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Pure motives II: applications and conjectures

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Mixed motives

and
cycle complexes, I

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Outline:

- Proofs of the localization and embedding theorems
- Cycle complexes
- Bivariant cycle cohomology

Proofs of the localization and embedding theorems

Statement of main results

The localization theorem

Theorem *The functor* C_* : $\mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM_-^{\mathsf{eff}}(k)$ *extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM_-^{\mathsf{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.

$\mathbf{R}C_*$ *identifies* $DM_-^{\mathsf{eff}}(k)$ *with the localization* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$, *where* $\mathcal{A}$ *is the localizing subcategory of* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$ *generated by complexes*

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

The embedding theorem

Theorem *There is a commutative diagram of exact tensor functors*

$$\begin{tikzcd} K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) \arrow[r, "L"] \arrow[d, "q"'] & D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \arrow[d, "\mathbf{R}C_*"] \\ DM^{\mathsf{eff}}_{\mathsf{gm}}(k) \arrow[r, "i"'] & DM^{\mathsf{eff}}_{-}(k) \end{tikzcd}$$

such that

1. i *is a full embedding with dense image.*

2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Now to work. We have:

Theorem (Global PST) *Let F^* be a complex of PSTs on $\mathbf{Sm}/k$: $F \in C^-(PST)$. Suppose that the cohomology presheaves $h^i(F)$ are homotopy invariant. Then*

(1) For $Y \in \mathbf{Sm}/k$, $\mathbb{H}^i(Y_{\mathsf{Nis}}, F^*_{\mathsf{Nis}}) \cong \mathbb{H}^i(Y_{\mathsf{Zar}}, F^*_{\mathsf{Zar}})$

(2) The presheaf $Y \mapsto \mathbb{H}^i(Y_{\mathsf{Nis}}, F^*_{\mathsf{Nis}})$ is homotopy invariant

(1) and (2) follows from the PST theorem using the spectral sequence:

$$E_2^{p,q} = H^p(Y_\tau, h^q(F)_\tau) \Longrightarrow \mathbb{H}^{p+q}(Y_\tau, F_\tau), \tau = \mathsf{Nis}, \mathsf{Zar}.$$

$\mathbb{A}^1$-homotopy

The inclusions i_0, i_1 : Spec $k \to \mathbb{A}^1$ give maps of PST's i_0, i_1 : $1 \to L(\mathbb{A}^1)$.

Definition Two maps of PST's $f, g : F \to G$ are $\mathbb{A}^1$-*homotopic* if there is a map

$$h : F \otimes L(\mathbb{A}^1) \to F$$

with $f = h \circ (\mathrm{id} \otimes i_0)$, $g = h \circ (\mathrm{id} \otimes i_1)$.

The usual definition gives the notion of $\mathbb{A}^1$-*homotopy equivalence*.

These notions extend to complexes by allowing chain homotopies.

Example $p^* : F \to C_n(F)$ is an $\mathbb{A}^1$-homotopy equivalence:

$n = 1$ is the crucial case since $C_1(C_{n-1}(F)) = C_n(F)$.

We have the homotopy inverse $i_0^* : C_1(F) \to F$.

To define a homotopy $h : C_1(F) \otimes L(\mathbb{A}^1) \to C_1(F)$ between $p^* i_0^*$ and id:

$$\begin{aligned} &\mathrm{Hom}(C_1(F) \otimes L(\mathbb{A}^1), C_1(F)) \\ &\qquad = \mathrm{Hom}(\mathcal{H}om(L(\mathbb{A}^1), F), \mathcal{H}om(L(\mathbb{A}^1) \otimes L(\mathbb{A}^1), F)) \end{aligned}$$

so we need a map $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$.

Taking $\mu(x, y) = xy$ works.

Lemma *The inclusion* $F = C_0(F) \to C_*(F)$ *is an* $\mathbb{A}^1$*-homotopy equivalence.*

Proof. Let F_* be the "constant" complex, $F_n := F$, $d_n = 0, \mathrm{id}$.

$F \to F_*$ is a chain homotopy equivalence.

$F_* \to C_*(F)$ is an $\mathbb{A}^1$-homotopy equivalence by the Example.

$\mathbb{A}^1$-homotopy and $\mathrm{Ext}_{\mathrm{Nis}}$

Lemma *Let F, G be in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, with G homotopy invariant. Then $\mathrm{id} \otimes p_* : F \otimes L(\mathbb{A}^1) \to F$ induces an isomorphism*

$$\mathrm{Ext}^n(F, G) \to \mathrm{Ext}^n(F \otimes L(\mathbb{A}^1), G).$$

Here Ext *is in* $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$.

Proof. For $F = L(X)$, we have

$$\mathrm{Ext}^n(L(X), G) \cong H^n(X_{\mathrm{Nis}}, G),$$

so the statement translates to:

$$p^* : H^n(X_{\mathrm{Nis}}, G) \to H^n(X \times \mathbb{A}^1_{\mathrm{Nis}}, G)$$

is an isomorphism. This follows from: G strictly homotopy invariant and the Leray spectral sequence.

In general: use the left resolution $\mathcal{L}(F) \to F$.

Proposition *Let* $f : F_* \to F'_*$ *be an* $\mathbb{A}^1$*-homotopy equivalence in* $C^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$*. Then*

$$\mathrm{Hom}_{D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))}(F_*, G[n]) \xrightarrow{f^*} \mathrm{Hom}_{D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))}(F'_*, G[n])$$

is an isomorphism for all $G \in HI(k)$*.*

Theorem ($\mathbb{A}^1$**-resolution)** *For* $G \in HI(k)$*,* F *a PST, we have*

$$\mathrm{Ext}^n(F_{\mathsf{Nis}}, G) \cong \mathrm{Hom}_{D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))}(C_*(F)_{\mathsf{Nis}}, G[n])$$

for all n*. Hence:*

$\mathrm{Ext}^i(F_{\mathsf{Nis}}, , G) = 0$ *for* $0 \le i \le n$ *and all* $G \in HI(k)$
$\Leftrightarrow h_i^{\mathsf{Nis}}(F) = 0$ *for* $0 \le i \le n$*.*

Proof. The $\mathbb{A}^1$-homotopy equivalence $F \to C_*(F)$ induces an $\mathbb{A}^1$-homotopy equivalence $F_{\mathsf{Nis}} \to C_*(F)_{\mathsf{Nis}}$.

Nisnevich acyclicity theorem A very important consequence of the $\mathbb{A}^1$-resolution theorem is

Theorem *Let F be a PST such that $F_{\mathsf{Nis}} = 0$. Then $C_*(F)_{\mathsf{Nis}}$ and $C_*(F)_{\mathsf{Zar}}$ are acyclic complexes of sheaves.*

Proof. We need to show that

$$h_i^{\mathsf{Nis}}(F) = 0 = h_i^{\mathsf{Zar}}(F)$$

for all i. The vanishing of the $h_i^{\mathsf{Nis}}(F)$ follows from the $\mathbb{A}^1$-resolution theorem.

Since $h_i(F)$ is a homotopy invariant PST, it follows from the PST theorem that

$$h_i^{\mathsf{Zar}}(F) = h_i^{\mathsf{Nis}}(F)$$

hence $h_i^{\mathsf{Zar}}(F) = 0$.

The localization theorem

Theorem *The functor C_* extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM_-^{\mathsf{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.

$\mathbf{R}C_*$ identifies $DM_-^{\mathsf{eff}}(k)$ with the localization $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$, where $\mathcal{A}$ is the localizing subcategory of $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$ generated by complexes

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

Proof. It suffices to prove

1. For each $F \in \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$, $F \to C_*(F)$ is an isomorphism in $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$.

2. For each $T \in DM_-^{\mathsf{eff}}(k)$, $B \in \mathcal{A}$, $\mathsf{Hom}(B, T) = 0$.

Indeed: (1) implies $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$ is surjective on isomorphism classes.

(2) implies $DM_-^{\mathsf{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$ is fully faithful, hence an equivalence.

(1) again implies the composition

$$D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A} \to DM_-^{\mathsf{eff}}(k)$$

sends F to $C_*(F)$.

To prove: 2. For each $T \in DM_{-}^{\mathsf{eff}}(k)$, $B \in \mathcal{A}$, $\mathsf{Hom}(B, T) = 0$.

$\mathcal{A}$ is generated by complexes $I(X) := L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X)$.

But $\mathsf{Hom}(L(Y), T) \cong \mathbb{H}^0(Y_{\mathsf{Nis}}, T)$ for $T \in D^{-}(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$ and

$$\mathbb{H}^*(X, T) \cong \mathbb{H}^*(X \times \mathbb{A}^1, T)$$

since T is in $DM_{-}^{\mathsf{eff}}(k)$, so $\mathsf{Hom}(I(X), T) = 0$.

To prove: 1. For each $F \in \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$, $F \to C_*(F)$ is an isomorphism in $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$.

First: $\mathcal{A}$ is a $\otimes$-ideal: $A \in \mathcal{A}$, $B \in D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \Longrightarrow A \otimes B \in \mathcal{A}$.

$\mathcal{A}$ is localizing, so can take $A = I(X)$, $B = L(Y)$. But then $A \otimes B = I(X \times Y)$.

Second: $F_* \to C_*(F)$ is a term-wise $\mathbb{A}^1$-homotopy equivalence and $F \to F_*$ is an iso in $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$, so it suffices to show:

For each $F \in \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$, $\mathrm{id} \otimes i_0 = \mathrm{id} \otimes i_1 : F \to F \otimes \mathbb{A}^1$ in $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$.

To show: For each $F \in \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$, $\mathsf{id} \otimes i_0 = \mathsf{id} \otimes i_1 : F \to F \otimes \mathbb{A}^1$ in $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$.

For this: $i_0 - i_1 : L(\mathsf{Spec}\, k) \to L(\mathbb{A}^1)$ goes to 0 after composition with $L(\mathbb{A}^1) \to L(\mathsf{Spec}\, k)$, so lifts to a map $\phi : L(\mathsf{Spec}\, k) \to I(\mathbb{A}^1)$.

Thus $\mathsf{id} \otimes i_0 - \mathsf{id} \otimes i_1 : F \to F \otimes L(\mathbb{A}^1)$ lifts to $\mathsf{id} \otimes \phi : F \to F \otimes I(\mathbb{A}^1) \in \mathcal{A}$.

The embedding theorem

Theorem *There is a commutative diagram of exact tensor functors*

$$\begin{tikzcd} K^b(\mathsf{Cor}_{\mathsf{fin}}(k)) \arrow[r, "L"] \arrow[d] & D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \arrow[d, "\mathbf{R}C_*"] \\ DM^{\mathsf{eff}}_{\mathsf{gm}}(k) \arrow[r, "i"'] & DM^{\mathsf{eff}}_{-}(k) \end{tikzcd}$$

such that

1. i *is a full embedding with dense image.*
2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Proof of the embedding theorem.

We already know that $\mathbf{R}C_*(L(X)) \cong C_*(L(X)) = C_*(X)$.

To show that $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}_{-}(k)$ exists:

$DM^{\mathrm{eff}}_{-}(k)$ is already pseudo-abelian. Using the localization theorem, we need to show that the two types of complexes we inverted in $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$ are already inverted in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

Type 1. $[X \times \mathbb{A}^1] \to [X]$. This goes to $L(X \times \mathbb{A}^1) \to L(X)$, which is a generator in $\mathcal{A}$.

Type 2. $([U \cap V] \to [U] \oplus [V]) \to [U \cup V]$. The sequence

$$0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(U \cup V) \to 0$$

is exact as Nisnevich sheaves (N.B. *not* as Zariski sheaves), hence the map is inverted in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

To show that i is a full embedding:

We need show show that $L^{-1}(\mathcal{A})$ is the thick subcategory generated by cones of maps of Type 1 and Type 2.

The proof uses results of Ne'eman on compact objects in triangulated categories.

To show that i has dense image: This uses the canonical left resolution $\mathfrak{L}(F) \to F$.

Cycle complexes

We introduce various cycle complexes and describe their main properties.

Our goal is to describe the morphisms in $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$ using algebraic cycles, more precisely, as the homology of a cycle complex.

Bloch's cycle complex

A *face* of $\Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$ is a closed subset defined by $t_{i_1} = \ldots = t_{i_s} = 0$.

Definition $X \in \mathbf{Sch}_k$. $z_r(X, n) \subset z_{r+n}(X \times \Delta^n)$ is the subgroup generated by the closed irreducible $W \subset X \times \Delta^n$ such that

$$\dim W \cap X \times F \leq r + \dim F$$

for all faces $F \subset \Delta^n$.

If X is equi-dimensional over k of dimension d, set

$$z^q(X, n) := z_{d-q}(X, n).$$

Let $\delta_i^n : \Delta^n \to \Delta^{n+1}$ be the inclusion to the face $t_i = 0$.

The cycle pull-back δ_i^{n*} is a well-defined map

$$\delta_i^{n*} : z_r(X, n+1) \to z_r(X, n)$$

Definition Bloch's cycle complex $z_r(X, *)$ is $z_r(X, n)$ in degree n, with differential

$$d_n := \sum_{i=0}^{n+1} (-1)^i \delta_i^{n*} : z_r(X, n+1) \to z_r(X, n)$$

Bloch's *higher Chow groups* are

$$\mathrm{CH}_r(X, n) := H_n(z_r(X, *)).$$

For X locally equi-dimensional over k, we have the complex $z^q(X, *)$ and the higher Chow groups $\mathrm{CH}^q(X, n)$.

A problem with functoriality

Even for $X \in \mathbf{Sm}/k$, the complex $z^q(X,*)$ is only functorial for *flat* maps, and covariantly functorial for proper maps (with a shift in q). This complex is NOT a complex of PST's.

This is corrected by a version of the the classical *Chow's moving lemma* for cycles modulo rational equivalence.

Products

There is an external product $z^q(X,*)\otimes z^{q'}(Y,*) \to z^{q+q'}(X\times_k Y,*)$, induced by taking products of cycles. For X smooth, this induces a cup product, using δ_X^*.

Properties of the higher Chow groups

(1) **Homotopy**
$p^* : z_r(X,*) \to z_{r+1}(X \times \mathbb{A}^1,*)$ is a quasi-isomorphism for $X \in \mathbf{Sch}_k$.

(2) **Localization amd Mayer-Vietoris**
For $X \in \mathbf{Sch}_k$, let $i : W \to X$ be a closed subset with complement $j : U \to X$. Then

$$z_r(W,*) \xrightarrow{i_*} z_r(X,*) \xrightarrow{j^*} z_r(U,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$. Similarly, if $X = U \cup V$, U, V open in X, the sequence

$$z_r(X,*) \to z_r(U,*) \oplus z_r(V,*) \to z_r(U \cap V,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$.

(3) **K-theory**

For X regular, there is a functorial *Chern character isomorphism*

$$ch : K_n(X)_{\mathbb{Q}} \to \oplus_q \mathrm{CH}^q(X, n)_{\mathbb{Q}}$$

identifying $\mathrm{CH}^q(X, n)_{\mathbb{Q}}$ with the weight q eigenspace $K_n(X)^{(q)}$ for the Adams operations.

(4) **Classical Chow groups**

$\mathrm{CH}^n(X, 0) = \mathrm{CH}^n(X)$.

(5) **Weight one**

For $X \in \mathbf{Sm}/k$, $\mathrm{CH}^1(X, 1) = H^0(X, \mathcal{O}_X^*)$, $\mathrm{CH}^1(X, 0) = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$, $\mathrm{CH}^1(X, n) = 0$ for $n > 1$.

The proof localization property uses a different type of moving lemma (Bloch's moving by blowing up faces).

Equi-dimensional cycles

Definition Fix $X \in \mathbf{Sch}_k$. For $U \in \mathbf{Sm}/k$ let $z_r^{\text{equi}}(X)(U) \subset z(X \times U)$ be the subgroup generated by the closed irreducible $W \subset X \times U$ such that $W \to U$ is equi-dimensional with fibers of dimension r (or empty).

Remark The standard formula for composition of correspondences makes $z_r^{\text{equi}}(X)$ a PST; in fact $z_r^{\text{equi}}(X)$ is a Nisnevich sheaf with transfers.

Definition The complex of *equi-dimensional cycles* is

$$z_r^{\mathrm{equi}}(X,*) := C_*(z_r^{\mathrm{equi}}(X))(\mathrm{Spec}\, k).$$

Explicitly: $z_r^{\mathrm{equi}}(X,n)$ is the subgroup of $z_{r+n}(X \times \Delta^n)$ generated by irreducible W such that $W \to \Delta^n$ is equi-dimensional with fiber dimension r. Thus:

There is a natural inclusion

$$z_r^{\mathrm{equi}}(X,*) \to z_r(X,*).$$

Note. $z_0^{\text{equi}}(X)(Y) \subset \mathcal{Z}(Y \times X)$ is the subgroup generated by integral closed subschemes $W \subset Y \times X$ just that $W \to Y$ is *quasi-finite* and dominant over some component of Y.

Write $C_*^c(X)$ for $C_*(z_0^{\text{equi}}(X))$.

Since $z_r^{\text{equi}}(X)$ is a Nisnevich sheaf with transfers, $C_*^c(X)$ defines an object $M_{\text{gm}}^c(X)$ of $DM_-^{\text{eff}}(k)$.

$X \mapsto M_{\text{gm}}^c(X)$ is covariantly functorial for *proper* maps and contravariantly functorial for *flat maps of relative dimension 0* (e.g. open immersions).

Similarly, we can define the PST $L(X)$ for $X \in \mathbf{Sch}_k$ by $L(X)(Y) =$ the cycles on $X \times Y$, finite over X. This gives the object

$$M_{\mathsf{gm}}(X) := C_*(X) := C_*(L(X))$$

of $DM_-^{\mathsf{eff}}(k)$, covariantly functorial in X, extending the definition of M_{gm} from $\mathbf{Sm}/k$ to $\mathbf{Sch}_k$.

Bivariant cycle cohomology

The cdh topology

Definition The cdh site is given by the pre-topology on $\mathbf{Sch}_k$ with covering families generated by

1. Nisnevich covers

2. $p \amalg i : Y \amalg F \to X$, where $i : F \to X$ is a closed immersion, $p : Y \to X$ is proper, and

$$p : Y \setminus p^{-1}F \to X \setminus F$$

is an isomorphism (abstract blow-up).

Remark If k admits resolution of singularities (for finite type k-schemes and for abstract blow-ups to smooth k-schemes), then each cdh cover admits a refinement consisting of *smooth* k-schemes.

Definition Take $X, Y \in \mathbf{Sch}_k$. The *bivariant cycle cohomology* of Y with coefficients in cycles on X are

$$A_{r,i}(Y, X) := \mathbb{H}^{-i}(Y_{\mathsf{cdh}}, C_*(z_r^{\mathsf{equi}}(X))_{\mathsf{cdh}}).$$

$A_{r,i}(Y, X)$ is contravariant in Y and covariant in X (for proper maps).

We have the natural map

$$h_i(z_r^{\mathsf{equi}}(X))(Y) := H_i(C_*(z_r^{\mathsf{equi}}(X))(Y)) \to A_{r,i}(Y, X).$$

Mayer-Vietoris and blow-up sequences

Since Zariski open covers and abstract blow-ups are covering families in the cdh topology, we have a Mayer-Vietoris sequence for $U, V \subset Y$:

$$
\begin{aligned}
\ldots \to A_{r,i}(U \cup V, X) \to A_{r,i}(U, X) \oplus A_{r,i}(V, X) \\
\to A_{r,i}(U \cap V, X) \to A_{r,i-1}(U \cup V, X) \to \ldots
\end{aligned}
$$

and for $p \amalg i : Y' \amalg F \to Y$:

$$
\begin{aligned}
\ldots \to A_{r,i}(Y, X) \to A_{r,i}(Y', X) \oplus A_{r,i}(F, X) \\
\to A_{r,i}(p^{-1}(F), X) \to A_{r,i-1}(Y, X) \to \ldots
\end{aligned}
$$

Additional properties of $A_{r,i}$ require some fundamental results on the behavior of homotopy invariant PST's with respect to cdh-sheafification. Additionally, we will need some essentially algebro-geometric results comparing different cycle complexes. These two types of results are:

1. Acyclicity theorems. We have already seen the Nisnevich acyclicity theorem:

Theorem *Let F be a PST F with $F_{\mathrm{Nis}} = 0$. Then the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic.*

We will also need the cdh version:

Theorem (cdh-acyclity) *Assume that k admits resolution of singularities. For F a PST with $F_{\mathrm{cdh}} = 0$, the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic.*

This result transforms sequences of PST's which become short exact after cdh-sheafification, into distinguished triangles after applying $C_*(-)_{\text{Zar}}$.

Using a hypercovering argument and Voevodsky's PST theorem, these results also show that cdh, Nis and Zar cohomology of a homotopy invariant PST all agree on smooth varieties:

Theorem (cdh-Nis-Zar) *Assume that k admits resolution of singularities. For $U \in \mathbf{Sm}/k$, $F^* \in C^-(PST)$ such that the cohomology presheaves of F are homotopy invariant,*

$$\mathbb{H}^n(U_{\text{Zar}}, F^*_{\text{Zar}}) \cong \mathbb{H}^n(U_{\text{Nis}}, F^*_{\text{Nis}}) \cong \mathbb{H}^n(Y_{\text{cdh}}, F^*_{\text{cdh}})$$

We will derive the important consequences of the cdh acyclicity theorem for bivariant cohomology in the next lecture.

2. Moving lemmas. The bivariant cohomology $A_{r,i}$ is defined using cdh-hypercohomology of z_r^{equi}, so comparing z_r^{equi} with other complexes leads to identification of $A_{r,i}$ with cdh-hypercohomology of the other complexes. These comparisions of z_r^{equi} with other complexes is based partly on a number of very interesting geometric constructions, due to Friedlander-Lawson and Suslin. We will not discuss these results here, except to mention where they come in.