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Uniqueness of bP^n

Charles Weibel
Rutgers University, Piscataway, USA

5 Lecture 5: Uniqueness of βP^n

The purpose of this lecture is to prove Theorem 5.1, which replaces the unproven "Theorem 2.1" in [MC/l]. As before, k is a field of characteristic 0, and P^n is the cohomology operation of bidegree $(2n(\ell-1), n(\ell-1))$ referred to in Lecture 2. (The n of this lecture is unrelated to the n in the norm residue homomorphism of the Bloch-Kato conjecture.)

Note that the Bockstein β is a derivation, βP^n is additive and commutes with simplicial suspension Σ by [RPO]. Thus by axiom 2.2(2), if $y \in H^{2n,n}(X, \mathbb{Z}/\ell)$ then $P^n(y) = y^\ell$ and:

$$\beta P^n(\Sigma y) = \Sigma\beta(P^n y) = \Sigma\beta(y^\ell) = \ell\Sigma\beta(y) = 0.$$

Thus βP^n satisfies (1) and (2) of the following uniqueness theorem.

Theorem 5.1. *Let $\phi\colon H^{2n+1,n}(-,\mathbb{Z}) \to H^{2n\ell+2,n\ell}(-,\mathbb{Z}/\ell)$ be a cohomology operation such that for all X and all $x \in H^{2n+1,n}(X,\mathbb{Z})$:*

1. *$\phi(bx) = b\phi(x)$ for $b \in \mathbb{Z}$;*

2. *If $x = \Sigma y$ for $y \in H^{2n,n}(X,\mathbb{Z})$ then $\phi(x) = 0$.*

Then ϕ is a multiple of βP^n.

Remark 5.1.1. In topology, βP^n is the image of a cohomology operation $H^{2n+1,n}(-,\mathbb{Z}) \to H^{2n\ell+2,n\ell}(-,\mathbb{Z})$. We will see the relevance of this in the next Lecture.

We saw in Lecture 2 (2.6) that the cohomology operations described in Theorem 5.1 correspond to elements of $H^{**}(BK_n, \mathbb{Z}/\ell)$. We saw in Lecture 4 (see 4.8) that a complete description of this cohomology is possible, but it is messy for $n > 1$. To cut down on the bookkeeping in the proof of 5.1, it is useful to introduce the notion of *scalar weight*.

The multiplicative action of the monoid $(\mathbb{Z}, \times)$ on the sheaf of $\mathbb{Z}$-modules $K_n = u\mathbb{L}^n$ and $BK_n = u\mathbb{L}^n[1]$ defines an action of $\mathbb{Z}/\ell^\times$ on $H^{**}(K_n, \mathbb{Z}/\ell)$ and $H^{**}(BK_n, \mathbb{Z}/\ell)$, at least when $n > 0$. The induced representations of $(\mathbb{Z}/\ell)^\times$ decompose the cohomology into the direct sum of its isotypical pieces, corresponding to the irreducible representations $\mathbb{Z}/\ell, \mu_\ell, \mu_\ell^{\otimes 2}, \dots, \mu_\ell^{\otimes \ell-2}$. We will say that the isotypical piece corresponding to $\mu_\ell^{\otimes s}$ has *scalar weight* s; an element $x \in H^{**}(K_n, \mathbb{Z}/\ell)$ has scalar weight s if $a \cdot x = a^s x$ for all $a \in \mathbb{Z}/\ell$.

Recall from 4.6(b) that $K_n \simeq S^\infty(V_n)$, and that $R_{\mathrm{tr}}(K_n) \simeq \oplus R_{\mathrm{tr}}(\widetilde{S}^i V_n) = \oplus S^i_{\mathrm{tr}}(\mathbb{L}^n)$.

Theorem 5.2. *Under the decomposition* $H^{**}(K_n, \mathbb{Z}/\ell) = \oplus H^{**}(\widetilde{S}^i V_n, \mathbb{Z}/\ell)$*, the summands* $H^{**}(\widetilde{S}^i V_n, \mathbb{Z}/\ell) \cong \mathrm{Hom}_{\mathbf{DM}}(S^i_{tr}\mathbb{L}^n_R, R(*)[*])$ *have scalar weight* $i \mod (\ell - 1)$.

Lemma 5.3. *If* $R(q)[p]$ *is a summand of* $S^m_{tr}(\mathbb{L}^n)$*, and* $m \equiv s \mod (\ell - 1)$ *for* $0 \le s < \ell - 1$*, then:*

(a) $q \ge ns$*, with equality iff* $m < \ell$

(b) $q \ge n(\ell - 1)$ *if* $s = 0$*, with equality iff* $m = \ell - 1$.

(c) $p \ge 2q \ge 2n$

Proof. Recall from Theorem 4.16 that $R(q)[p] = \mathbb{L}^q[b]$ for $b \ge 0$, so $p = 2q + b$. Hence (a) and (b) imply (c). If $m < \ell$ then $S^m_{\mathrm{tr}}(\mathbb{L}^n) = \mathbb{L}^{mn}$ and $q = mn$ by 4.7.1. This yields the 'if' parts. To prove the 'only if' parts of (1) and (2), suppose that $m \ge \ell$ and write $m = \sum m_i \ell^i$, noting that $\sum m_i > m_0$, $\sum m_i \equiv m \mod (\ell - 1)$. We also have $q \ge (\sum m_i)n + (\ell - 1)$ by Proposition 4.8. Since $\sum m_i \ge s$, we have $q > ns$. If $s = 0$ then $\sum m_i \ge \ell - 1$ and we have $q \ge (n + 1)(\ell - 1)$. □

Remark 5.3.1. These are the equations (2.6), (2.7) and (2.8) of [MC/l], strengthened to inequalities when $m \ge \ell$.

We now turn to the cohomology of $K_n \wedge \dots \wedge K_n$ in scalar weight 1. The following presentation is due to Voevodsky and is taken from [MC/l].

Lemma 5.4. *The scalar weight $s = 1$ part of $H^{p,q}(K_n^{\wedge r}, \mathbb{Z}/\ell)$ vanishes if $q < n\ell$ and $r \geq 2$, and also if $q = n\ell$ and $p < 2n\ell$.*

Proof. ([MC/l, 2.7 and 2.8]) By 4.13, 4.11 and 5.2 it suffices to consider $x_1 \otimes \cdots \otimes x_r$ where the x_i are in $\mathrm{Hom}(S^{a_i}_{\mathrm{tr}}\mathbb{L}^n, R(q_i)[p_i])$, $\Sigma p_i = p$, $\Sigma q_i = q$ and $\Sigma a_i \equiv 1 \mod (\ell - 1)$. If $q < n\ell$ then $a_i \not\equiv 0$ by 5.3(b) and we must have $\Sigma a_i \geq \ell$, which is excluded by 5.3(a) as $q \geq n\Sigma a_i$. This establishes the case $q < n\ell$. When $q = n\ell$, the vanishing comes from 5.3(c). □

We now analyze the motivic cohomology $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$, where BK_n is the simplicial classifying space $[r] \mapsto (K_n)^r$.

$$* \Longleftarrow K_n \equiv K_n \times K_n \cdots$$

There is a standard spectral sequence for the cohomology of a simplicial space with coefficients in $\mathbb{Z}/\ell(n\ell)$; for BK_n it has

$$E_1^{r,s} = \widetilde{H}^{s,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell) \Longrightarrow H^{r+s,n\ell}(BK_n, \mathbb{Z}/\ell). \tag{5.5}$$

For $n \geq 1$, the spectral sequence converges because $K_n^{\wedge r}$ is nr-fold T-connected so that $E_1^{r,s} = 0$ for $r > \ell$. This in turn follows from the fact [RPO, 3.7] that K_n is n-fold T-connected, together the Künneth formula 4.13. (This is the argument of [MC/l, 2.6].)

The relevant part of the spectral sequence looks like this, using Lemma 5.4:

$$\begin{array}{lllll}
 & 0 & & & \\
 & 0 & H^{2n\ell+1,n\ell}(K_n) \to & & \\
s = 2n\ell & 0 & H^{2n\ell,n\ell}(K_n) \;\to & H^{2n\ell,n\ell}(K_n \wedge K_n) \to & H^{2n\ell,n\ell}(K_n^{\wedge 3}) \to \\
 & 0 & \quad 0 & \multicolumn{2}{l}{\text{(nothing in scalar weight 1 below here)}}
\end{array}$$

Recall from [RPO, 3.7] that $H^{2n,n}(K_n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ on the fundamental class α.

Example 5.5.1. $\gamma = \{(\alpha\otimes 1 + 1\otimes\alpha)^\ell - \alpha^\ell\otimes 1 - 1\otimes\alpha^\ell\}/\ell = \alpha^{\ell-1}\otimes\alpha + \cdots + \alpha\otimes\alpha^{\ell-1}$ is an element of $H^{2n\ell,n\ell}(K_n{\wedge}K_n, \mathbb{Z}/\ell)$. A calculation shows that $d_1^{2,2n\ell}$ maps γ to zero in $H^{2n\ell,n\ell}(K_n{\wedge}K_n{\wedge}K_n, \mathbb{Z}/\ell)$. (Formally this follows from $\delta(\alpha^\ell) = -\ell\gamma$ and $\delta^2 = 0$).

Lemma 5.6. ([MC/l, 2.9]) *Let D_r denote the subset of elements of scalar weight one in $H^{2n\ell,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell)$. Then D_r is the $\mathbb{Z}/\ell$-vector space generated by monomials of the form $\alpha^{i_1}{\wedge}\dots{\wedge}\alpha^{i_r}$, where $i_1+\cdots+i_r = \ell$ and each $i_j > 0$.*

Proof. The monomials are linearly independent by the Künneth formula 4.13. Lemma 5.4 implies that D_r is generated by elements of the form $x_1\otimes\cdots\otimes x_r$ whre the $x_i \in \mathrm{Hom}(S^{a_i}\mathbb{L}^n, R(q_i)[p_i])$. By 5.3(b), at most one x_i can have scalar weight 0, and only if $r=2$.

If $r = 2$ and $a_1 \equiv 0$ then $a_2 \equiv 1$. Then by 3.4(a, b) we must have $q_1 = n(\ell-1)$ and $q_2 = n$, and $p_i = 2q_i$. By 3.4(b) and (4.7) we must have $x_1 = \alpha^{\ell-1}$ and $x_2 = \alpha$. Thus we are reduced to the case in which all $a_i \not\equiv 0 \mod (\ell-1)$. By 5.3(a, c) and $q = n\ell$ we must have $\Sigma a_i = \ell$, $q_i = na_i$ and $p_i = 2q_i$. Since $S^{a_i}_{\mathrm{tr}}(\mathbb{L}^n) = \mathbb{L}^{na_i}$ by (4.7) we must have $x_i = \alpha^{a_i}$ up to scalars. □

Proof of Theorem 5.1: (Voevodsky) We regard ϕ as an element of $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$. Condition 5.1(1) says that ϕ has scalar weight one. Condition 5.1(2) says that ϕ (like βP^n) is in the kernel of the map

$$H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}\ell) \to H^{2n\ell+2,n\ell}(\Sigma K_n, \mathbb{Z}\ell) = H^{2n\ell+1,n\ell}(K_n, \mathbb{Z}/\ell)$$

defined by the inclusion of ΣK_n as the degree one part of BK_n. That is, ϕ and βP^n lie in the kernel of the edge map in the spectral sequence (5.5).

Voevodsky observes in [MC/l] that, by a formal calculation of Lazard [3, 1.21], the kernel of $E_1^{2,2n\ell} \to E_1^{3,2n\ell}$ is $\mathbb{Z}/\ell$ in scalar weight one, on the cycle γ displayed in 5.5.1. Since βP^n represents a nonzero element of $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$ by [1] and [RPO], it follows that every such element ϕ must be a multiple of βP^n. □