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Motivic sheaves

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Recall from Lecture 2:

$\mathrm{Cor}_{fin}(k)$:= the category with objects $[X]$ for $X \in \mathbf{Sm}/k$ and morphisms

$$\mathrm{Hom}_{\mathrm{Cor}_{fin}(k)}([X],[Y]) := c(X,Y) \subset \mathcal{Z}(X \times Y)$$

the *finite over X* correspondences.

$\mathrm{Cor}_{fin}(k)$ is a tensor category with $[X] \oplus [Y] = [X \amalg Y]$, $[X] \otimes [Y] = [X \times Y]$, etc.

Form a triangulated tensor category by taking the homotopy category $K^b(\mathrm{Cor}_{fin}(k))$.

Localize $K^b(\mathsf{Cor}_{fin}(k))$ to impose

1. Homotopy invariance: $[X \times \mathbb{A}^1] \cong [X]$

2. Mayer-Vietoris: For $X = U \cup V$, the complex

$$[U \cap V] \to [U] \oplus [V] \to [X]$$

is isomorphic to 0.

Then take the pseudo-abelian hull to form the category of *effective geometric motives* $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$.

Sending X to the image of $[X]$ in $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$ gives a symmetric monoidal functor

$$M_{\mathsf{gm}} : \mathbf{Sm}/k \to DM^{\mathsf{eff}}_{\mathsf{gm}}(k).$$

Form the category of *geometric motives* by inverting $\mathbb{Z}(1) := \widetilde{M_{\mathrm{gm}}}(\mathbb{P}^1)[-2]$:

$$DM_{\mathrm{gm}}(k) := DM_{\mathrm{gm}}^{\mathrm{eff}}(k)[\mathbb{Z}(1))^{\otimes -1}].$$

The canonical functor $i : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_{\mathrm{gm}}(k)$ is a full embedding.

Note. We will see later that, just as for $M_{\sim}(k)$, $DM_{\mathrm{gm}}(k)$ is a *rigid* tensor (triangulated) category: we invert $\mathbb{Z}(1)$ so that every object has a dual.

Elementary constructions in $DM_{\mathsf{gm}}^{\mathsf{eff}}(k)$

Motivic cohomology

Definition For $X \in \mathbf{Sm}/k$, $q \in \mathbb{Z}$, set

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Z}(q)[p]).$$

Compare with $\mathrm{CH}^r(X) = \mathrm{Hom}_{CHM(k)}(\mathbb{1}(-r), \mathfrak{h}(X))$. In fact, for all $X \in \mathbf{Sm}/k$, there is a natural isomorphism

$$\begin{aligned}\mathrm{CH}^r(X) &= \mathrm{Hom}_{CHM(k)}(\mathbb{1}(-r), \mathfrak{h}(X)) \\ &= \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Z}(r)[2r]) = H^{2r}(X, \mathbb{Z}(r)).\end{aligned}$$

In particular, sending $\mathfrak{h}(X)$ to $M_{\mathrm{gm}}(X)$ for $X \in \mathbf{SmProj}/k$ gives a full embedding

$$CHM(k)^{\mathrm{op}} \hookrightarrow DM_{\mathrm{gm}}(k)$$

Products Define the cup product

$$H^p(X,\mathbb{Z}(q)) \otimes H^{p'}(X,\mathbb{Z}(q')) \to H^{p+p'}(X,\mathbb{Z}(q+q'))$$

by sending $a \otimes b$ to

$$M_{\mathsf{gm}}(X) \xrightarrow{\delta} M_{\mathsf{gm}}(X) \otimes M_{\mathsf{gm}}(X) \xrightarrow{a\otimes b} \mathbb{Z}(q)[p] \otimes \mathbb{Z}(q')[p'] \cong \mathbb{Z}(q+q')[p+p'].$$

This makes $\oplus_{p,q} H^p(X,\mathbb{Z}(q))$ a graded commutative ring with unit 1 the map $M_{\mathsf{gm}}(X) \to \mathbb{Z}$ induced by $p_X : X \to \operatorname{Spec} k$.

Homotopy property

Applying $\mathrm{Hom}_{DM_{\mathrm{gm}}}(-, \mathbb{Z}(q)[p])$ to the isomorphism $p_* : M_{\mathrm{gm}}(X \times \mathbb{A}^1) \to M_{\mathrm{gm}}(X)$ gives the *homotopy property* for $H^*(-, \mathbb{Z}(*))$:

$$p^* : H^p(X, \mathbb{Z}(q)) \xrightarrow{\sim} H^p(X \times \mathbb{A}^1, \mathbb{Z}(q)).$$

Mayer-Vietoris

For $U, V \subset X$ open subschemes we can apply $\mathrm{Hom}_{DM_{\mathsf{gm}}}(-, \mathbb{Z}(q)[p])$ to the distinguished triangle

$$M_{\mathsf{gm}}(U\cap V) \to M_{\mathsf{gm}}(U) \oplus M_{\mathsf{gm}}(V) \to M_{\mathsf{gm}}(U\cup V) \to M_{\mathsf{gm}}(U\cap V)[1].$$

This gives the *Mayer-Vietoris exact sequence* for $H^*(-, \mathbb{Z}(*))$:

$$\begin{aligned} \ldots \to H^{p-1}(U \cup V, \mathbb{Z}(q)) \to H^p(U \cap V, \mathbb{Z}(q)) \\ \to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \to H^p(U \cup V, \mathbb{Z}(q)) \to \ldots \end{aligned}$$

The end of the road?

It is difficult to go much further using only the techniques of geometry and homological algebra.

One would like to have:

- Chern classes of line bundles and a projective bundle formula
- A Gysin isomorphism
- A computation of the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as algebraic cycles.

Voevodsky achieves this by viewing $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a subcategory of a derived category of "Nisnevich sheaves with transfer".

Motivic sheaves

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Outline:

- Sites and sheaves
- Categories of motivic complexes
- The Suslin complex
- The main results: the localization and embedding theorems

Sites and sheaves

We give a quick review of the theory of sheaves on a Grothendieck site.

Presheaves A *presheaf* P on a small category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Morphisms of presheaves are natural transformations of functors. This defines the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, $PreShv^{\mathcal{A}}(\mathcal{C})$.

Remark We require $\mathcal{C}$ to be small so that the collection of natural transformations $\vartheta : F \to G$, for presheaves F, G, form a set. It would suffice that $\mathcal{C}$ be essentially small (the collection of isomorphism classes of objects form a set).

Structural results

Theorem *(1) If $\mathcal{A}$ is an abelian category, then so is $PreShv^{\mathcal{A}}(\mathcal{C})$, with kernel and cokernel defined objectwise: For $f : F \to G$,*

$$ker(f)(x) = \mathrm{ker}(f(x) : F(x) \to G(x));$$
$$\mathrm{coker}(f)(x) = \mathrm{coker}(f(x) : F(x) \to G(x)).$$

(2) For $\mathcal{A} = \mathbf{Ab}$, $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.

The second part is proved by using a result of Grothendieck, noting that $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has the set of generators $\{\mathbb{Z}_X \mid X \in \mathcal{C}\}$, where $\mathbb{Z}_X(Y)$ is the free abelian group on $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Pre-topologies

Definition Let $\mathcal{C}$ be a category. A *Grothendieck pre-topology* τ on $\mathcal{C}$ is given by defining, for $X \in \mathcal{C}$, a collection $\mathrm{Cov}_\tau(X)$ of *covering families* of X: a covering family of X is a set of morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{C}$. These satisfy:

A1. $\{\mathrm{id}_X\}$ is in $\mathrm{Cov}_\tau(X)$ for each $X \in \mathcal{C}$.

A2. For $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ and $g : Y \to X$ a morphism in $\mathcal{C}$, the fiber products $U_\alpha \times_X Y$ all exist and $\{p_2 : U_\alpha \times_X Y \to Y\}$ is in $\mathrm{Cov}_\tau(Y)$.

A3. If $\{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(X)$ and if $\{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ is in $\mathrm{Cov}_\tau(U_\alpha)$ for each α, then $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$ is in $\mathrm{Cov}_\tau(X)$.

A category with a (pre) topology is a *site*

Sheaves on a site

For S presheaf of abelian groups on $\mathcal{C}$ and $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$, we have the "restriction" morphisms

$$\begin{aligned} f_\alpha^* &: S(X) \to S(U_\alpha) \\ p_{1,\alpha,\beta}^* &: S(U_\alpha) \to S(U_\alpha \times_X U_\beta) \\ p_{2,\alpha,\beta}^* &: S(U_\beta) \to S(U_\alpha \times_X U_\beta). \end{aligned}$$

Taking products, we have the sequence of abelian groups

$$0 \to S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \xrightarrow{\prod p_{1,\alpha,\beta}^* - \prod p_{2,\alpha,\beta}^*} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta). \quad (1)$$

Definition A presheaf S is a *sheaf* for τ if for each covering family $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, the sequence (1) is exact. The category $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ of sheaves of abelian groups on $\mathcal{C}$ for τ is the full subcategory of $PreShv^{\mathbf{Ab}}(\mathcal{C})$ with objects the sheaves.

Proposition *(1) The inclusion* $i : Shv_{\mathcal{T}}^{\mathbf{Ab}}(\mathcal{C}) \to PreShv_{\mathcal{T}}^{\mathbf{Ab}}(\mathcal{C})$ *admits a left adjoint: "sheafification".*

(2) $Shv_{\mathcal{T}}^{\mathbf{Ab}}(\mathcal{C})$ *is an abelian category: For* $f : F \to G$, $\ker(f)$ *is the presheaf kernel.* $\mathrm{coker}(f)$ *is the sheafification of the presheaf cokernel.*

(3) $Shv_{\mathcal{T}}^{\mathbf{Ab}}(\mathcal{C})$ *has enough injectives.*

Categories of motivic complexes

Nisnevich sheaves The sheaf-theoretic construction of mixed motives is based on the notion of a *Nisnevich sheaf with transfer*.

Definition Let X be a k-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Using Nisnevich covers as covering families gives us the *small Nisnevich site on* X, X_{Nis}. The *big Nisnevich site over* k, with underlying category $\mathbf{Sm}/k$, is defined similarly.

Notation $\mathrm{Sh}^{\mathrm{Nis}}(X) :=$ Nisnevich sheaves of abelian groups on X, $\mathrm{Sh}^{\mathrm{Nis}}(k) :=$ Nisnevich sheaves of abelian groups on $\mathbf{Sm}/k$

For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/k$ or X_{Nis}, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf.

For a category $\mathcal{C}$, we have the category of presheaves of abelian groups on $\mathcal{C}$, i.e., the category of functors $\mathcal{C}^{\mathsf{op}} \to \mathbf{Ab}$.

Definition (1) The category $\mathsf{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\mathsf{Cor}_{fin}(k)$.

(2) The category of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$, $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))$, is the full subcategory of $\mathsf{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}/k$, the restriction of F to X_{Nis} is a sheaf.

Note. A PST F is a presheaf on $\mathbf{Sm}/k$ together with *transfer maps*

$$\mathsf{Tr}(a) : F(Y) \to F(X)$$

for every finite correspondence $a \in \mathsf{Cor}_{fin}(X, Y)$, with:

$\mathsf{Tr}(\Gamma_f) = f^*$, $\mathsf{Tr}(a \circ b) = \mathsf{Tr}(b) \circ \mathsf{Tr}(a)$, $\mathsf{Tr}(a \pm b) = \mathsf{Tr}(a) \pm \mathsf{Tr}(b)$.

Definition Let F be a presheaf of abelian groups on $\mathbf{Sm}/k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}/k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism.

We call F *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\text{Nis}}, F_{\text{Nis}})$ is homotopy invariant.

Theorem (PST) *Let F be a homotopy invariant PST on $\mathbf{Sm}/k$. Then*

(1) The cohomology presheaves $X \mapsto H^q(X_{\text{Nis}}, F_{\text{Nis}})$ are PST's

(2) F_{Nis} is strictly homotopy invariant.

(3) $F_{\text{Zar}} = F_{\text{Nis}}$ and $H^q(X_{\text{Zar}}, F_{\text{Zar}}) = H^q(X_{\text{Nis}}, F_{\text{Nis}})$.

Remarks (1) uses the fact that for finite map $Z \to X$ with X Hensel local and Z irreducible, Z is also Hensel local. (2) and (3) rely on Voevodsky's generalization of Quillen's proof of Gersten's conjecture, viewed as a "moving lemma using transfers". For example:

Lemma (Voevodsky's moving lemma) *Let X be in $\mathbf{Sm}/k$, S a finite set of points of X, $j_U : U \to X$ an open subscheme. Then there is an open neighborhood $j_V : V \to X$ of S in X and a finite correspondence $a \in c(V, U)$ such that, for all homotopy invariant PST's F, the diagram*

$$\begin{array}{ccc} F(X) & \xrightarrow{j_U^*} & F(U) \\ {\scriptstyle j_V^*}\downarrow & & \downarrow{\scriptstyle Tr(a)} \\ F(V) & = & F(V) \end{array}$$

commutes.

One consequence of the lemma is

(1) If X is semi-local, then $F(X) \to F(U)$ is a split injection.

Variations on this construction prove:

(2) If X is semi-local and smooth then $F(X) = F_{\text{Zar}}(X)$ and $H^n(X_{\text{Zar}}, F_{\text{Zar}}) = 0$ for $n > 0$.

(3) If U is an open subset of $\mathbb{A}^1_k$, then $F_{\text{Zar}}(U) = F(U)$ and $H^n(U, F_{\text{Zar}}) = 0$ for $n > 0$.

(4) If $j : U \to X$ has complement a smooth k-scheme $i : Z \to X$, then $coker F(X_{\text{Zar}}) \to j_* F(U_{\text{Zar}})$ (as a sheaf on Z_{Zar}) depends only on the Nisnevich neighborhood of Z in X.

(1)-(4) together with some cohomological techniques prove the theorem.

The category of motivic complexes

Definition Inside the derived category $D^{-}(\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))$, we have the full subcategory $DM_{-}^{\mathsf{eff}}(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.

Proposition *$DM_{-}^{\mathsf{eff}}(k)$ is a triangulated subcategory of $D^{-}(\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))$.*

This follows from

Lemma *Let $HI(k) \subset \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))$ be the full subcategory of homotopy invariant sheaves. Then $HI(k)$ is an abelian subcategory of $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))$, closed under extensions in $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))$.*

Proof of the lemma. Given $f : F \to G$ in $HI(k)$, $ker(f)$ is the presheaf kernel, hence in $HI(k)$.

The presheaf $coker(f)$ is homotopy invariant, so by the PST theorem $coker(f)_{\mathrm{Nis}}$ is homotopy invariant.

Given $0 \to A \to E \to B \to 0$ exact in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))$ with $A, B \in HI(k)$.

Consider $p : X \times \mathbb{A}^1 \to X$. The PST theorem implies $R^1 p_* A = 0$, so

$0 \to p_* A \to p_* E \to p_* B \to 0$ is exact as sheaves on X.

Thus $p_* E = E$, so E is homotopy invariant.

The Suslin complex

The Suslin complex

Let $\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_{i=0}^n t_i - 1$.

$n \mapsto \Delta^n$ defines the cosimplicial k-scheme Δ^*.

Definition Let F be a presheaf on $\mathrm{Cor}_{fin}(k)$. Define the presheaf $C_n(F)$ by

$$C_n(F)(X) := F(X \times \Delta^n)$$

The *Suslin complex* $C_*(F)$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : C_n(F) \to C_{n-1}(F).$$

For $X \in \mathbf{Sm}/k$, let $C_*(X)$ be the complex of sheaves $C_n(X)(U) := \mathrm{Cor}_{fin}(U \times \Delta^n, X)$.

Remarks (1) If F is a sheaf with transfers on $\mathbf{Sm}/k$, then $C_*(F)$ is a complex of sheaves with transfers.

(2) The homology presheaves $h_i(F) := \mathcal{H}^{-i}(C_*(F))$ are homotopy invariant. Thus, by Voevodsky's PST theorem, the associated Nisnevich sheaves $h_i^{\mathsf{Nis}}(F)$ are strictly homotopy invariant. We thus have the functor

$$C_* : \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)) \to DM_-^{\mathsf{eff}}(k).$$

(3) For X in $\mathbf{Sch}_k$, we have the sheaf with transfers $L(X)(Y) = \mathsf{Cor}_{fin}(Y, X)$ for $Y \in \mathbf{Sm}/k$.

For $X \in \mathbf{Sm}/k$, $L(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $\mathrm{Hom}(-, X)$.

We have the canonical isomorphisms $\mathrm{Hom}(L(X), F) = F(X)$ and $C_*(X) = C_*(L(X))$.

In fact: For $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))$ there is a canonical isomorphism

$$\mathrm{Ext}^n_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))}(L(X), F) \cong H^n(X_{\mathrm{Nis}}, F)$$

Statement of main results

The localization theorem

Theorem *The functor C_* extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))) \to DM_-^{\mathsf{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))$.

$\mathbf{R}C_*$ *identifies $DM_-^{\mathsf{eff}}(k)$ with the localization $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))/\mathcal{A}$, where $\mathcal{A}$ is the localizing subcategory of $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))$ generated by complexes*

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

The tensor structure

We define a tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))$:

Set $L(X) \otimes L(Y) := L(X \times Y)$.

For a general F, we have the canonical surjection

$$\bigoplus_{(X, s \in F(X))} L(X) \to F.$$

Iterating gives the canonical left resolution $\mathcal{L}(F) \to F$. Define

$$F \otimes G := H_0^{\mathrm{Nis}}(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

The unit for $\otimes$ is $L(\mathrm{Spec}\, k)$.

There is an *internal Hom* in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))$:

$$\mathcal{H}om(L(X), G)(U) = G(U \times X);$$
$$\mathcal{H}om(F, G) := H^0_{\mathrm{Nis}}(\mathcal{H}om(\mathcal{L}(F), G)).$$

Tensor structure in DM_{-}^{eff}

The tensor structure on $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k))$ induces a tensor structure $\otimes^{L}$ on $D^{-}(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{fin}(k)))$.

We make $DM_{-}^{\mathsf{eff}}(k)$ a tensor triangulated category via the localization theorem:

$$M \otimes N := \mathbf{R}C_{*}(i(M) \otimes^{L} i(N)).$$

The embedding theorem

Theorem *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc}
\mathcal{H}^b(\mathrm{Cor}_{fin}(k)) & \xrightarrow{L} & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{fin}(k))) \\
\downarrow & & \downarrow {\scriptstyle \mathbf{R}C_*} \\
DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow[i]{} & DM^{\mathrm{eff}}_{-}(k)
\end{array}$$

such that

1. i is a full embedding with dense image.

2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Corollary *For X and $Y \in \mathbf{Sm}/k$,* $\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X))$.

Suslin homology

Definition For $X \in \mathbf{Sm}/k$, define the *Suslin homology* of X as

$$H_i^{\mathsf{Sus}}(X) := H_i(C_*(X)(\operatorname{Spec} k)).$$

Theorem *Let U, V be open subschemes of $X \in \mathbf{Sm}/k$. Then there is a long exact Mayer-Vietoris sequence*

$$\ldots \to H_{n+1}^{\mathsf{Sus}}(U \cup V) \to H_n^{\mathsf{Sus}}(U \cap V) \\ \to H_n^{\mathsf{Sus}}(U) \oplus H_n^{\mathsf{Sus}}(V) \to H_n^{\mathsf{Sus}}(U \cup V) \to \ldots$$

Proof. By the embedding theorem, we have

$$H_n^{\mathsf{Sus}}(Y) = \operatorname{Hom}_{DM_-^{\mathsf{eff}}(k)}(M_{\mathsf{gm}}(\operatorname{Spec} k), M_{\mathsf{gm}}(Y)[-n]).$$

for all $Y \in \mathbf{Sm}/k$, $n \in \mathbb{Z}$. Also, $[U \cap V] \to [U] \oplus [V] \to [U \cup V]$ extends to a distinguished triangle in $DM_-^{\mathsf{eff}}(k)$.