The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

SMR/1840-4

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Quadratic forms and their invariants

Alexander Vishik
University of Nottingham, UK

Lecture 1

Quadratic forms and their invariants

Let k be a field of characteristic different from 2.

Let V be some finite dimensional vector space over k. Quadratic form on V is a map $q : V \to k$ which is a diagonal part of some symmetric bilinear form $B_q : V_q \times V_q \to k$. That is, $q(v) = B_q(v, v)$. It is easy to see that under our characteristic assumption B_q can be reconstructed from q uniquely.

The form is called *nondegenerate* if the respective symmetric bilinear form is, in other words, if no vector in V is orthogonal to the whole V: $V^{\perp} = 0$.

Under our assumptions, each quadratic form is *diagonalisable*, that is, one can choose the coordinates $x_1, \ldots, x_n$ on V so that $q((x_1, \ldots, x_n)) = a_1x_1^2 + \ldots + a_nx_n^2$ for certain $a_1, \ldots, a_n \in k^*$. We will denote such form $\langle a_1, \ldots, a_n\rangle$, and sometimes will call a_i-the *eigenvalues*.

Warning: in the contrast to the case of linear transformation, these "eigenvalues" are not defined uniquely, so in some other orthogonal coordinates the same form can be presented by $\langle b_1, \ldots, b_n\rangle$ for completely different set $b_1, \ldots, b_n \in k^*$. Try this on the example $\langle 1, -1\rangle$ and $\langle a, -a\rangle$, where $a \in k^*$ (hint: show that both of them are isomorphic to the form xy).

On the set of quadratic forms have two operations: $+$ and $\cdot$

$(q_1, V_1) + (q_2, V_2) := (q_1 \perp q_2, V_1 \oplus V_2)$, where $(q_1 \perp q_2)((v_1, v_2)) = q_1(v_1) + q_2(v_2)$, and

$(q_1, V_1) \cdot (q_2, V_2) := (q_1 \otimes q_2, V_1 \otimes V_2)$, where $(q_1 \otimes q_2)(v_1 \otimes v_2) = q_1(v_1) \cdot q_2(v_2)$.

Definition 0.1 *Define $\widetilde{W}(k)$ - the* Grothendieck-Witt ring *of k as the Grothendieck group (*group completion*) of the monoid of isomorphism classes of nondegenerate quadratic forms over k with respect to operation $+$. Notice, that the operations $+$ and $\cdot$ naturally descend to $\widetilde{W}(k)$ and supply it with the structure of the commutative ring.*

Why to study quadratic forms?

Let me give you several reasons why quadratic forms can be interesting.

1) Connected to K-theory.

More precisely, to Milnors K-theory and motivic cohomology.

Consider form $\mathbb{H} =< 1, -1 >$ called *elementary hyperbolic form*. It is an easy observation, that for arbitrary quadratic form q, $\mathbb{H} \cdot q = \mathbb{H} \perp \ldots \perp \mathbb{H}$ (the number of copies $= dim(q)$). Thus the image of the map $\mathbb{Z} \cdot \mathbb{H} \to \widetilde{W}(k)$ is an ideal in $\widetilde{W}(k)$.

Definition 0.2 *Define $W(k)$ - the* Witt ring *of k as the quotien $\widetilde{W}(k)/\mathbb{Z}\cdot\mathbb{H}$.*

Inside $W(k)$ one has the ideal I of even-dimensional forms (notice that the dimension modulo 2 is well-defined on $W(k)$). This ideal gives rise to the multiplicative filtration

$$W(k) \supset I \supset I^2 \supset I^3 \supset \ldots,$$

and one can consider the associated graded ring

$$gr_{I^\bullet}(W(k)) := W(k)/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots .$$

This ring is basically of "the same size" as $W(k)$, but with the operations $+$ and $\cdot$ somewhat damaged (some information is lost).

Milnors "Conjecture" on quadratic forms relates our graded ring with the ring called *Milnors K-theory*, where the latter is defined as follows. Consider k^* as an abelian group = $\mathbb{Z}$-module. Let $\mathrm{T}_{\mathbb{Z}}(k^*)$ be the tenzor algebra of this module over $\mathbb{Z}$, that is:

$$\mathrm{T}_{\mathbb{Z}}(k^*) = \mathbb{Z} \oplus (k^*) \oplus (k^* \otimes_{\mathbb{Z}} k^*) \oplus (k^* \otimes_{\mathbb{Z}} k^* \otimes_{\mathbb{Z}} k^*) \oplus \ldots .$$

Definition 0.3 *Milnor K-theory of k is defined as a quotient of the tenzor algebra above by the explicite quaratic relations:*

$$\mathrm{K}_*^M(k) := \mathrm{T}_{\mathbb{Z}}(k^*)/(a \otimes (1-a), a \in k^* \backslash 1).$$

Milnors conjecture on quadratic forms states that $\mathrm{K}_*^M(k)/2$ is naturally isomorphic to $gr_{I^\bullet}(W(k))$. And Milnors K-theory is a particular case of *motivic cohomology*. So, our ring can be also interpreted as $\oplus_n \mathrm{H}_{\mathcal{M}}^{n;n}(\mathrm{Spec}(k), \mathbb{Z}/2)$ (notice, that in algebraic geometry, in contrast to topology, the cohomology are numbered by two integers, as opposed to one). If one uses also *Beilinson-Lichtenbaum "Conjecture"* (which follows from the Milnors one, and so is settled), one can see that the knowledge of quadratic forms over k gives one the comlete knowledge of motivic cohomology of a point with $\mathbb{Z}/2$-coefficients.

2) Related to stable homotopy groups of spheres. In a sense, it is just the sharpened version of the reason 1).

One of the most important questions in topology (central to the mathematics as a whole) is the study of *stable homotopy groups of spheres*. Homotopy groups of spheres $\pi_n(S^m)$ count the continuous maps $S^n \to S^m$ up to homotopy (two maps are called homotopic, if you can continuously "pull" one into the other). There is a *suspension operation* Σ such that $\Sigma(S^n) = S^{n+1}$; being a functor, it acts also on the homotopy classes of maps and provides a group homomorphism $\pi_n(S^n) \to \pi_{n+1}(S^{m+1})$. The *stable homotopy groups* are defined as

$$\pi_n^s(S^0) := \lim_{N\to\infty} \pi_{n+N}(S^N).$$

Computation of these groups was performed only for small number of n.

In algebraic geometry both homotpy and homology groups are numerated by two integers (the world here is more complicated - there are two suspensions).

It was proven by F.Morel that the Grothendieck-Witt ring of quadratic forms over k describes the $(0,0)$ stable homotopy group of spheres:

$$\widetilde{W}(k) \overset{naturally}{\cong} \pi_{0,0}^s(S^0).$$

So, studying quadratic forms we study the homotopy groups of spheres, and the experience obtained here in the end could prove usefull back in the topological world.

3) Quadrics give examples of *homogeneous variaties*.

To each quadratic form q one can asign the respective projective quadric $Q \subset \mathbb{P}(V_q)$ given by the equation $q = 0$. If q is nondegenerate, the respective quadratic hypersurface will be *smooth* (no singularities). The group of orthogonal linear transformation preserving the form q (denoted $O(q)$) acts naturally on Q, and the action is *transitive* in certain sense. Thus, Q is a *projective homogeneous variety* for the algebraic group $O(q)$. Other homogeneous varieties for other algebraic groups behave in many respects similar to the ones for the orthogonal group. Hence, studying the quadrics we get certain insight into the behavior of other homogeneous varieties. Useful to mention, that all such varieties are somewhat trivial over algebraically closed field, and so here we are dealing with the pure extract of the effects which distinguish arbitrary field from the algebraically closed one (can be then used to extend the results on some other more complicated varieties from the case of algebraically closed field to that of arbitrary one).

Connection to K-theory

If you just have some arbitrary form at your disposal it is not very easy to see much K-theory in it. But some forms are better than others, and with the good forms the connection is well-visible. The best such forms are *Pfister forms.*

Pfister forms

Definition 0.4 *Let $a \in k^*$. The 1-fold Pfister form $\langle\langle a\rangle\rangle$ is the 2-dimensional form $\langle 1, -a\rangle$.*

If now $a_1, \ldots, a_n \in k^*$, then n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is the product $\langle\langle a_1\rangle\rangle \otimes \ldots \otimes \langle\langle a_n\rangle\rangle$.

Examples:

$n = 1$ $\langle\langle a\rangle\rangle = \mathrm{Nrm}_{k\sqrt{a}/k}$ - the norm map from the the quadratic extension.

$n = 2$ $\langle\langle a, b\rangle\rangle = \mathrm{Nrd}_{Quat(\{a,b\},k)/k}$ - the *reduced norm* map in the *Quaternion algebra.*

$n = 3$ $\langle\langle a, b, c\rangle\rangle = \mathrm{Nrd}_{\mathbb{O}(\{a,b,c\},k)/k}$ - the *reduced norm* map in the *Octonian algebra.*

In all three cases we have an algebra structure on the underlying vector space of quadratic form, that is a bilinear operation $* : V \times V \to V$ such that

$$q(x * y) = q(x) \cdot q(y)$$

(although, for $n = 2$ the operation is not commutative, and for $n = 3$ not even associative).

For $n > 3$ it is still possible to define such an operation $*$, but it will not be bilinear, but only linear in the 1-st coordinate, and rational in the 2-nd. And Pfister forms are the only forms for which such multiplicativity holds (if you demand this property not just over k but also over all extensions F/k).

The quadratic form q is called *isotropic* if it represents zero nontrivially (that is, there is $v \neq 0$, such that $q(v) = 0$). This property is equivalent to the fact that $\mathbb{H}$ is a direct summand in our form: $q = \mathbb{H} \perp q'$. For each quadratic form q there is unique anisotropic form q_{an} such that $q = \mathbb{H} \perp \ldots \perp \mathbb{H} \perp q_{an}$, and the number of hyperbolic summands $i_W(q)$ is called the *Witt index* (of course, it is also uniquely determined). The forms for which $dim(q_{an}) \leqslant 1$ (almost nothing left) are called *completely split.* Notice that the form is anisotropic if and only if the respective projective quadric Q has no k-rational points at all.

The Main Property of Pfister forms is:

$$\text{Pfister form is isotropic} \;\Leftrightarrow\; \text{it is completely split}$$

Sometimes two sets of parameters $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ define the same (isomorphic) Pfister form. It appeares that this happens iff there is an equality of the respective *pure symbols* $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$ as elements of $\mathrm{K}_n^M(k)/2$ (pure symbol $\{c_1, \ldots, c_m\}$ is just the product $\{c_1\} \cdot \ldots \cdot \{c_m\}$ of elements of degree 1 in $\mathrm{K}_*^M(k)$, where K_1^M is naturally identified with k^*).

Thus, the Pfister form depends only on pure symbol (which is also reconstructed from the form uniquely) , and we can denote it as $\langle\langle\alpha\rangle\rangle$, for pure symbol $\alpha \in \mathrm{K}_n^M(k)/2$.

The *Milnor map* in the isomorphism from the Milnor "Conjecture"

$$\mathrm{K}_*^M(k)/2 \xrightarrow{\phi} gr_{I^\bullet}(W(k))$$

is defined as identity on 0-degree component (isomorpic to $\mathbb{Z}/2$), is given by $\phi(\{a\}) = \langle\langle a\rangle\rangle(mod. I^2)$ on the component of degree 1, and then uniquely extended as a homomorphism of algebras (the left algebra is generated by the first degree component, and it is not difficult to see that ϕ respects our explicite quadratic relations $a \otimes (1-a)$). Thus under the Milnor map the pure symbols goes to the respective Pfister forms (modulo I^{n+1}).

In a meantime, we observe that to each Pfister form we can assign two invariants:

$$\text{foldness} = n, \text{ and pure symbol } \alpha \in \mathrm{K}_n^M(k)/2,$$

from which the form itself can be reconstructed.

But Pfister forms live only in dimension of the form 2^n. What about other dimensions? In any dimension there is a "substitute" for the Pfister form, which, may be, not as good as the Pfister form itself, but still is the best thing one can find there. These are so-called *excellent forms*. To construct an excellent form of dimension d, one has start by presenting d in the form $2^{r_1} - 2^{r_2} + 2^{r_3} - \ldots \pm 2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$ (one can easily check that there is $1-1$ correspondence between $\mathbb{N}$ and such sequences). then for each $1 \leqslant i \leqslant s$ one has to choose pure symbol $\alpha_i \in \mathrm{K}_{r_i}^M(k)/2$ in such a way that α_s divides α_{s-1} divides ... divides α_1. Notice that $\beta = \{b_1, \ldots, b_l\}$ divides $\alpha = \{a_1, \ldots, a_m\}$ in $\mathrm{K}_*^M(k)/2$ if and only if our symbols have other presentations: $\beta = \{c_1, \ldots, c_l\}$ and $\alpha = \{c_1, \ldots, c_l, d_{l+1}, \ldots, d_m\}$.

In particular, if β divides α, then $\langle\langle\beta\rangle\rangle$ is naturally a subform of $\langle\langle\alpha\rangle\rangle$ (since $\langle 1\rangle$ is a subform of $\langle\langle d_{l+1}, \ldots, d_m\rangle\rangle$). In particular, in our situation, $\langle\langle\alpha_1\rangle\rangle \supset \langle\langle\alpha_2\rangle\rangle \supset \ldots \supset \langle\langle\alpha_s\rangle\rangle$. Using this fact and the decreesing induction on r one can define the form $\langle\langle\alpha_r\rangle\rangle - \langle\langle\alpha_{r+1}\rangle\rangle + \ldots \pm \langle\langle\alpha_s\rangle\rangle$ as a subform (and a direct summand) of $\langle\langle\alpha_r\rangle\rangle$ orthogonal to $\langle\langle\alpha_{r+1}\rangle\rangle - \ldots \mp \langle\langle\alpha_s\rangle\rangle$. It follows from the definition that the dimension of the obtained form will be exactly $d = 2^{r_1} - 2^{r_2} + \ldots \pm 2^{r_s}$.

Examples:

$d = 2^n$: then *excellent form* is just the *Pfister form*

$d = 5$: the form $\langle 1, -c, ac, bc, -abc\rangle$ is excellent, $a, b, c, \in k^*$. $r_1 = 3, r_2 = 2, r_3 = 0$, $\alpha_1 = \{a, b, c\}$, $\alpha_2 = \{a, b\}$, $\alpha_3 = 1 = \{\emptyset\}$.

$d = 6$: the form $\langle\langle a\rangle\rangle \cdot \langle -b, -c, bc\rangle$ is excellent. $r_1 = 3, r_2 = 1$, $\alpha_1 = \{a, b, c\}$, $\alpha_2 = \{a\}$.

We observe that each excellent form produces invariants (which determine it, in turn): numbers $r_1, \ldots, r_s$, and pure symbols $\alpha_1 \in \mathrm{K}^M_{r_1}(k)/2$,. . .,$\alpha_s \in \mathrm{K}^M_{r_s}(k)/2$.

So, as the first approximation we can expect that each quadratic form produces a series of invariants living in the groups of the type $K_0, K_1, K_2, \ldots$, where invariants of type K_0 are *discrete invariants* taking values in the discrete groups (collection of integers), and the invariants of type K_1, K_2, etc. ... are taking values in more and more "continuous groups" (where we count K_2 more continuous than K_1).