The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

SMR/1840-27

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Symmetric products of motives

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4 Lecture 4: Symmetric products of motives

This lecture summarizes the main points of [V07]. If X is a normal quasiprojective variety over a field of characteristic 0, the symmetric product $S^m X = X^m/\Sigma_m$ is also normal (where Σ_m is the symmetric group). More generally, if G is a subgroup of Σ_m then $S^G X = X^m/G$ determines a functor from the category **Norm** of normal quasiprojective varieties to itself. If X_+ denotes the disjoint union of X and $\mathrm{Spec}(X)$ then there is a natural split sequence of pointed objects (which extends to simplicial objects as well):

$$(4.1) \qquad S^{m-1}(X_+) \to S^m(X_+) \to (S^m X)_+.$$

It is easy to see that $\widetilde{S}^m(X_+) = (S^m X)_+$ is a functor on $\mathbf{Norm}_+$, and that it extends to finite correspondences, giving us a functor S^m_{tr} from **Cor** — or even **Cor**(**Norm**) — to itself, characterized by the formula:

$$S^m_{\mathrm{tr}}(R_{\mathrm{tr}}X) = R_{\mathrm{tr}}\widetilde{S}^m(X_+) = R_{\mathrm{tr}}(S^m X).$$

This functor extends to simplicial objects and commutes with direct sums.

Lemma 4.2. *For any (simplicial) normal X, Y we have*

$$S^m_{tr}(R_{tr}X \oplus R_{tr}Y) \cong \bigoplus_{i+j=m} S^i_{tr}(R_{tr}X) \otimes S^j_{tr}(R_{tr}Y).$$

Proof. Immediate from $R_{\mathrm{tr}}(X \coprod Y) = R_{\mathrm{tr}}X \oplus R_{\mathrm{tr}}Y$ and $S^m(X \coprod Y) = \coprod S^i(X) \times S^j(Y)$. □

Examples 4.3. (a) Since $S^m(S^0) \cong \{0, 1, \dots, m\}$ we have $\widetilde{S}^m(S^0) = S^0$ and $S^m_{\mathrm{tr}}(R) \cong R$.

(b) Since $R_{\mathrm{tr}}(\mathbb{P}^1) \cong R \oplus \mathbb{L}^1$ and $S^m\mathbb{P}^1 \cong \mathbb{P}^m$, Lemma 4.2 yields $S^m_{\mathrm{tr}}(\mathbb{L}^1) \cong \mathbb{L}^m$.

We write $S^\infty(X_+)$ for the colimit of the pointed spaces $S^n(X_+)$. From (4.1) one gets:

Proposition 4.4. *For any simplicial object $V_\bullet$ of $\mathbf{Norm}_+$ there is an isomorphism*

$$R_{tr}(S^\infty V_\bullet) \cong \bigoplus_{m=0}^{\infty} S^m_{tr} R_{tr}(V_\bullet).$$

The following result is a translation of the Suslin-Voevodsky result [10] that finite correspondences of degree $m \geq 0$ from X to Y correspond to morphisms from X to $S^m(Y)$, together with the fact that a connected simplicial H-space has a homotopy inverse.

Theorem 4.5. *Let $V_\bullet$ be a simplicial object of* $\mathbf{Norm}_+$. *If the simplicial sets* $\mathrm{Hom}(X, V_\bullet)$ *are connected for all X, then the morphism $S^\infty(V_\bullet) \to u\mathbb{Z}_{tr}(V_\bullet)$ is a global weak equivalence of spaces (functors on Sm/k).*

Examples 4.6. (a) When V is S^0 (which is *not* connected), the morphism in 4.5 is $\mathbb{N} \to \mathbb{Z}$.

(b) If $n \geq 1$, 4.4 and 4.5 yield $K_n = u\mathbb{Z}_{\mathrm{tr}}(V_\bullet) \cong S^\infty(V_\bullet)$ and $R_{\mathrm{tr}}(K_n) \cong \oplus_{m=1}^\infty S^m_{\mathrm{tr}}(\mathbb{L}^n)$.

(c) The pointed space $K_1 = u\mathbb{L}^1$ represents $H^{2,1}(-, \mathbb{Z})$, where $\mathbb{L}^1 = \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1/\mathbb{A}^1 - 0)$. Since $S^m_{\mathrm{tr}}(\mathbb{L}^1) \cong \mathbb{L}^m$ by 4.3(b), Proposition 4.4 yields:

$$R_{\mathrm{tr}}(K_1) \simeq R_{\mathrm{tr}} S^\infty(\mathbb{A}^1/\mathbb{A}^1 - 0) \simeq \oplus \mathbb{L}^m.$$

So cohomology operations $H^{2,1}(X, \mathbb{Z}) \to H^{p,q}(X, R)$ are classified by the elements of

$$H^{p,q}(K_1, R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}} K_1, R(q)[p]) \cong \prod_{m=1}^{\infty} H^{p-2m,q-m}(k, R).$$

These correspond to homogeneous polynomials $f(t) = \Sigma a_i t^i$ of bidegree (p, q) in $H^{**}(k, R)[t]$ with $a_0 = 0$ and bidegree$(t) = (2, 1)$, as described in Example 2.1(b). The operations $x \mapsto f(x)$ are nontrivial on $x \in H^{2,1}(\mathbb{P}^N, R)$ for large N, since $H^{*,*}(\mathbb{P}^N, R) = H^{*,*}(k, R)[x]/(x^{N+1})$; see [4, 15.5]

Now suppose that R is either $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$, so that $(\ell - 1)!$ is a unit of R. If $m < \ell$, the symmetrizing idempotent $e = (\Sigma\sigma)/m!$ of $R\,[\Sigma_m]$ acts on $R_{\mathrm{tr}}(X^m)$ and it is easy to see that the canonical map $R_{\mathrm{tr}}(X^m) \to S^m_{\mathrm{tr}}(R_{\mathrm{tr}} X) = R_{\mathrm{tr}}(S^m X)$ induces an isomorphism

$$S^m_{\mathrm{tr}}(R_{\mathrm{tr}} X) \cong e \cdot R_{\mathrm{tr}}(X^m), \qquad m < \ell. \tag{4.7}$$

Example 4.7.1. Fix $m < \ell$. If the interchange τ on $T \otimes T$ is equivalent to the identity (e.g., $T = \mathbb{L}^a[2b]$); then $S^m_{\mathrm{tr}}(T) \cong T^{\otimes m}$. If $\tau \simeq -1$ (e.g., $T = \mathbb{L}^a[2b+1]$), then $S^m_{\mathrm{tr}}(T) \cong 0$.

We will now describe $S^m_{\mathrm{tr}}(M)$ in terms of S^ℓ_{tr}. If G is any subgroup of Σ_m, the wreath product

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

acts on $\{1, \dots, mn\}$ by decomposing it into n blocks of m elements, with G acting on the blocks and Σ_n permuting the blocks. Thus $G \wr \Sigma_n \subset \Sigma_{mn}$. It is easy to see that

$$S^n(S^G(X_+)) = S^{G \wr \Sigma_n}(X_+).$$

Similarly, if H is a subgroup of Σ_n and we embed $\Sigma_m \times \Sigma_n$ in Σ_{m+n} then $S^{G\times H}(X_+) = S^G(X_+) \times S^H(X_+)$ and $S^{G\times H}_{\mathrm{tr}}(R_{\mathrm{tr}}X) = S^G_{\mathrm{tr}}(R_{\mathrm{tr}}X) \otimes S^H_{\mathrm{tr}}(R_{\mathrm{tr}}X)$.

Proposition 4.8. *If $m = m_0 + m_1\ell + \cdots + m_r\ell^r$ with $0 \le m_i < \ell$, the subgroup*

$$G = \Sigma_{m_0} \times (\Sigma_\ell \wr \Sigma_{m_1}) \times ((\Sigma_\ell \wr \Sigma_\ell) \wr \Sigma_{m_2}) \cdots \times ((\Sigma_\ell{}^{\wr r}) \wr \Sigma_{m_r})$$

of Σ_m contains a Sylow ℓ-subgroup of Σ_m. If $R = \mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$ then for every simplicial V and $M = R_{tr}(V)$, $S^m_{tr}(M)$ is a direct summand of

$$S^G_{tr}(M) = (S^{m_0}_{tr}M) \otimes S^{m_1}_{tr}(S^\ell_{tr}M) \otimes S^{m_2}_{tr}(S^\ell_{tr}(S^\ell_{tr}M)) \otimes \cdots \otimes S^{m_r}_{tr}((S^\ell_{tr})^r M).$$

Proof. (Voevodsky, [V07]) The display is $S^G_{\mathrm{tr}}(M)$ by the above remarks, and the map π from $S^G(V) = V^m/G$ to $S^mV = V^m/\Sigma_m$ is finite of degree $d = [\Sigma_m \colon G]$. It is well known (and easy to check) that G contains a Sylow ℓ-subgroup of Σ_m, so $\ell \nmid d$. The transpose π^t is a finite correspondence, and the composition $\pi \circ \pi^t$ is multiplication by d on $R_{\mathrm{tr}}(\widetilde{S}^mV) = S^m_{\mathrm{tr}}(M)$. ☐

Theorem 4.9. *When $R = \mathbb{Z}/\ell$, $S^\ell_{tr}(\mathbb{L}^n)$ is $\mathbb{A}^1$-equivalent to*

$$\mathbb{L}^{n\ell} \oplus \bigoplus_{i=1}^{n-1} \left\{ \mathbb{L}^{n+i(\ell-1)} \oplus \mathbb{L}^{n+i(\ell-1)}[1] \right\}.$$

Proof. (Sketch) Let C be the cyclic group of order ℓ and $G = C \rtimes (\mathbb{Z}/\ell)^\times \subseteq \Sigma_\ell$. Using the methods of [RPO], Voevodsky [V07] computes $R_{\rm tr}(V-0)/C$, where V is the direct sum of n copies of the reduced regular representation $\mathbb{A}^{\ell-1}$ of C. Next, he observes that $S^C_{\rm tr}\mathbb{L}^n$ is $\mathbb{A}^1$-equivalent to $\mathbb{L}^n \otimes R_{\rm tr}(V-0)/C[1]$. Taking $(\mathbb{Z}/\ell)^\times$-invariants, it follows that $S^G_{\rm tr}(\mathbb{L}^n)$ is $\mathbb{A}^1$-equivalent to the motive displayed in 4.9. Since $[\Sigma_\ell\colon G] = (\ell-2)!$, $S^\ell_{\rm tr}(\mathbb{L}^n)$ is a summand. Using the computation of $B\mu_\ell$ and $B\Sigma_\ell$ in [RPO], one shows that each summand of $S^G_{\rm tr}(\mathbb{L}^n)$ belongs to $S^\ell_{\rm tr}(\mathbb{L}^n)$. □

Corollary 4.10. *When $R = \mathbb{Z}/\ell$ and $a > 0$, $S^\ell_{tr}(\mathbb{L}^a[b])[1] \to S^\ell_{tr}(\mathbb{L}^a[b+1])$ is a split injection for all b, and we have:*

$$S^\ell_{tr}(\mathbb{L}^a[1]) = \bigoplus_{i=1}^{a} \{\mathbb{L}^{a+i(\ell-1)}[1] \oplus \mathbb{L}^{a+i(\ell-1)}[2]\};$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[1])[b-1] \oplus \bigoplus_{i=1}^{k} \{\mathbb{L}^{a\ell}[2i\ell+1] \oplus \mathbb{L}^{a\ell}[2i\ell+2]\}, \quad b = 2k+1;$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[b-1])[1] \oplus \mathbb{L}^{a\ell}[b\ell], \quad b \geq 2 \text{ even.}$$

Proof. Set $T = \mathbb{L}^a[b]$. Voevodsky shows in [V07] that the cone of $(S^\ell_{\rm tr}T)[1] \to S^\ell_{\rm tr}(T[1])$ is : $T^{\otimes\ell}[2]$ for b even, and $T^{\otimes\ell}[\ell]$ for b odd. In the odd case, the boundary map is zero for weight reasons. In the even case, the boundary map is an element of $\operatorname{Hom}(T^{\otimes\ell}, S^\ell_{\rm tr}T) = \mathbb{Z}/\ell$. Using the topological realization functor, the topological calculations of Cartan [1] show that the boundary map is also zero. The result now follows by induction on b. □

Remark 4.10.1. The above formulas are incorrect for $a = 0$, where $\mathbb{L}^0 = R$; here we have $S^\ell_{tr}(R[1]) = 0$, and $S^\ell_{tr}(R[2]) \cong R[2\ell]$.

A *proper Tate motive* is a direct sum of motives of the form $\mathbb{L}^a[b]$ with $b \geq 0$. The category of proper Tate motives over a field R is idempotent complete, and closed in **DM** under $\otimes$.

Theorem 4.11. *When $R = \mathbb{Z}/\ell$, $S^\infty_{tr}(\mathbb{L}^n)$ is a proper Tate motive. For each a there are only finitely many terms of weight a.*

Proof. Combining 4.4, 4.8, 4.9 and 4.10 yields the theorem. ☐

Proposition 4.12. (Pure Künneth formula) *Let X and Y be pointed simplicial schemes such that $R_{tr}(Y)$ is a direct sum of motives $R(q_\alpha)[p_\alpha]$. Assume that for each q there are only finitely many α with $q_\alpha = q$. Then the Künneth homomorphism is an isomorphism:*

$$H^{**}(X, R) \otimes_{H^{**}(k,R)} H^{**}(Y, R) \to H^{**}(X \times Y, R).$$

Proof. By (2.3), $H^{n,i}(X \times Y, R) = \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(X \times Y), R(i)[n])$. Now $R_{\mathrm{tr}}(X \times Y)$ is the direct sum of the $R^{\mathrm{tr}}(X)(q_\alpha)[p_\alpha]$, and we claim that

$$\mathrm{Hom}(R_{\mathrm{tr}}(X)(q)[p], R(i)[n]) = \begin{cases} H^{n-p,i-q}(X, R) & \text{if } q \le i; \\ 0 & \text{if } q > i. \end{cases}$$

The case $X = \mathrm{Spec}(k)$ shows that $H^{**}(Y, R)$ is a free $H^{**}(k, R)$-module on finitely many generators γ_α in bidegrees (p_α, q_α), and the result follows.

To verify the claim, we may suppose that $p = 0$. Suppose first that $q \le i$. By the Cancellation Theorem [4, 16.25] we have $\mathrm{Hom}(M(q), R(i)) = \mathrm{Hom}(M, R(i - q))$ for any M in **DM**. In particular, $\mathrm{Hom}(R_{\mathrm{tr}}(X)(q), R(i)[n]) = \mathrm{Hom}(R_{\mathrm{tr}}(X), R(i-q)[n]) = H^{n,i-q}(X, R)$. Similarly, the case when $q > i$ reduces to the case $i = 0$, $q > 0$. Here $R_{\mathrm{tr}}(X)(q)$ is a summand of $R_{\mathrm{tr}}(X \times \mathbb{P}^q)$ and $H^{p,0}(-, R) = H^p_{\mathrm{Zar}}(-, R)$, so the result follows from $H^*_{\mathrm{Zar}}(X, R) \cong H^*_{\mathrm{Zar}}(X \times \mathbb{P}^q, R)$; see [RPO, 3.5]. ☐

Recall from 2.5 that $K_n = u\mathbb{L}^n$ represents $H^{2n,n}(-, \mathbb{Z})$, and that $\mathrm{char}(k) = 0$.

Corollary 4.13. *For all $n > 0$ the Künneth maps are isomorphisms:*

$$H^{**}(K_n, \mathbb{Z}/\ell) \otimes_{H^{**}} \cdots \otimes_{H^{**}} H^{**}(K_n, \mathbb{Z}/\ell) \xrightarrow{\simeq} H^{**}(K_n \times \cdots \times K_n, \mathbb{Z}/\ell).$$

This replaces the unproven "Lemma 2.3" in [MC/l]. Note that 4.13 is equivalent to: $\tilde{H}^{**}(K_n, \mathbb{Z}/\ell) \otimes_{H^{**}} \cdots \otimes_{H^{**}} \tilde{H}^{**}(K_n, \mathbb{Z}/\ell) \xrightarrow{\simeq} \tilde{H}^{**}(K_n \wedge \cdots \wedge K_n, \mathbb{Z}/\ell)$.