Lecture 4. Universal K-functors.

1. Preliminaries: left exact categories of right exact 'spaces'.

We start with left exact stuctures formed by localizations of 'spaces' represented by svelte categories. Then the obtained facts are used to define natural left exact structures on the category of 'spaces' represented by right exact categories.

The following proposition is a refinement of [R3, 1.4.1].

1.1. Proposition. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q (i.e. its inverse image functor $C_Y \xrightarrow{q^*} C_X$) is a localization. Then*

(a) The canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ is a localization.

(b) If q is a continuous localization, then $\widetilde{q}$ is a continuous localization.

(c) If $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system, then $\Sigma_{\widetilde{q}^*}$ has the same property.

1.2. Corollary. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q is a localization, and let $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ be a canonical morphism. Suppose the category C_Y has finite limits (resp. finite colimits). Then $\widetilde{q}^*$ is a left (resp. right) exact localization, if the localization q^* is left (resp. right) exact.*

Proof. By 1.1(a), $\widetilde{q}^*$ is a localization functor.

Suppose that the category C_Y has finite limits and the localization functor $C_Y \xrightarrow{q^*} C_X$ is left exact. Then it follows from [GZ, I.3.4] that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a right multiplicative system. The latter implies, by 1.1(c), that $\Sigma_{\widetilde{q}^*}$ is a right multiplicative system. Therefore, by [GZ, I.3.1], the localization functor $\widetilde{q}^*$ is left exact. ∎

The following assertion is a refinement of [R3, 1.4.2].

1.3. Proposition. *Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be morphisms of 'spaces' such that p^* and q^* are localization functors. Then the square*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{q_1} & X \coprod_{p,q} Y \end{array}$$

is cartesian.

1.4. Left exact structures on the category of 'spaces'. Let $\mathfrak{L}$ denote the class of all localizations of 'spaces' (i.e. morphisms whose inverse image functors are localizations). We denote by $\mathfrak{L}_\ell$ (resp. $\mathfrak{L}_\mathfrak{r}$) the class of localizations $X \xrightarrow{q} Y$ of 'spaces' such that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{L}_{\mathfrak{e}}$ the intersection of $\mathfrak{L}_\ell$ and $\mathfrak{L}_{\mathfrak{r}}$ (i.e. the class of localizations q such that Σ_{q^*} is a multiplicative system) and by $\mathfrak{L}^{\mathfrak{c}}$ the class of continuous (i.e. having a direct image functor) localizations of 'spaces'. Finally, we set $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}} = \mathfrak{L}^{\mathfrak{c}} \cap \mathfrak{L}_{\mathfrak{e}}$; i.e. $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ is the class of continuous localizations $X \xrightarrow{q} Y$ such that Σ_{q^*} is a multiplicative system.

1.4.1. Proposition. *Each of the classes of morphisms* $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_{\mathfrak{r}}$, $\mathfrak{L}_{\mathfrak{e}}$, $\mathfrak{L}^{\mathfrak{c}}$, *and* $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ *are structures of a left exact category on the category* $|Cat|^o$ *of 'spaces'.*

Proof. It is immediate that each of these classes is closed under composition and contains all isomorphisms of the category $|Cat|^o$. It follows from 1.1 that each of the classes is stable under cobase change. In other words, the arrows of each class can be regarded as cocovers of a copretopology. It remains to show that these copretopologies are subcanonical. Since $\mathfrak{L}$ is the finest copretopology, it suffices to show that $\mathfrak{L}$ is subcanonical.

The copretopology $\mathfrak{L}$ being subcanonical means precisely that for any localization $X \xrightarrow{q} Y$, the square

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ q \downarrow & & \downarrow q_1 \\ Y & \xrightarrow{q_2} & Y \coprod_{q,q} Y \end{array}$$

is cartesian. But, this follows from 1.3. ∎

1.5. Observation. Each object of the left exact category $(|Cat|^o, \mathfrak{L}^{\mathfrak{c}})$ is injective.

In fact, a 'space' X is an injective object of $(|Cat|^o, \mathfrak{L}^{\mathfrak{c}})$ iff each morphism $X \xrightarrow{q} Y$ is split; i.e. there is a morphism $Y \xrightarrow{\mathfrak{t}} X$ such that $\mathfrak{t} \circ q = id_X$. Since the morphism q is continuous, it has a direct image functor, q_*, which is fully faithful, because q^* is a localization functor. The latter means precisely that the adjunction arrow $q^*q_* \longrightarrow Id_{C_X}$ is an isomorphism. Therefore, the morphism $Y \xrightarrow{\mathfrak{t}} X$ whose inverse image functor coincides with q_* satisfies the equality $\mathfrak{t} \circ q = id_X$.

1.6. Left exact structures on the category of right (or left) exact 'spaces'. A *right exact 'space'* is a pair $(X, \mathfrak{E}_X)$, where X is a 'space' and $\mathfrak{E}_X$ is a right exact structure on the category C_X. We denote by $\mathfrak{Esp}_{\mathfrak{r}}$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ and morphisms from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ are given by morphisms $X \xrightarrow{f} Y$ of 'spaces' whose inverse image functor, f^*, is *'exact'*; i.e. f^* maps deflations to deflations and preserves pull-backs of deflations.

Dually, a *left exact 'space'* is a pair $(Y, \mathfrak{I}_Y)$, where $(C_Y, \mathfrak{I}_Y)$ is a left exact category. We denote by $\mathfrak{Esp}_\ell$ the category whose objects are left exact 'spaces' $(Y, \mathfrak{I}_Y)$ and morphisms $(Y, \mathfrak{I}_Y) \longrightarrow (Z, \mathfrak{I}_Z)$ are given by morphisms $Y \longrightarrow Z$ whose inverse image functors are *'coexact'*, which means that they preserve inflations their push-forwards.

1.6.1. Note. The categories $\mathfrak{Esp}_{\mathfrak{r}}$ and $\mathfrak{Esp}_\ell$ are naturally isomorphic to each other: the isomorphism is given by the dualization functor $(X, \mathfrak{E}_X) \longmapsto (X^o, \mathfrak{E}^{op}_X)$. Therefore, every assertion about the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' translates into an assertion about the category $\mathfrak{Esp}_\ell$ of left exact 'spaces' and vice versa.

1.6.2. Proposition. *The category* $\mathfrak{Esp}_{\mathfrak{r}}$ *has fibered coproducts.*

1.6.3. Canonical left exact structures on the category $\mathfrak{Esp}_\mathfrak{r}$. Let $\mathfrak{L}_{\mathfrak{es}}$ denote the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that q^* is a localization functor and each arrow of $\mathfrak{E}_X$ is isomorphic to an arrow $q^*(\mathfrak{e})$ for some $\mathfrak{e} \in \mathfrak{E}_Y$.

If Σ_{q^*} is a left or right multiplicative system, then this condition means that $\mathfrak{E}_X$ is the smallest right exact structure containing $q^*(\mathfrak{E}_Y)$.

1.6.3.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces'.*

1.6.3.2. Corollary. *Each of the classes of morphisms of 'spaces' $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ (cf. 1.4, 1.4.1) induces a structure of a left exact category on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_\ell$ induces the class $\mathfrak{L}^{\mathfrak{es}}_\ell$ of morphisms of the category $\mathfrak{Esp}_\mathfrak{r}$ formed by all arrows $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ from $\mathfrak{L}_{\mathfrak{es}}$ such that the morphism of 'spaces' $X \xrightarrow{q} Y$ belongs to $\mathfrak{L}_\ell$. Similarly, we define the classes $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^\mathfrak{c}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e};\mathfrak{c}}_{\mathfrak{es}}$. ■

1.6.3.3. The left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$. For a right exact 'space' $(X, \mathfrak{E}_X)$, let $Sq(X, \mathfrak{E}_X)$ denote the class of all cartesian squares in the category C_X some of the arrows of which (at least two) belong to $\mathfrak{E}_X$.

The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that its inverse image functor, $\mathfrak{q}^*$, is equivalent to a localization functor and each square of $Sq(X, \mathfrak{E}_X)$ is isomorphic to some square of $\mathfrak{q}^*(Sq(Y, \mathfrak{E}_Y))$.

1.6.3.4. Proposition. *The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' which is coarser than $\mathfrak{L}_{\mathfrak{es}}$ and finer than $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$.*

Proof. The argument is left to the reader. ■

1.7. Relative right exact 'spaces'. The category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' has initial objects and no final object. Final objects appear if we fix a right exact 'space' $\mathcal{S} = (S, \mathfrak{E}_S)$ and consider the category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$ instead of $\mathfrak{Esp}_\mathfrak{r}$. The category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$ has a natural final object and cokernels of all morphisms. It also inherits left exact structures from $\mathfrak{Esp}_\mathfrak{r}$, in particular those defined above (see 1.6.3.2). Therefore, our theory of derived functors (satellites) can be applied to functors from $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$.

1.8. The category of right exact k-'spaces'. For a commutative unital ring k, we denote by $\mathfrak{Esp}^\mathfrak{r}_k$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X is a k-linear additive category and morphisms are morphisms of right exact 'spaces' whose inverse image functors are k-linear.

Each of the left exact structures $\mathfrak{L}_{\mathfrak{es}}$, $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^\mathfrak{c}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e};\mathfrak{c}}_{\mathfrak{es}}$ induces a left exact structure on the category $\mathfrak{Esp}^\mathfrak{r}_k$ of right exact k-'spaces'. We denote them by respectively $\mathfrak{L}_{\mathfrak{es}}(k)$, $\mathfrak{L}^{\mathfrak{es}}_\ell(k)$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}(k)$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}(k)$, $\mathfrak{L}^\mathfrak{c}_{\mathfrak{es}}(k)$, and $\mathfrak{L}^{\mathfrak{e};\mathfrak{c}}_{\mathfrak{es}}(k)$.

2. The group K_0 of a right (or left) exact 'space'.

2.1. The group $\mathbb{Z}_0|C_X|$. For a svelte category C_X, we denote by $|C_X|$ the set of isomorphism classes of objects of C_X, by $\mathbb{Z}|C_X|$ the free abelian group generated by $|C_X|$, and by $\mathbb{Z}_0(C_X)$ the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M] - [N]$ for all arrows $M \longrightarrow N$ of the category C_X. Here $[M]$ denotes the isomorphism class of an object M.

2.2. Proposition. *(a) The maps* $X \longmapsto \mathbb{Z}|C_X|$ *and* $X \longmapsto \mathbb{Z}_0(C_X)$ *extend naturally to presheaves of* $\mathbb{Z}$*-modules on the category of 'spaces'* $|Cat|^o$ *(i.e. to functors from* $(|Cat|^o)^{op}$ *to* $\mathbb{Z} - mod$*).*

(b) If the category C_X *has an initial (resp. final) object* x*, then* $\mathbb{Z}_0(C_X)$ *is the subgroup of* $\mathbb{Z}|C_X|$ *generated by differences* $[M] - [x]$*, where* $[M]$ *runs through the set* $|C_X|$ *of isomorphism classes of objects of* C_X*.*

Proof. The argument is left to the reader. ■

2.3. Note. Evidently, $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$ and $\mathbb{Z}_0(C_X) \simeq \mathbb{Z}_0(C_X^{op})$.

2.4. The group K_0 of a right exact 'space'. Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. We denote by $K_0(X, \mathfrak{E}_X)$ the quotient of the group $\mathbb{Z}_0|C_X|$ by the subgroup generated by the expressions $[M'] - [L'] + [L] - [M]$ for all cartesian squares

$$\begin{array}{ccc} M' & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ L' & \xrightarrow{f} & L \end{array}$$

whose vertical arrows are deflations.

We call $K_0(X, \mathfrak{E}_X)$ the *group* K_0 of the right exact 'space' $(X, \mathfrak{E}_X)$.

2.4.1. Example: the group K_0 of a 'space'. Any 'space' X is identified with the *trivial* right exact 'space' $(X, Iso(C_X))$. We set $K_0(X) = K_0(X, Iso(C_X))$. That is $K_0(X)$ coincides with the group $\mathbb{Z}_0(C_X)$.

2.4.2. Proposition. *Let* $(X, \mathfrak{E}_X)$ *be a right exact 'space' such that the category* C_X *has initial objects. Then* $K_0(X, \mathfrak{E}_X)$ *is isomorphic to the quotient of the group* $\mathbb{Z}_0(X)$ *by the subgroup generated by the expressions* $[M] - [L] - [N]$ *for all conflations* $N \longrightarrow M \longrightarrow L$*.*

2.5. Proposition. *(a) The map* $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ *extends to a contravariant functor,* K_0*, from the category* $\mathfrak{Esp}_\mathfrak{r}$ *of right exact 'spaces' (cf. 6.8) to the category* $\mathbb{Z}-mod$ *of abelian groups.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ *be a morphism of* $\mathfrak{Esp}_\mathfrak{r}$ *having the following property:*

(†) *if* M' *and* L' *are non-isomorphic objects of* C_X *which can be connected by non-oriented sequence of arrows (i.e. they belong to one connected component of the associated groupoid), then there exist objects* M *and* L *of* C_Y *which have the same property and such that* $f^*(M) \simeq M'$*,* $f^*(L) \simeq L'$*.*

Then $K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$ *is a group epimorphism.*

In particular, the functor K_0 *maps 'exact' localizations to epimorphisms.*

3. Higher K-groups of right exact 'spaces'.

3.1. The relative functors K_0 and their derived functors. Fix a right exact 'space' $\mathcal{Y} = (Y, \mathfrak{E}_\mathcal{Y})$. The functor $(\mathfrak{Esp}_\mathfrak{r})^{op} \xrightarrow{K_0} \mathbb{Z} - mod$ induces a functor

$$(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y})^{op} \xrightarrow{K_0^\mathcal{Y}} \mathbb{Z} - mod$$

defined by

$$K_0^{\mathcal{Y}}(\mathcal{X},\xi) = K_0^{\mathcal{Y}}(\mathcal{X},\mathcal{X}\xrightarrow{\xi}\mathcal{Y}) = Cok(K_0(\mathcal{Y})\xrightarrow{K_0(\xi)}K_0(\mathcal{X}))$$

and acting correspondingly on morphisms.

The main advantage of the functor $K_0^{\mathcal{Y}}$ is that its domain, the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ has a final object, cokernels of morphisms, and natural left exact structures induced by left exact structures on $\mathfrak{Esp}_{\mathfrak{r}}$. Fix a left exact structure $\mathfrak{I}$ on $\mathfrak{Esp}_{\mathfrak{r}}$ (say, one of those defined in 6.8.3.2) and denote by $\mathfrak{I}_{\mathcal{Y}}$ the left exact structure on $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ induced by $\mathfrak{I}$. Notice that, since the category $\mathbb{Z}-mod$ is complete (and cocomplete), there is a well defined satellite endofunctor of $\mathcal{H}om((\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y})^{op},\mathbb{Z}-mod)$, $F\longmapsto\mathcal{S}_{\mathfrak{I}_{\mathcal{Y}}}F$. So that for every functor F from $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y})^{op}$ to $\mathbb{Z}-mod$, there is a unique up to isomorphism universal ∂^*-functor $(\mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}}F,\mathfrak{d}_i \mid i\geq 0)$.

In particular, there is a universal contravariant ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}} = (K_i^{\mathcal{Y},\mathfrak{I}},\mathfrak{d}_i \mid i\geq 0)$ from the right exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y},\mathfrak{I}_{\mathcal{Y}})$ of right exact 'spaces' over $\mathcal{Y}$ to the category $\mathbb{Z}-mod$ of abelian groups; that is $K_i^{\mathcal{Y},\mathfrak{I}} = \mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}}K_0^{\mathcal{Y},\mathfrak{I}}$ for all $i\geq 0$.

We call the groups $K_i^{\mathcal{Y},\mathfrak{I}}(\mathcal{X},\xi)$ *universal* K-groups of the right exact 'space' $(\mathcal{X},\xi)$ over $\mathcal{Y}$ with respect to the left exact structure $\mathfrak{I}$.

3.2. 'Exactness' properties. In general, the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is not 'exact'. The purpose of this section is to find some natural left exact structures $\mathfrak{I}$ on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ of right exact 'spaces' over $\mathcal{Y}$ and its subcategory $\mathfrak{Esp}^*_{\mathfrak{r}}/\mathcal{Y}$ (cf. 7.1.7) for which the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is 'exact'.

3.2.1. Proposition. *Let $(X,\xi)\xrightarrow{q}(X',\xi')$ be a morphism of the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ such that $X\xrightarrow{q}X'$ belongs to $\mathfrak{L}_{\mathfrak{es}}$ (cf. 6.8.3) and has the following property:*

(#) if $M\xrightarrow{\mathfrak{s}}L$ is a morphism of $C_{X'}$ such that $q^*(\mathfrak{s})$ is invertible, then the element $[M]-[L]$ of the group $K_0(X')$ belongs to the image of the map $K_0(X'')\xrightarrow{K_0(\mathfrak{c}_q)}K_0(X')$, where $(X',\xi')\xrightarrow{\mathfrak{c}_q}(X'',\xi'')$ is the cokernel of the morphism $(X,\xi)\xrightarrow{q}(X',\xi')$.

Suppose, in addition, that one of the following two conditions holds:

(i) the category $C_{X'}$ has an initial object;

(ii) for any pair of arrows $N\xrightarrow{f}L\xleftarrow{\mathfrak{s}}M$, of the category $C_{X'}$ such that $\mathfrak{q}^*(\mathfrak{s})$ is invertible, there exists a commutative square

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle\mathfrak{t}}\downarrow & & \downarrow{\scriptstyle\mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array}$$

such that $\mathfrak{q}^*(\mathfrak{t})$ is invertible.

Then for every conflation $(X,\xi)\xrightarrow{\mathfrak{q}}(X',\xi')\xrightarrow{\mathfrak{c}_\mathfrak{q}}(X'',\xi'')$ of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y},\mathfrak{I}_{\mathcal{Y}})$ the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'')\xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_\mathfrak{q})}K_0^{\mathcal{Y}}(X',\xi')\xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})}K_0^{\mathcal{Y}}(X,\xi)\longrightarrow 0$$

of morphisms of abelian groups is exact.

3.2.2. Proposition. *The class* $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$ *of all morphisms* $(X,\xi) \xrightarrow{q} (X',\xi')$ *of* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ *such that* $X \xrightarrow{q} X'$ *belongs to* $\mathfrak{L}_{\mathfrak{es}}$ *and satisfies the condition (#) of 3.2.1, is a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$*.*

3.2.2.1. Proposition. *The class* $\mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}}$ *of all morphisms* $(X,\xi) \xrightarrow{q} (X',\xi')$ *og* $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$ *such that the functor* $C_{X'} \xrightarrow{q^*} C_X$ *satisfies the condition (ii) of 3.2.1, is a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$*.*

3.2.3. Proposition. *Let* $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ *be a right exact 'space', and let* $\mathfrak{I}$ *be a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ *which is coarser than* $\mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}}$ *(cf. 3.2.2). Then the universal* ∂^**-functor* $K_\bullet^{\mathcal{Y}} = (K_i^{\mathcal{Y}}, \mathfrak{d}_i \mid i \geq 0)$ *from the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ *to the category* $\mathbb{Z}-$*mod of abelian groups is 'exact'; i.e. for any conflation* $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} (X'',\xi'')$*, the associated long sequence*

$$\ldots \xrightarrow{K_1^{\mathcal{Y}}(\mathfrak{q})} K_1^{\mathcal{Y}}(X,\xi) \xrightarrow{\mathfrak{d}_0} K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_{\mathfrak{q}})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

is exact.

Proof. Since the left exact structure $\mathfrak{I}_{\mathcal{Y}}$ is coarser than $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$, it satisfies the condition (#) of 3.2.1. Therefore, by 3.2.1, for any conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} (X'',\xi'')$ of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$, the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_{\mathfrak{q}})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact. Therefore, by [Lecture III, 3.5.4.1], the universal ∂^*-functor $K_\bullet^{\mathcal{Y}} = (K_i^{\mathcal{Y}}, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})^{op}$ to $\mathbb{Z} - mod$ is 'exact'. ■

The following proposition can be regarded as a machine for producing universal 'exact' K-functors.

3.2.4. Proposition. *Let* $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ *be a right exact 'space',* $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ *a left exact category with final objects, and* $\mathfrak{F}$ *a functor* $C_{\mathfrak{S}} \longrightarrow \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ *which maps conflations of* $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ *to conflations of the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}})$*. Then there exists a (unique up to isomorphism) universal* ∂^**-functor* $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \mathfrak{d}_i \mid i \geq 0)$ *from the right exact category* $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ *to* $\mathbb{Z} - mod$ *whose zero component,* $K_0^{\mathfrak{S},\mathfrak{F}}$*, is the composition of the functor* $\mathcal{C}_{\mathfrak{S}}^{op} \xrightarrow{\mathfrak{F}^{op}} \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}^{op}$ *and the functor* $K_0^{\mathcal{Y}}$*.*

The ∂^**-functor* $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ *is 'exact'.*

Proof. The existence of the ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ follows, by [Lecture III, 3.3.2], from the completeness (– existence of limits of small diagrams) of the category $\mathbb{Z} - mod$ of abelian groups. The main thrust of the proposition is the 'exactness' of $K_\bullet^{\mathfrak{S},\mathfrak{F}}$.

By hypothesis, the functor $\mathfrak{F}$ maps conflations to conflations. Therefore, it follows from 3.2.1 that for any conflation $\mathfrak{X} \longrightarrow \mathfrak{X}' \longrightarrow \mathfrak{X}''$ of the left exact category $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$, the sequence of abelian groups $K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}'') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}) \longrightarrow 0$ is exact. By [Lecture III, 3.5.4.1], this implies the 'exactness' of the ∂^*-functor $K_\bullet^{\mathcal{Y}}$. ∎

3.3. The 'absolute' case. Let $|Cat_*|^o$ denote the subcategory of the category $|Cat|^o$ of 'spaces' whose objects are 'spaces' represented by categories with initial objects and morphisms are those morphisms of 'spaces' whose inverse image functors map initial objects to initial objects. The category $|Cat_*|^o$ is pointed: it has a canonical zero (that is both initial and final) object, x, which is represented by the category with one (identical) morphism. Thus, the initial objects of the category $|Cat|^o$ of all 'spaces' are zero objects of the subcategory $|Cat_*|^o$.

Each morphism $X \xrightarrow{\mathfrak{f}} Y$ of the category $|Cat_*|^o$ has a cokernel, $Y \xrightarrow{\mathfrak{c}_\mathfrak{f}} \mathcal{C}(\mathfrak{f})$, where the category $C_{\mathcal{C}(\mathfrak{f})}$ representing the 'space' $\mathcal{C}(\mathfrak{f})$ is the kernel $Ker(\mathfrak{f}^*)$ of the functor $\mathfrak{f}^*$. By definition, $Ker(\mathfrak{f}^*)$ is the full subcategory of the category C_Y generated by all objects of C_Y which the functor $\mathfrak{f}^*$ maps to initial objects. The inverse image functor $\mathfrak{c}_\mathfrak{f}^*$ of the canonical morphism $\mathfrak{c}_\mathfrak{f}$ is the natural embedding $Ker(\mathfrak{f}^*) \longrightarrow C_Y$.

Let $\mathfrak{Esp}_\mathfrak{r}^*$ denote the category formed by right exact 'spaces' with initial objects and those morphisms of right exact 'spaces' whose inverse image functor is 'exact' and maps initial objects to initial objects. The category $\mathfrak{Esp}_\mathfrak{r}^*$ is pointed and the forgetfull functor

$$\mathfrak{Esp}_\mathfrak{r}^* \xrightarrow{\ \mathfrak{J}^*\ } |Cat_*|^o, \quad (X, \mathfrak{E}_X) \longmapsto X,$$

is a left adjoint to the canonical full embedding $|Cat_*|^o \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}_\mathfrak{r}^*$ which assigns to every 'space' X the right exact category $(X, Iso(C_X))$. Both functors, $\mathfrak{J}^*$ and $\mathfrak{J}_*$, map zero objects to zero objects.

Let x be a zero object of the category $\mathfrak{Esp}_\mathfrak{r}^*$. Then $\mathfrak{Esp}_\mathfrak{r}^*/x$ is naturally isomorphic to $\mathfrak{Esp}_\mathfrak{r}^*$ and the relative K_0-functor K_0^x coincides with the functor K_0.

3.3.1. The left exact structure $\mathfrak{L}_{\mathfrak{es}}^*$. We denote by $\mathfrak{L}_{\mathfrak{es}}^*$ the canonical left exact structure $\mathfrak{L}_{\mathfrak{es}}^x$; it does not depend on the choice of the zero object x. It follows from the definitions above that $\mathfrak{L}_{\mathfrak{es}}^*$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ having the following properties:

(a) $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is a localization functor (which is 'exact'), and every arrow of $\mathfrak{E}_X$ is isomorphic to an arrow of $\mathfrak{q}^*(\mathfrak{E}_Y)$.

(b) If $M \xrightarrow{s} M'$ is an arrow of C_Y such that $\mathfrak{q}^*(s)$ is an isomorphism, then $[M]-[M']$ is an element of $KerK_0(\mathfrak{q})|$.

3.3.2. Proposition. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, and $C_\mathfrak{S} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_\mathfrak{r}^*$ a functor which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{L}_{\mathfrak{es}}^*)$. Then there exists a (unique up to isomorphism) universal ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to $\mathbb{Z}-mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor $\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{r}^*)^{op}$ and the functor K_0.*

The ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'. In particular, the ∂^*-functor $K_\bullet = (K_i, \mathfrak{d}_i| \ i \geq 0)$ from $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{L}_{\mathfrak{es}}^*)$ to $\mathbb{Z}-mod$ is 'exact'.

Proof. The assertion is a special case of 3.2.4. ∎

3.4. Universal K-theory of abelian categories. Let $\mathfrak{Esp}_k^{\mathfrak{a}}$ denote the category whose objects are 'spaces' X represented by k-linear abelian categories and morphisms $X \xrightarrow{f} Y$ are represented by k-linear exact functors.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^* \qquad (1)$$

which assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the right exact (actually, exact) 'space' $(X, \mathfrak{E}_X^{st})$, where $\mathfrak{E}_X^{st}$ is the *standard* (i.e. the finest) right exact structure on the category C_X, and maps each morphism $X \xrightarrow{f} Y$ to the morphism $(X, \mathfrak{E}_X^{st}) \xrightarrow{f} (Y, \mathfrak{E}_Y^{st})$ of right exact 'spaces'. One can see that the functor $\mathfrak{F}$ maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ (represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

3.4.1. Proposition. *Let C_X and C_Y be k-linear abelian categories endowed with the standard exact structure. Any exact localization functor $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ satisfies the conditions (a) and (b) of 3.3.1.*

Proof. In fact, each morphism $\mathfrak{q}^*(M) \xrightarrow{\widetilde{h}} \mathfrak{q}^*(N)$ is of the form $\mathfrak{q}^*(h)\mathfrak{q}^*(s)^{-1}$ for some morphisms $M' \xrightarrow{h} N$ and $M' \xrightarrow{s} M$ such that $\mathfrak{q}^*(s)$ is invertible. The morphism h is a (unique) composition $\mathfrak{j} \circ \mathfrak{e}$, where $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since the functor $\mathfrak{q}^*$ is exact, $\mathfrak{q}^*(\mathfrak{j})$ is a monomorphism and $\mathfrak{q}^*(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{h}$ is an epimorphism iff $\mathfrak{q}^*(\mathfrak{j})$ is an isomorphism. This shows that the condition (a) holds.

Let $M \xrightarrow{s} M'$ be a morphism and

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{s} M' \longrightarrow Cok(s) \longrightarrow 0$$

the associated with s exact sequence. Representing s as the composition, $\mathfrak{j}\circ\mathfrak{e}$, of a monomorphism $\mathfrak{j}$ and an epimorphism $\mathfrak{e}$, we obtain two short exact sequences,

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{\mathfrak{e}} N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \xrightarrow{\mathfrak{j}} M' \longrightarrow Cok(s) \longrightarrow 0,$$

hence $[M] = [Ker(s)] + [N]$ and $[M'] = [N] + [Cok(s)]$, or $[M'] = [M] + [Ker(s)] - [Cok(s)]$ in $K_0(Y)$. It follows from the exactness of the functor $\mathfrak{q}^*$ that the morphism $\mathfrak{q}^*(s)$ is an isomorphism iff $Ker(s)$ and $Cok(s)$ are objects of the category $Ker(\mathfrak{q}^*)$. Therefore, in this case, it follows that $[M'] = [M]$ modulo $\mathbb{Z}|Ker(\mathfrak{q}^*)|$ in $K_0(Y)$. ∎

3.4.2. Proposition. *(a) The class $\mathfrak{L}^{\mathfrak{a}}$ of all morphisms $X \xrightarrow{q} Y$ of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ such that $C_Y \xrightarrow{q^*} C_X$ is a localization functor, is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{a}}$.*

(b) The functor $\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^*$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{L}_{\mathfrak{es}}^*)$. Moreover, $\mathfrak{L}^{\mathfrak{a}} = \mathfrak{F}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$, that is the left exact structure $\mathfrak{L}^{\mathfrak{a}}$ is induced by the left exact structure $\mathfrak{L}_{\mathfrak{es}}^*$ via the functor $\mathfrak{F}$.

3.4.3. The universal Grothendieck K-functor. The composition $K_0^{\mathfrak{a}}$ of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}^*_{\mathfrak{r}})^{op} \xrightarrow{K_0^*} \mathbb{Z}-mod$ assigns to each object X of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ the abelian group $K_0^*(X, \mathfrak{E}^{st}_X)$. It follows from 2.4.2 that the group $K_0^*(X, \mathfrak{E}^{st}_X)$ coincides with the Grothendieck group of the abelian category C_X. Therefore, we call $K_0^{\mathfrak{a}}$ the *Grothendieck K_0-functor.*

3.4.4. Proposition. *There exists a universal ∂^*-functor $K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z}-mod$ whose zero component is the Grothendieck functor K_0. The universal ∂^*-functor $K_\bullet^{\mathfrak{a}}$ is 'exact'; that is for any exact localization $X \xrightarrow{\mathfrak{q}} X'$, the canonical long sequence*

$$\ldots \xrightarrow{K_1^{\mathfrak{a}}(\mathfrak{q})} K_1^{\mathfrak{a}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X'') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathfrak{a}}(X') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X) \longrightarrow 0 \qquad (3)$$

is exact.

Proof. By 3.4.2(b), the functor $\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ which maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. Therefore, $\mathfrak{F}$ maps conflations to conflations.

The assertion follows now from 3.3.2.1 applied to the functor $\mathfrak{F}$. ∎

3.4.5. The universal ∂^*-functor $K_\bullet^{\mathfrak{a}}$ and the Quillen's K-theory. For a 'space' X represented by a svelte k-linear abelian category C_X, we denote by $K_i^{\mathfrak{Q}}(X)$ the i-th Quillen's K-group of the category C_X. For each $i \geq 0$, the map $X \longmapsto K_i^{\mathfrak{Q}}(X)$ extends naturally to a functor

$$(\mathfrak{Esp}^{\mathfrak{a}}_k)^{op} \xrightarrow{K_i^{\mathfrak{Q}}} \mathbb{Z}-mod$$

It follows from the Quillen's localization theorem [Q, 5.5] that for any exact localization $X \xrightarrow{\mathfrak{q}} X'$ and each $i \geq 0$, there exists a *connecting morphism* $K_{i+1}^{\mathfrak{Q}}(X) \xrightarrow{\mathfrak{d}_i^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X'')$, where $C_{X''} = Ker(\mathfrak{q}^*)$, such that the sequence

$$\ldots \xrightarrow{K_1^{\mathfrak{Q}}(\mathfrak{q})} K_1^{\mathfrak{Q}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X'') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathfrak{Q}}(X') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X) \longrightarrow 0 \qquad (4)$$

is exact. It follows (from the proof of the Quillen's localization theorem) that the connecting morphisms $\mathfrak{d}_i^{\mathfrak{Q}}(\mathfrak{q})$, $i \geq 0$, depend functorially on the localization morphism $\mathfrak{q}$. In other words, $K_\bullet^{\mathfrak{Q}} = (K_i^{\mathfrak{Q}}, \mathfrak{d}_i^{\mathfrak{Q}} \mid i \geq 0)$ is an 'exact' ∂^*-functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z}-mod$ of abelian groups.

Naturally, we call the ∂^*-functor $K_\bullet^{\mathfrak{Q}}$ the *Quillen's K-functor.*

Since $K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ is a universal ∂^*-functor from $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to $\mathbb{Z}-mod$, the identical isomorphism $K_0^{\mathfrak{Q}} \longrightarrow K_0^{\mathfrak{a}}$ extends uniquely to a ∂^*-functor morphism

$$K_\bullet^{\mathfrak{Q}} \xrightarrow{\varphi_\bullet^{\mathfrak{Q}}} K_\bullet^{\mathfrak{a}}. \qquad (5)$$

4. The universal K-theory of exact categories. Let $\mathfrak{Esp}^{\mathfrak{e}}_k$ denote the subcategory of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$ whose objects are 'spaces' represented by svelte exact k-linear categories and inverse image of morphisms are k-linear functors.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^{*} \qquad (1)$$

which maps objects and morphisms of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ to the corresponding objects and morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}^{*}$.

4.1. Proposition. *The functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^{*}$ preserves cocartesian squares and maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ to the zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^{*}$.*

Proof. The argument is similar to that of 7.5.2(b). Details are left to the reader. ■

4.2. Corollary. *The class of morphisms $\mathfrak{L}_k^{\mathfrak{e}} = \mathfrak{F}_{\mathfrak{r}}^{-1}(\mathfrak{L}_{\mathfrak{es}}^{*})$ is a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ and $\mathfrak{F}_r$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}_k^{\mathfrak{e}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^{*}, \mathfrak{L}_{\mathfrak{es}}^{*})$.*

The composition $K_0^{\mathfrak{e}}$ of the inclusion functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^{*})^{op}$$

and the functor $(\mathfrak{Esp}_{\mathfrak{r}}^{*})^{op} \xrightarrow{K_0^{*}} \mathbb{Z}-mod$ assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the abelian group $K_0^{*}(X, \mathfrak{E}_X^{st})$ which coincides with the Quillen's group K_0 of the exact category $(C_X, \mathfrak{E}_X)$.

4.3. Proposition. *There exists a universal ∂^*-functor $K_\bullet^{\mathfrak{e}} = (K_i^{\mathfrak{e}}, \mathfrak{d}_i^{\mathfrak{e}} \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}^{\mathfrak{e}})^{op}$ to the category $\mathbb{Z}-mod$ whose zero component is the functor $K_0^{\mathfrak{e}}$. The universal ∂^*-functor $K_\bullet^{\mathfrak{e}}$ is 'exact'; that is for any exact localization $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ which belongs to $\mathfrak{L}^{\mathfrak{e}}$, the canonical long sequence*

$$\begin{array}{ccccccc} K_1^{\mathfrak{e}}(X.\mathfrak{E}_X) & \xleftarrow{K_1^{\mathfrak{e}}(\mathfrak{q})} & K_1^{\mathfrak{e}}(X', \mathfrak{E}_{X'}) & \xleftarrow{K_1^{\mathfrak{e}}(\mathfrak{c}_{\mathfrak{q}})} & K_0^{\mathfrak{e}}(X'', \mathfrak{E}_{X''}) & \xleftarrow{\mathfrak{d}_1^{\mathfrak{e}}(\mathfrak{q})} & \ldots \\ \mathfrak{d}_0^{\mathfrak{e}}(\mathfrak{q}) \downarrow & & & & & & \\ K_0^{\mathfrak{e}}(X'', \mathfrak{E}_{X''}) & \xrightarrow{K_0^{\mathfrak{e}}(\mathfrak{c}_{\mathfrak{q}})} & K_0^{\mathfrak{e}}(X', \mathfrak{E}_{X'}) & \xrightarrow{K_0^{\mathfrak{e}}(\mathfrak{q})} & K_0^{\mathfrak{e}}(X, \mathfrak{E}_X) & \longrightarrow & 0 \end{array} \qquad (4)$$

is exact.

Proof. The functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^{*}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}^{\mathfrak{e}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^{*}, \mathfrak{L}_{\mathfrak{es}}^{*})$ which maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^{*}$. Therefore, $\mathfrak{F}$ maps conflations to conflations. It remains to apply 3.3.2. ■

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