The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

SMR/1840-21

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Mixed motives and cycle complexes

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Recall from lecture 3:

- The category PST: presheaves on $\mathrm{Cor}_{\mathrm{fin}}(k)$ and the category $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$.

- The full subcategory $DM^{\mathrm{eff}}_{-}(k) \subset D^{-}(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$: complexes of sheaves with homotopy invariant cohomology sheaves.

- The Suslin complex: $C_* : \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)) \to DM^{\mathrm{eff}}_{-}(k)$.

$$\ldots \to C_n(F)(X) := F(X \times \Delta^n) \xrightarrow{\sum_{i=0}^{n}(-1)^i \delta_i^*} C_{n-1}(F)(X) = F(X \times \Delta^{n-1}) \to \ldots$$

- For $X \in \mathbf{Sm}/k$, the representable sheaf $L(X)$: $L(X)(Y) = c_{fin}(Y, X) = \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{fin}}(k)}([Y], [X])$.

Thus, we have the functor

$$M : \mathbf{Sm}/k \to DM_{-}^{\mathsf{eff}}(k)$$

$M(X) := C_*(L(X))$

We also recall two main structural results:

Theorem (PST) *Let F be a homotopy invariant PST on $\mathbf{Sm}/k$. Then*

(1) The cohomology presheaves $X \mapsto H^q(X_{\mathsf{Nis}}, F_{\mathsf{Nis}})$ are PST's

(2) F_{Nis} is strictly homotopy invariant.

(3) $F_{\mathsf{Zar}} = F_{\mathsf{Nis}}$ and $H^q(X_{\mathsf{Zar}}, F_{\mathsf{Zar}}) = H^q(X_{\mathsf{Nis}}, F_{\mathsf{Nis}})$.

Proposition *$DM_-^{\mathsf{eff}}(k)$ is a triangulated subcategory of $D^-(\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.*

This latter result follows from

Lemma *Let* $HI(k) \subset \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$ *be the full subcategory of homotopy invariant sheaves. Then* $HI(k)$ *is an abelian subcategory of* $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$*, closed under extensions in* $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$*.*

which in turn is a consequence of the PST theorem.

Theorem (Global PST) *Let F^* be a complex of PSTs on $\mathbf{Sm}/k$: $F \in C^-(PST)$. Suppose that the cohomology presheaves $h^i(F)$ are homotopy invariant. Then*

(1) For $Y \in \mathbf{Sm}/k$, $\mathbb{H}^i(Y_{\mathrm{Nis}}, F^*_{\mathrm{Nis}}) \cong \mathbb{H}^i(Y_{\mathrm{Zar}}, F^*_{\mathrm{Zar}})$

(2) The presheaf $Y \mapsto \mathbb{H}^i(Y_{\mathrm{Nis}}, F^*_{\mathrm{Nis}})$ is homotopy invariant

(1) and (2) follows from the PST theorem using the spectral sequence:

$$E_2^{p,q} = H^p(Y_\tau, h^q(F)_\tau) \Longrightarrow \mathbb{H}^{p+q}(Y_\tau, F_\tau), \tau = \mathrm{Nis}, \mathrm{Zar}.$$

Statement of main results

The localization theorem

Theorem *The functor* $C_* : \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM_-^{\mathsf{eff}}(k)$ *extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM_-^{\mathsf{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.

$\mathbf{R}C_*$ *identifies* $DM_-^{\mathsf{eff}}(k)$ *with the localization* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$, *where* $\mathcal{A}$ *is the localizing subcategory of* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$ *generated by complexes*

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

The tensor structure

We define a tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$:

Set $L(X) \otimes L(Y) := L(X \times Y)$.

For a general F, we have the canonical surjection

$$\bigoplus_{(X, s \in F(X))} L(X) \to F.$$

Iterating gives the canonical left resolution $\mathcal{L}(F) \to F$. Define

$$F \otimes G := H_0^{\mathrm{Nis}}(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

The unit for $\otimes$ is $L(\mathrm{Spec}\, k)$.

There is an *internal Hom* in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$:

$$\mathcal{H}om(L(X), G)(U) = G(U \times X);$$
$$\mathcal{H}om(F, G) := H^0_{\mathrm{Nis}}(\mathcal{H}om(\mathcal{L}(F), G)).$$

Tensor structure in DM_-^{eff}

The tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ induces a tensor structure $\otimes^L$ on $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

We make $DM_-^{\mathrm{eff}}(k)$ a tensor triangulated category via the localization theorem:

$$M \otimes N := \mathbf{R}C_*(i(M) \otimes^L i(N)).$$

We have the functor $L : \mathsf{Cor}_{\mathsf{fin}}(k) \to \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$ sending X to the representable sheaf $L(X)$. This extends to a functor on the homotopy categories

$$L : K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) \to K^b(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)));$$

composing with

$$K^b(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to K^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \\ \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$$

gives

$$L : K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))).$$

We also have the canonical localization functor

$$q : K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$$

and the localization functor

$$\mathbf{R}C_* : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM^{\mathsf{eff}}_-(k).$$

The embedding theorem

Theorem *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc} K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) & \xrightarrow{L} & D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathbf{R}C_*} \\ DM^{\mathsf{eff}}_{\mathsf{gm}}(k) & \xrightarrow[i]{} & DM^{\mathsf{eff}}_{-}(k) \end{array}$$

such that

1. *i is a full embedding with dense image.*
2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Corollary *For* X *and* $Y \in \mathbf{Sm}/k$*,*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X)).$$

Proof. For a sheaf F, and $Y \in \mathbf{Sm}/k$,

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}})}(L(Y), F) = F(Y)$$

Thus the Hom in the derived category, for F a complex of sheaves, is:

$$\mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}))}(L(Y), F[n]) = \mathbb{H}^n(Y_{\mathrm{Nis}}, F).$$

Thus (using the embedding theorem and localization theorem)

$$\begin{aligned}
&\mathsf{Hom}_{DM^{\mathsf{eff}}_{\mathsf{gm}}(k)}(M_{\mathsf{gm}}(Y), M_{\mathsf{gm}}(X)[n]) \\
&\qquad = \mathsf{Hom}_{DM^{\mathsf{eff}}_{-}(k)}(C_*(Y), C_*(X)[n]) \\
&\qquad = \mathsf{Hom}_{D^{-}(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}))}(L(Y), C_*(X)[n]) \\
&\qquad = \mathbb{H}^n(Y_{\mathsf{Nis}}, C_*(X)).
\end{aligned}$$

PST theorem $\Longrightarrow \mathbb{H}^n(Y_{\mathsf{Zar}}, C_*(X)) = \mathbb{H}^n(Y_{\mathsf{Nis}}, C_*(X))$.

Suslin homology

Definition For $X \in \mathbf{Sm}/k$, define the *Suslin homology* of X as

$$H_i^{\mathrm{Sus}}(X) := H_i(C_*(X)(\operatorname{Spec} k)).$$

Theorem *Let U, V be open subschemes of $X \in \mathbf{Sm}/k$. Then there is a long exact Mayer-Vietoris sequence*

$$\ldots \to H_{n+1}^{\mathrm{Sus}}(U \cup V) \to H_n^{\mathrm{Sus}}(U \cap V) \to H_n^{\mathrm{Sus}}(U) \oplus H_n^{\mathrm{Sus}}(V) \to H_n^{\mathrm{Sus}}(U \cup V) \to \ldots$$

Proof. By the embedding theorem $[U \cap V] \to [U] \oplus [V] \to [U \cup V]$ maps to the distinguished triangle in $DM_-^{\mathrm{eff}}(k)$:

$$C_*(U \cap V)_{\mathrm{Nis}} \to C_*(U)_{\mathrm{Nis}} \oplus C_*(V)_{\mathrm{Nis}} \to C_*(U \cup V)_{\mathrm{Nis}} \to$$

This yields a long exact sequence upon applying $\mathrm{Hom}_{DM^{\mathrm{eff}}}(M_{\mathrm{gm}}(Y), -)$ for any $Y \in \mathbf{Sm}/k$.

By the corollary to the embedding theorem, this gives the long exact sequence

$$\ldots \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U \cap V)) \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U)) \oplus \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(V)) \\ \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U \cap V) \to \mathbb{H}^{-n+1}(Y_{\mathrm{Nis}}, C_*(U \cap V)) \to \ldots$$

Now just take $Y = \mathrm{Spec}\, k$, since

$$\mathbb{H}^{-n}(\mathrm{Spec}\, k_{\mathrm{Nis}}, C_*(X)) = H_n(C_*(X)(\mathrm{Spec}\, k)) = H_n^{\mathrm{Sus}}(X).$$

In fact, the embedding theorem implies that for all $Y \in \mathbf{Sm}/k$, the homology sheaves $h_n^{\mathsf{Zar}}(Y)$ associated to the presheaf $U \mapsto H_n(C_*(Y)(U))$ are the same as the sheaves associated to the presheaf

$$U \mapsto \mathsf{Hom}_{DM_-^{\mathsf{eff}}(k)}(M_{\mathsf{gm}}(U), M_{\mathsf{gm}}(Y)[-n]).$$

Thus

- $h_n^{\mathsf{Zar}}(Y \times \mathbb{A}^1) \to h_n^{\mathsf{Zar}}(Y)$ is an isomorphism

- for $Y = U \cup V$, have a long exact Mayer-Vietoris sequence

$$\ldots \to h_p^{\mathsf{Zar}}(U \cap V) \to h_p^{\mathsf{Zar}}(U) \oplus h_p^{\mathsf{Zar}}(V) \to h_p^{\mathsf{Zar}}(Y) \to h_{p-1}^{\mathsf{Zar}}(U \cap V) \to \ldots$$

Fundamental constructions in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$

We discuss the projective bundle formula the blow-up formula and the Gysin isomorphism, realizing these as morphisms and isomorphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}$.

Weight one motivic cohomology

$\mathbb{Z}(1)[2]$ is the reduced motive of $\mathbb{P}^1$, and $M_{\mathrm{gm}}(\mathbb{P}^1)$ is represented in DM_{-}^{eff} by the Suslin complex $C_*(\mathbb{P}^1)$. The homology sheaves of $C_*(\mathbb{P}^1)$ and $C_*(\operatorname{Spec} k)$ are given by:

Lemma *$h_0^{\mathrm{Zar}}(\mathbb{P}^1) = \mathbb{Z}$, $h_1^{\mathrm{Zar}}(\mathbb{P}^1) = \mathbb{G}_m$ and $h_n^{\mathrm{Zar}}(\mathbb{P}^1) = 0$ for $n \geq 2$. $h_0^{\mathrm{Zar}}(\operatorname{Spec} k) = \mathbb{Z}$, $h_n^{\mathrm{Zar}}(\operatorname{Spec} k) = 0$ for $n \geq 1$.*

Sketch of proof: $C_n(\operatorname{Spec} k)(Y) = c(Y \times \Delta^n, \operatorname{Spec} k) = H^0(Y_{\mathrm{Zar}}, \mathbb{Z})$. Thus

$$h_p(C_*(\operatorname{Spec} k)(Y)) = \begin{cases} H^0(Y_{\mathrm{Zar}}, \mathbb{Z}) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

We have $h_p^{\mathsf{Zar}}(\mathbb{A}^1) = h_p^{\mathsf{Zar}}(\operatorname{Spec} k)$ and we have a Meyer-Vietoris sequence

$$\ldots \to h_p^{\mathsf{Zar}}(\mathbb{A}^1) \oplus h_p^{\mathsf{Zar}}(\mathbb{A}^1) \to h_p^{\mathsf{Zar}}(\mathbb{P}^1) \to h_{p-1}^{\mathsf{Zar}}(\mathbb{A}^1 \setminus 0) \to \ldots$$

giving

$$h_p^{\mathsf{Zar}}(\mathbb{P}^1) = h_p^{\mathsf{Zar}}(\operatorname{Spec} k) \oplus h_{p-1}^{\mathsf{Zar}}(\mathbb{G}_m).$$

where $h_{p-1}^{\mathsf{Zar}}(\mathbb{G}_m) := h_{p-1}^{\mathsf{Zar}}(\mathbb{A}^1 \setminus 0)/h_{p-1}^{\mathsf{Zar}}(1)$.

So we need to see that

$$h_p^{\mathsf{Zar}}(\mathbb{A}^1 \setminus 0) = \begin{cases} \mathbb{G}_m \oplus \mathbb{Z} \cdot [1] & \text{for } p = 0 \\ 0 & \text{else.} \end{cases}$$

For this, let $Y = \mathrm{Spec}\,\mathcal{O}$ for $\mathcal{O} = \mathcal{O}_{X,x}$ some $x \in X \in \mathbf{Sm}/k$

$$h_p^{\mathrm{Zar}}(\mathbb{A}^1 \setminus 0)_{X,x} = H_p(C_*(\mathbb{A}^1 \setminus 0)(Y)).$$

For $W \subset Y \times \Delta^n \times (\mathbb{A}^1 \setminus 0)$ finite and surjective over $Y \times \Delta^n$, W has a monic defining equation

$$F_W(y,t,x) = x^N + \sum_{i=1}^{N-1} F_i(y,t)x^i + F_0(y,t)$$

with $F_0(y,t)$ a unit in $\mathcal{O}[t_0,\ldots,t_n]/\sum_i t_i - 1$.

Map $h_0^{\mathsf{Zar}} \to \mathbb{Z}$ by $W \mapsto \deg_Y W$.

Define $\mathsf{cl}_Y : \mathbb{G}_m(Y) \to H_0(C_*(\mathbb{A}^1 \setminus 0)(Y))_{\deg 0}$ by

$$\mathsf{cl}_Y(u) := [\Gamma_u - \Gamma_1],$$

$\Gamma_u \subset Y \times \mathbb{A}^1 \setminus 0$ the graph of $u : Y \to \mathbb{A}^1 \setminus 0$.

One shows cl_Y is a group homomorphism by using the cycle T on $Y \times \Delta^1 \times \mathbb{A}^1 \setminus 0$ defined by

$$t(x - uv)(x - 1) + (1 - t)(x - u)(x - v),$$

$$dT = (\Gamma_{uv} - \Gamma_1) - (\Gamma_u - \Gamma_1) - (\Gamma_v - \Gamma_1).$$

To show cl_Y is surjective: If $W \subset Y \times \mathbb{A}^1 \setminus 0$ is finite over Y, we have the unit $u := (-1)^N F_W(y, 0)$ with $F_W(y, x)$ the monic defining equation for W, $N = \deg_Y W$. The function

$$F(y, x, t) := tF_W(y, x) + (1 - t)(x - u)(x - 1)^{N-1}$$

defines a finite cycle T on $Y \times \Delta^1 \times \mathbb{A}^1 \setminus 0$ with

$$dT = W - \Gamma_u - (N - 1)\Gamma_1 = (W - \deg_Y W \cdot \Gamma_1) - (\Gamma_u - \Gamma_1).$$

To show that cl_Y is injective: show sending W to $(-1)^N F_W(y, 0)$ passes to H_0. This can be done by noting that there are no non-constant maps $f : \mathbb{A}^1 \to \mathbb{A}^1 \setminus 0$.

The proof that $h_p^{\mathsf{Zar}}(Y) = 0$ for $p > 0$ is similar.

This computation implies that

$$\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$$

in $DM_-^{\mathsf{eff}}(k)$. Indeed:

$$\begin{aligned}\mathbb{Z}(1)[2] &\cong \mathrm{Cone}(C_*(\mathbb{P}^1) \to C_*(\mathrm{Spec}\, k))[-1] \\ &\cong h_1^{\mathsf{Zar}}(\mathbb{P}^1)[1] = \mathbb{G}_m[1]\end{aligned}$$

This yields:

Proposition *For $X \in \mathbf{Sm}/k$, we have*

$$H^n(X, \mathbb{Z}(1)) = \begin{cases} H^0_{\mathrm{Zar}}(X, \mathcal{O}^*_X) & \text{for } n = 1 \\ \mathrm{Pic}(X) := H^1_{\mathrm{Zar}}(X, \mathcal{O}^*_X) & \text{for } n = 2 \\ 0 & \text{else.} \end{cases}$$

Proof.. Since $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ in $DM^{\mathrm{eff}}_-(k)$, the corollary to the embedding theorem gives:

$$\begin{aligned} \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}}(M_{\mathrm{gm}}(X), \mathbb{Z}(1)[n]) &\cong \mathbb{H}^n_{\mathrm{Nis}}(X, \mathbb{Z}(1)) \\ &\cong \mathbb{H}^n_{\mathrm{Zar}}(X, \mathbb{Z}(1)) \cong H^{n-1}_{\mathrm{Zar}}(X, \mathbb{G}_m). \end{aligned}$$

Chern classes of line bundles

Definition Let $L \to X$ be a line bundle on $X \in \mathbf{Sm}_k$.

We let $c_1(L) \in H^2(X, \mathbb{Z}(1))$ be the element corresponding to $[L] \in H^1_{\mathsf{Zar}}(X, \mathcal{O}_X^*)$.

Weighted spheres Before we compute the motive of $\mathbb{P}^n$, we need:

Lemma *There is a canonical isomorphism* $M_{\text{gm}}(\mathbb{A}^n \setminus 0) \to \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

Proof. For $n = 1$, we have $M_{\text{gm}}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, by definition of $\mathbb{Z}(1)$. The Mayer-Vietoris distinguished triangle

$$M_{\text{gm}}(\mathbb{A}^1 \setminus 0) \to M_{\text{gm}}(\mathbb{A}^1) \oplus M_{\text{gm}}(\mathbb{A}^1) \to M_{\text{gm}}(\mathbb{P}^1) \to M_{\text{gm}}(\mathbb{A}^1 \setminus 0)[1]$$

defines an isomorphism $t : M_{\text{gm}}(\mathbb{A}^1 \setminus 0) \to \mathbb{Z}(1)[1] \oplus \mathbb{Z}$.

For general n, write $\mathbb{A}^n \setminus 0 = \mathbb{A}^n \setminus \mathbb{A}^{n-1} \cup \mathbb{A}^n \setminus \mathbb{A}^1$. By induction, Mayer-Vietoris and homotopy invariance, this gives the distinguished triangle

$$(\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})$$

$$\to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \oplus (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})$$

$$\to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)$$

$$\to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})[1]$$

yielding the result.

Projective bundle formula Let $E \to X$ be a rank $n+1$ vector bundle over $X \in \mathbf{Sm}/k$, $q : \mathbb{P}(E) \to X$ the resulting $\mathbb{P}^{n-1}$ bundle, $\mathcal{O}(1)$ the tautological quotient bundle.

Define $\alpha_j : M_{\text{gm}}(\mathbb{P}(E)) \to M_{\text{gm}}(X)(j)[2j]$ by

$$M_{\text{gm}}(\mathbb{P}(E)) \xrightarrow{\delta} M_{\text{gm}}(\mathbb{P}(E)) \otimes M_{\text{gm}}(\mathbb{P}(E)) \xrightarrow{q \otimes c_1(\mathcal{O}(1))^j} M_{\text{gm}}(X)(j)[2j]$$

Theorem $\oplus_{j=0}^{n} \alpha_j : M_{\text{gm}}(\mathbb{P}(E)) \to \oplus_{j=0}^{n} M_{\text{gm}}(X)(j)[2j]$ *is an isomorphism.*

Proof. The map is natural in X. Mayer-Vietoris reduces to the case of a trivial bundle, then to the case $X = \operatorname{Spec} k$, so we need to prove:

Lemma $\oplus_{j=0}^{n}\alpha_j : M_{\mathrm{gm}}(\mathbb{P}^n) \to \oplus_{j=0}^{n}\mathbb{Z}(j)[2j]$ *is an isomorphism.*

Proof. Write $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus 0)$. $M_{\mathrm{gm}}(\mathbb{A}^n) = \mathbb{Z}$. $\mathbb{P}^n \setminus 0$ is an $\mathbb{A}^1$ bundle over $\mathbb{P}^{n-1}$, so induction gives

$$M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) = \oplus_{j=0}^{n-1}\mathbb{Z}(j)[2j].$$

Also $M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

The Mayer-Vietoris distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{A}^n) \oplus M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)[1]$$

gives the result.

Gysin isomorphism

Definition For $i : Z \to X$ a closed subset, let $M_{\mathrm{gm}}(X/X \setminus Z) \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ be the image in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of the complex $[X \setminus Z] \xrightarrow{j} [X]$, with $[X]$ in degree 0.

Note. The Mayer-Vietoris property for $M_{\mathrm{gm}}(-)$ yields a Zariski excision property: If Z is closed in U, an open in X, then $M_{\mathrm{gm}}(U/U \setminus Z) \to M_{\mathrm{gm}}(X/X \setminus Z)$ is an isomorphism.

In fact, Voevodsky's moving lemma shows that $M_{\mathrm{gm}}(X/X \setminus Z)$ depends only on the Nisnevich neighborhood of Z in X: this is the *Nisnevich excision* property.

Motivic cohomology with support Let $Z \subset X$ be a closed subset, $U := X \setminus Z$. Setting

$$H^p_Z(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM^{\mathrm{eff}}_{-}(k)}(M_{\mathrm{gm}}(X/U), \mathbb{Z}(q)[p])$$

gives the long exact sequence for cohomology with support:

$$\ldots \to H^p_Z(X, \mathbb{Z}(q)) \xrightarrow{i_*} H^p(X, \mathbb{Z}(q)) \xrightarrow{j^*} H^p(U, \mathbb{Z}(q)) \to H^{p+1}_Z(X, \mathbb{Z}(q)) \to$$

Theorem (Gysin isomorphism) *Let* $i : Z \to X$ *be a closed embedding in* $\mathbf{Sm}/k$ *of codimension* n, $U = X \setminus Z$. *Then there is a natural isomorphism in* $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$

$$M_{\mathrm{gm}}(X/U) \cong M_{\mathrm{gm}}(Z)(n)[2n].$$

In particular:

$$\begin{aligned} H_Z^p(X, \mathbb{Z}(q)) &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X/U), \mathbb{Z}(q)[p]) \\ &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Z)(n)[2n], \mathbb{Z}(q)[p]) \\ &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Z), \mathbb{Z}(q-n)[p-2n]) \\ &= H^{p-2n}(Z, \mathbb{Z}(q-n)). \end{aligned}$$

Gysin distinguished triangle

Theorem *Let $i : Z \to X$ be a codimension n closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. There is a canonical distinguished triangle in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$:*

$$M_{\mathrm{gm}}(U) \xrightarrow{j_*} M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(n)[2n] \to M_{\mathrm{gm}}(U)[1]$$

Proof. By definition of $M_{\mathrm{gm}}(X/U)$, we have the canonical distinguished triangle in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$:

$$M_{\mathrm{gm}}(U) \xrightarrow{j_*} M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(X/U) \to M_{\mathrm{gm}}(U)[1]$$

then insert the Gysin isomorphism $M_{\mathrm{gm}}(X/U) \cong M_{\mathrm{gm}}(Z)(n)[2n]$.

Applying $\mathrm{Hom}(-, \mathbb{Z}(q)[p])$ to the Gysin distinguished triangle gives the long exact Gysin sequence

$$\ldots \to H^{p-2n}(Z, \mathbb{Z}(q-n)) \xrightarrow{i_*} H^p(X, \mathbb{Z}(q))$$
$$\xrightarrow{j^*} H^p(U, \mathbb{Z}(q))$$
$$\xrightarrow{\partial} H^{p-2n+1}(Z, \mathbb{Z}(q-n)) \to$$

which is the same as the sequence for cohomology with supports, using the Gysin isomorphism

$$H^{p-2n}(Z, \mathbb{Z}(q-n)) \cong H^p_Z(X, \mathbb{Z}(q)).$$

Now for the proof of the Gysin isomorphism theorem:

We first prove a special case:

Lemma *Let* $E \to Z$ *be a vector bundle of rank* n *with zero section* s. *Then* $M_{\mathrm{gm}}(E/E \setminus s(Z)) \cong M_{\mathrm{gm}}(Z)(n)[2n]$.

Proof. Since $M_{\mathrm{gm}}(E) \to M_{\mathrm{gm}}(Z)$ is an isomorphism by homotopy, we need to show

$$M_{\mathrm{gm}}(E \setminus s(Z)) \cong M_{\mathrm{gm}}(Z) \oplus M_{\mathrm{gm}}(Z)(n)[2n-1].$$

Let $\mathbb{P} := \mathbb{P}(E \oplus \mathcal{O}_Z)$, and write $\mathbb{P} = E \cup (\mathbb{P} \setminus s(Z))$. Mayer-Vietoris gives the distinguished triangle

$$M_{\mathrm{gm}}(E \setminus s(Z)) \to M_{\mathrm{gm}}(E) \oplus M_{\mathrm{gm}}(\mathbb{P} \setminus s(Z)) \\ \to M_{\mathrm{gm}}(\mathbb{P}) \to M_{\mathrm{gm}}(E \setminus s(Z))[1]$$

Since $\mathbb{P} \setminus s(Z) \to \mathbb{P}(E)$ is an $\mathbb{A}^1$ bundle, the projective bundle formula gives the isomorphism we wanted.

Deformation to the normal bundle

For $i : Z \to X$ a closed immersion in $\mathbf{Sm}/k$, let

$$p : (X \times \mathbb{A}^1)_{Z\times 0} \to X \times \mathbb{A}^1$$

be the blow-up of $X \times \mathbb{A}^1$ along $Z \times 0$. Set

$$Def(i) := (X \times \mathbb{A}^1)_{Z\times 0} \setminus p^{-1}[X \times 0].$$

We have $\tilde{i} : Z \times \mathbb{A}^1 \to Def(i)$, $q : Def(i) \to \mathbb{A}^1$.

The fiber $\tilde{i}_1$ is $i : Z \to X$, the fiber $\tilde{i}_0$ is $s : Z \to N_{Z/X}$.

Lemma *The maps*

$$M_{\mathrm{gm}}(N_{Z/X}/N_{Z/X} \setminus s(Z)) \to M_{\mathrm{gm}}(Def(i)/[Def(i) \setminus Z \times \mathbb{A}^1])$$
$$M_{\mathrm{gm}}(X/X \setminus Z) \to M_{\mathrm{gm}}(Def(i)/[Def(i) \setminus Z \times \mathbb{A}^1])$$

are isomorphisms.

Proof. By Nisnevich excision, we reduce to the case $Z \times 0 \to Z \times \mathbb{A}^n$. In this case, $Z \times \mathbb{A}^1 \to Def(i)$ is just $(Z \times 0 \to Z \times \mathbb{A}^n) \times \mathbb{A}^1$, whence the result.

Proof of the theorem.

$$M_{\mathrm{gm}}(X/X \setminus Z) \cong M_{\mathrm{gm}}(N_{Z/X}/N_{Z/X} \setminus s(Z)) \cong M_{\mathrm{gm}}(Z)(n)[2n]$$