The following observations is useful (and are used) for analysing diagrams.

A.4.3. Proposition. *(a) Let $M \stackrel{g}{\longrightarrow} N$ be a morphism with a trivial kernel. Then a morphism $L \stackrel{f}{\longrightarrow} M$ has a kernel iff the composition $g \circ f$ has a kernel, and these two kernels are naturally isomorphic one to another.*

(b) Let

$$\begin{array}{ccc} L & \stackrel{f}{\longrightarrow} & M \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ \widetilde{M} & \stackrel{\phi}{\longrightarrow} & N \end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f, and the left square of the commutative diagram

$$\begin{array}{ccccccc} Ker(f) & \stackrel{\sim}{\longrightarrow} & Ker(\phi\gamma) & \stackrel{\mathfrak{k}(f)}{\longrightarrow} & L & \stackrel{f}{\longrightarrow} & M \\ & & {\scriptstyle \widetilde{\gamma}}\downarrow & cart & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ & & Ker(\phi) & \stackrel{\mathfrak{k}(\phi)}{\longrightarrow} & \widetilde{M} & \stackrel{\phi}{\longrightarrow} & N \end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram A.4.1(1) specializes to the diagram

$$\begin{array}{ccccc} Ker(gf) & \stackrel{\widetilde{f}}{\longrightarrow} & x & \stackrel{id_x}{\longrightarrow} & x \\ {\scriptstyle \mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & & \downarrow{\scriptstyle i_N} \\ L & \stackrel{f}{\longrightarrow} & M & \stackrel{g}{\longrightarrow} & N \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $Ker(f)$. The assertion follows from A.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from A.4.1. ■

A.4.4. Corollary. *Let C_X be a category with an initial object x. Let $L \stackrel{f}{\longrightarrow} M$ be a strict epimorphism and $M \stackrel{g}{\longrightarrow} N$ a morphism such that $Ker(g) \stackrel{\mathfrak{k}(g)}{\longrightarrow} M$ exists and is a monomorphism. Then the composition $g \circ f$ is a trivial morphism iff g is trivial.*

A.4.4.1. Note. The following example shows that the requirement "$Ker(g) \longrightarrow M$ is a monomorphism" in A.4.4 cannot be omitted.

Let C_X be the category Alg_k of associative unital k-algebras, and let $\mathfrak{m}$ be an ideal of the ring k such that the epimorphism $k \longrightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \longrightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \longrightarrow k/\mathfrak{m}$ is a trivial morphism.

A.5. The coimage of a morphism. Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with which one can associate a pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{i_M} M$. Since $i_N = f \circ i_M$, the morphism f equalizes the pair $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

A.5.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by A.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ■

A.5.2. Note. By A.4.1, all squares of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & & \\ {\scriptstyle id}\downarrow & cart & \downarrow & & \\ Ker(j_f p_f) & \xrightarrow{\widetilde{p}_f} & Ker(j_f) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & cart & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (2)$$

are cartesian.

If C_X is an additive category and $M \xrightarrow{f} L$ is an arrow of C_X having a kernel and a coimage, then the canonical morphism $Coim(f) \xrightarrow{j_f} L$ is a monomorphism. Quite a few non-additive categories have this property.

A.5.3. Example. Let C_X be the category Alg_k of associative unital k-algebras. Since cokernels of pairs of arrows exist in Alg_k, any algebra morphism has a coimage. It follows from 1.4.1 that the coimage of an algebra morphism $A \xrightarrow{\varphi} B$ is $A/K(\varphi)$, where $K(\varphi)$ is the kernel of ϕ in the usual sense (i.e. in the category of non-unital algebras). The canonical decomposition $\varphi = j_\varphi \circ p_\varphi$ coincides with the standard presentation of φ as the composition of the projection $A \longrightarrow A/K(\varphi)$ and the monomorphism $A/K(\varphi) \longrightarrow B$. In particular, φ is strict epimorphism of k-algebras iff it is isomorphic to the associated coimage map $A \xrightarrow{p_\varphi} Coim(\varphi) = A/K(\varphi)$.