Lecture 5. Comparison theorems for higher K-theory: reduction by resolution, additivity, devissage. Towards some applications.

In the first four sections, we fix a left exact subcategory of the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ of right exact 'spaces', or a left exact subcategory of the left exact category $\mathfrak{Esp}^{\mathfrak{r}}_k$ of k-linear right exact 'spaces' endowed with the induced left exact structure. The higher K-functors are computed as satellites of the restriction of the functor K_0 to this left exact subcategory.

1. Reduction by resolution.

1.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and C_Y its fully exact subcategory such that*

(a) If $M' \longrightarrow M \longrightarrow M''$ is a conflation with $M \in ObC_Y$, then $M' \in ObC_Y$.

(b) For any $M'' \in ObC_X$, there exists a deflation $M \longrightarrow M''$ with $M \in ObC_Y$.

Then the morphism $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.

Proof. The first part of the argument of 1.1 shows that if C_Y is a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the condition (b) and F_0 is a functor from $\mathfrak{Esp}^{op}_{\mathfrak{r}}$ to a category with filtered limits such that $F_0(Y, \mathfrak{E}_Y) \longrightarrow F_0(X, \mathfrak{E}_X)$ is an isomorphism, then $S^n_- F_0(Y, \mathfrak{E}_Y) \longrightarrow S^n_- F_0(X, \mathfrak{E}_X)$ is an isomorphism for all $n \geq 0$.

The condition (a) is used only in the proof that $K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X)$ is an isomorphism. ∎

1.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_{\mathfrak{Z}}, \mathfrak{E}_{\mathfrak{Z}})$ be right exact categories with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_{\mathfrak{Z}}, \mathfrak{E}_{\mathfrak{Z}})$. Let C_Y be the full subcategory of C_X generated by T-acyclic objects (that is objects V such that $T_i(V)$ is an initial object of $C_{\mathfrak{Z}}$ for $i \geq 1$). Assume that for every $M \in ObC_X$, there is a deflation $P \longrightarrow M$ with $P \in ObC_Y$, and that $T_n(M)$ is an initial object of $C_{\mathfrak{Z}}$ for n sufficiently large. Then the natural map $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.*

Proof. The assertion is deduced from 1.1 in the usual way (see [Q]). ∎

1.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects; and let*

$$\begin{array}{ccccc}
Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
{\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2
\end{array} \qquad (3)$$

be a commutative diagram (determined by its lower right square) such that $Ker(\mathfrak{k}'')$ and $Ker(\beta_2)$ are trivial. Then

(a) The upper row of (3) is 'exact', and the morphism β'_1 is the kernel of α'_1.

(b) Suppose, in addition, that the arrows f', α_1 and α_2 in (3) are deflations and $(C_X, \mathfrak{E}_X)$ has the following property:

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

Then the upper row of (3) is a conflation.

1.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 1.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ which has the following properties:*

(a) If $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ and N, M are objects of C_Y, then L belongs to C_Y too.

(b) For any deflation $M \longrightarrow \mathcal{L}$ with $\mathcal{L} \in ObC_Y$, there exist a deflation $\mathcal{M} \longrightarrow \mathcal{L}$ with $\mathcal{M} \in ObC_Y$ and a morphism $\mathcal{M} \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ \swarrow & & \searrow \\ M & \longrightarrow & \mathcal{L} \end{array}$$

commutes.

(c) If P, $\mathcal{M}$ are objects of C_Y and $P \longrightarrow x$ is a morphism to initial object, then $P \coprod \mathcal{M}$ exists (in C_X) and the sequence $P \longrightarrow P \coprod \mathcal{M} \longrightarrow \mathcal{M}$ (where the left arrow is the canonical coprojection and the right arrow corresponds to the $\mathcal{M} \xrightarrow{id} \mathcal{M}$ and the composition of $P \longrightarrow x \longrightarrow \mathcal{M}$) is a conflation.

Let C_{Y_n} be a full subcategory of C_X generated by all objects L having a C_Y-resolution of the length $\leq n$. And set $C_{Y_\infty} = \bigcup_{n \geq 0} C_{Y_n}$. Then C_{Y_n} is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$ for all $n \leq \infty$ and the natural morphisms

$$K_\bullet(Y, \mathfrak{E}_Y) \xrightarrow{\sim} K_\bullet(Y_1, \mathfrak{E}_{Y_1}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} K_\bullet(Y_\infty, \mathfrak{E}_{Y_\infty})$$

are isomorphisms for all $n \geq 0$.

1.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 1.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the conditions (a) and (c) of 1.4. Let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $(C_X, \mathfrak{E}_X)$, and let $\mathcal{P}' \longrightarrow M'$, $\mathcal{P}'' \longrightarrow M''$ be C_Y-resolutions of the length $n \geq 1$. Suppose that resolution $\mathcal{P}'' \longrightarrow M'$ is projective. Then there exists a C_Y-resolution $\mathcal{P} \longrightarrow M$ of the length n such that $\mathcal{P}_i = \mathcal{P}'_i \coprod \mathcal{P}''_i$ for all $i \geq 1$ and the splitting 'exact' sequence $\mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}''$ is an 'exact' sequence of complexes.*

2. Additivity of 'characteristic' filtrations.

2.1. Characteristic 'exact' filtrations and sequences.

2.1.1. The right exact 'spaces' $(X_n, \mathfrak{E}_{X_n})$. For a right exact exact 'space' $(X, \mathfrak{E}_X)$, let C_{X_n} be the category whose objects are sequences $M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_0$ of n morphisms of $\mathfrak{E}_X$, $n \geq 1$, and morphisms between sequences are commutative diagrams

$$\begin{array}{ccccccc} M_n & \longrightarrow & M_{n-1} & \longrightarrow & \ldots & \longrightarrow & M_0 \\ f_n \downarrow & & f_{n-1} \downarrow & & \ldots & & \downarrow f_0 \\ M'_n & \longrightarrow & M'_{n-1} & \longrightarrow & \ldots & \longrightarrow & M'_0 \end{array}$$

Notice that if x is an initial object of the category C_X, then $x \longrightarrow \ldots \longrightarrow x$ is an initial object of C_{X_n}.

We denote by $\mathfrak{E}_{X_n}$ the class of all morphisms (f_i) of the category C_{X_n} such that $f_i \in \mathfrak{E}_X$ for all $0 \leq i \leq n$.

2.1.1.1. Proposition. *(a) The pair $(C_{X_n}, \mathfrak{E}_{X_n})$ is a right exact category.*

(b) The map which assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the right exact 'space' $(X_n, \mathfrak{E}_{X_n})$ extends naturally to an 'exact' endofunctor of the left exact category $(\mathfrak{Esp}_r, \mathfrak{L}_{\mathfrak{es}})$ of right 'exact' 'spaces' which induces an 'exact' endofunctor $\mathcal{P}_n$ of its exact subcategory $(\mathfrak{Esp}_r^*, \mathfrak{L}_{\mathfrak{es}}^*)$.

Proof. The argument is left to the reader. ∎

2.1.2. Proposition. (Additivity of 'characteristic' filtrations) *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $f_n^* \xrightarrow{\mathfrak{t}_n} f_{n-1}^* \xrightarrow{\mathfrak{t}_{n-1}} \ldots \xrightarrow{\mathfrak{t}_1} f_0^*$ a sequence of deflations of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ such that the functors $\mathfrak{k}_i^* = Ker(\mathfrak{t}_i^*)$ are 'exact' for all $1 \leq i \leq n$. Then $K_\bullet(f_n) = K_\bullet(f_0) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_i)$.*

Proof. The argument uses facts on kernels (see Appendix A to Lecture 3). ∎

2.1.3. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $g^* \longrightarrow f^* \longrightarrow h^*$ a conflation of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. Then $K_\bullet(f) = K_\bullet(g) + K_\bullet(h)$.*

2.1.4. Corollary. (Additivity for 'characteristic' 'exact' sequences) *Let*

$$\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^* \longrightarrow \ldots \longrightarrow \mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$$

be an 'exact' sequence of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ which map initial objects to initial objects. Suppose that $\mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$ is a deflation and $\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^*$ is the kernel of $\mathfrak{f}_{n-1}^* \longrightarrow \mathfrak{f}_{n-2}^*$. Then the morphism $\sum_{0 \leq i \leq n} (-1)^i K_\bullet(\mathfrak{f}_i)$ from $K_\bullet(X, \mathfrak{E}_X)$ to $K_\bullet(Y, \mathfrak{E}_Y)$ is equal to zero.

Proof. The assertion follows from 2.1.3 by induction.

A more conceptual proof goes along the lines of the argument of 2.1.2. Namely, we assign to each right exact category $(C_Y, \mathfrak{E}_Y)$ the right exact category $(C_{Y_n^{\mathfrak{e}}}, \mathfrak{E}_{Y_n^{\mathfrak{e}}})$ whose objects are 'exact' sequences $\mathcal{L} = (L_n \longrightarrow L_{n-1} \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0)$, where $L_1 \longrightarrow L_0$ is a deflation and $L_n \longrightarrow L_{n-1}$ is the kernel of $L_{n-1} \longrightarrow L_{n-2}$. This assignment defines an endofunctor $\mathfrak{P}_n^{\mathfrak{e}}$ of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$ of right exact 'spaces' with initial objects, and maps $\mathcal{L} \longmapsto L_i$ determine morphisms $\mathfrak{P}_n^{\mathfrak{e}} \longrightarrow Id_{\mathfrak{Esp}_{\mathfrak{r}}^*}$. The rest of the argument is left to the reader. ∎

3. Infinitesimal 'spaces'. Devissage.

3.1. The Gabriel multiplication in right exact categories. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. Let $\mathbb{T}$ and $\mathbb{S}$ be subcategories of the category C_X.

The *Gabriel product* $\mathbb{S} \bullet \mathbb{T}$ is the full subcategory of C_X whose objects M fit into a conflation $L \xrightarrow{g} M \xrightarrow{h} N$ such that $L \in Ob\mathbb{S}$ and $N \in Ob\mathbb{T}$.

3.1.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ of the category C_X, there is the inclusion*

$$\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}.$$

Proof. An exercise on kernels and cartesian squares. ■

3.1.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be an exact category. Then the Gabriel multiplication is associative.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. By 3.1.1, we have the inclusion $\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. The opposite inclusion holds by duality, because $(\mathcal{A} \bullet \mathcal{B})^{op} = \mathcal{B}^{op} \bullet \mathcal{A}^{op}$. ■

3.2. The infinitesimal neighborhoods of a subcategory. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathbb{O}_X$ the full subcategory of C_X generated by all initial objects of C_X. For any subcategory $\mathcal{B}$ of C_X, we define subcategories $\mathcal{B}^{(n)}$ and $\mathcal{B}_{(n)}$, $0 \le n \le \infty$, by setting $\mathcal{B}^{(0)} = \mathbb{O}_X = \mathcal{B}_{(0)}$, $\mathcal{B}^{(1)} = \mathcal{B} = \mathcal{B}_{(1)}$, and

$$\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)} \bullet \mathcal{B} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}^{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}^{(n)};$$

$$\mathcal{B}_{(n)} = \mathcal{B} \bullet \mathcal{B}_{(n-1)} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}_{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}_{(n)}$$

It follows that $\mathcal{B}^{(n)} = \mathcal{B}_{(n)}$ for $n \le 2$ and, by 3.1.1, $\mathcal{B}_{(n)} \subseteq \mathcal{B}^{(n)}$ for $3 \le n \le \infty$.

We call the subcategory $\mathcal{B}^{(n+1)}$ the *upper n^{th} infinitesimal neighborhood* of $\mathcal{B}$ and the subcategory $\mathcal{B}_{(n+1)}$ the *lower n^{th} infinitesimal neighborhood* of $\mathcal{B}$. It follows that $\mathcal{B}^{(n+1)}$ is the strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of arrows

$$M_0 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{j}_2} \dots \xrightarrow{\mathfrak{j}_n} M_n = M$$

with the property: $M_0 \in Ob\mathcal{B}$, and for each $n \ge i \ge 1$, there exists a deflation $M_i \xrightarrow{\mathfrak{e}_i} N_i$ with $N_i \in Ob\mathcal{B}$ such that $M_{i-1} \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} N_i$ is a conflation.

Similarly, $\mathcal{B}_{(n+1)}$ is a strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$$

such that M_0 and $Ker(\mathfrak{e}_i)$ are objects of $\mathcal{B}$ for $1 \le i \le n$.

3.2.1. Note. It follows that $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(n+1)}$ for all $n \ge 0$, if $\mathcal{B}$ contains an initial object of the category C_X.

3.3. Fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathcal{E}_X)$. A subcategory $\mathcal{A}$ of C_X is a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$ if $\mathcal{A} \bullet \mathcal{A} = \mathcal{A}$.

3.3.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategory $\mathcal{B}$ of C_X, the subcategory $\mathcal{B}^{(\infty)}$ is the smallest fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be a fully exact subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, i.e. $\mathcal{A} = \mathcal{A} \bullet \mathcal{A}$. Then $\mathcal{B}^{(\infty)} \subseteq \mathcal{A}$, iff $\mathcal{B}$ is a subcategory of $\mathcal{A}$.

On the other hand, it follows from 3.1.1 and the definition of the subcategories $\mathcal{B}^{(n)}$ (see 3.2) that $\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m)} \subseteq \mathcal{B}^{(m+n)}$ for any nonnegative integers n and m. In particular, $\mathcal{B}^{(\infty)} = \mathcal{B}^{(\infty)} \bullet \mathcal{B}^{(\infty)}$, that is $\mathcal{B}^{(\infty)}$ is a fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$. ∎

3.4. Cofiltrations. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. A *cofiltration of the length* n+1 of an object M is a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0. \tag{1}$$

The cofiltration (1) is said to be *equivalent* to a cofiltration

$$M = \widetilde{M}_m \xrightarrow{\widetilde{\mathfrak{e}}_n} \dots \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\widetilde{\mathfrak{e}}_1} \widetilde{M}_0$$

if $m = n$ and there exists a permutation σ of $\{0, \dots, n\}$ such that $Ker(\mathfrak{e}_i) \simeq Ker(\widetilde{\mathfrak{e}}_{\sigma(i)})$ for $1 \le i \le n$ and $M_0 \simeq \widetilde{M}_0$.

The following assertion is a version (and a generalization) of Zassenhouse's lemma.

3.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ have the following property:*

(‡) *for any pair of deflations $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then any two cofiltrations of an object M have equivalent refinements.

3.5. Devissage.

3.5.1. Proposition. (Devissage for K_0.) *Let $((X, \mathfrak{E}_X), Y)$ be an infinitesimal 'space' such that $(X, \mathfrak{E}_X)$ has the following property (which appeared in 3.4.1):*

(‡) *for any pair of deflations $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then the natural morphism

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X) \tag{1}$$

is an isomorphism.

3.5.2. The ∂^*-functor $K_\bullet^{\mathfrak{sq}}$. Let $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$ denote the left exact structure on the category $\mathfrak{Esp}^{\rtimes}$ of $\mathfrak{Esp}_{\mathfrak{r}}$ (cf. 3.9.4) induced by the (defined in 6.8.3.3) left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'. Let $K_i^{\mathfrak{sq}}(X, \mathfrak{E}_X)$ denote the i-th satellite of the functor K_0 with respect to the left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$.

3.5.3. Proposition. *Let $((X, \mathfrak{E}_X), Y)$ be an infinitesimal 'space' such that the right exact 'space' $(X, \mathfrak{E}_X)$ has the property (‡) of 3.4.1, the category C_X has final objects, and all morphisms to final objects are deflations. Then the natural morphism*

$$K_i^{\mathfrak{sq}}(Y, \mathfrak{E}_Y) \longrightarrow K_i^{\mathfrak{sq}}(X, \mathfrak{E}_X) \tag{8}$$

is an isomorphism for all $i \geq 0$.

Proof. The assertion follows from a general devissage theorem for universal ∂^*-functors whose zero component satisfy devissage property (like K_0, by 3.5.1). ■

4. An application: K-groups of 'spaces' with Gabriel-Krull dimension.

4.1. Gabriel-Krull filtration. We recall the notion of the *Gabriel filtration* of an abelian category as it is presented in [R, 6.6]. Let C_X be an abelian category. The *Gabriel filtration of X* assigns to every cardinal α a Serre subcategory C_{X_α} of C_X which is constructed as folows:

Set $C_{X_0} = \mathbb{O}$.

If α is not a limit cardinal, then C_{X_α} is the smallest Serre subcategory of C_X containing all objects M such that the localization $q^*_{\alpha-1}(M)$ of M at $C_{X_{\alpha-1}}$ has a finite length.

If β is a limit cardinal, then C_{X_β} is the smallest Serre subcategory containing all subcategories C_{X_α} for $\alpha < \beta$.

Let C_{X_ω} denote the smallest Serre subcategory containing all the subcategories C_{X_α}. Clearly the quotient category C_X/C_{X_ω} has no simple objects.

An object M is said to have the *Gabriel-Krull dimension* β, if β is the smallest cardinal such that M belongs to C_{X_β}.

The 'space' X has a Gabriel-Krull dimension if $X = X_\omega$.

Every locally noetherian abelian category (e.g. the category of quasi-coherent sheaves on a noetherian scheme, or the category of left modules over a left noetherian associative algebra) has a Gabriel-Krull dimension.

It follows that for any limit ordinal β, we have $K_\bullet(X_\beta) = \bigcup_{\alpha<\beta} K_\bullet(X_\alpha)$. Therefore, $K_\bullet(X_\omega) = \bigcup_{\alpha \in \mathfrak{Or}_{\mathfrak{n}}} K_\bullet(X_\alpha)$, where $\mathfrak{Or}_{\mathfrak{n}}$ denotes the set of non-limit ordinals.

4.2. Reduction via localization. If α is a non-limit ordinal, we have the exact localization $C_{X_\alpha} \xrightarrow{q^*_{\alpha-1}} C_{X_\alpha}/C_{X_{\alpha-1}} = C_{X^q_\alpha}$, hence the corresponding long exact sequence

$$\dots \longrightarrow K_{n+1}(X^q_\alpha) \xrightarrow{\mathfrak{d}^\alpha_n} K_n(X_{\alpha-1}) \longrightarrow K_n(X_\alpha) \longrightarrow K_n(X^q_\alpha) \longrightarrow \dots \longrightarrow K_0(X^q_\alpha) \quad (1)$$

of K-groups.

4.3. Reduction by devissage. Suppose that the category C_X is *noetherian*, i.e. all objects of C_X are noetherian. Then the quotient category $C_{X^q_\alpha} = C_{X_\alpha}/C_{X_{\alpha-1}}$ is noetherian. Notice that the Krull dimension of X^q_α equals to zero; hence all objects of the category $C_{X^q_\alpha}$ have a finite length. Let $C_{X^q_{\alpha,\mathfrak{s}}}$ denote the full subcategory of $C_{X^q_\alpha}$ generated by semisimple objects. By devissage, the natural morphism $K_\bullet(X^q_{\alpha,\mathfrak{s}}) \longrightarrow K_\bullet(X^q_\alpha)$ is an isomorphism. If C_Y is a svelte abelian category whose objects are semisimple of finite length, then $K_\bullet(Y) = \coprod_{\mathcal{Q}\in\mathbf{Spec}(Y)} K_\bullet(\mathbf{Sp}(D_\mathcal{Q}))$, where $D_\mathcal{Q}$ is the *residue* skew field of the point $\mathcal{Q}$ of the spectrum of Y, which is the skew field $C_Y(M,M)^o$ of the endomorphisms of the simple object M such that $\mathcal{Q} = [M]$. In particular,

$$K_\bullet(X^q_\alpha) = \coprod_{\mathcal{Q}\in\mathbf{Spec}(X^q_\alpha)} K_\bullet(\mathbf{Sp}(D_\mathcal{Q}))$$

for every non-limit ordinal α.

5. First definitions of K-theory and G-theory of noncommutative schemes.

The purpose of this section is to sketch the first notions which allow extension of K-theory and G-theory to noncommutative schemes and more general locally affine 'spaces'. We consider here only the class of so-called *semiseparated* locally affine 'spaces' and schemes which includes the main examples of noncommutative schemes and locally affine 'spaces', starting from quantum flag varieties and noncommutative Grassmannians. Commutative semiseparated schemes are schemes $\mathcal{X}$ whose diagonal moprhism $\mathcal{X} \xrightarrow{\Delta_X} \mathcal{X} \times \mathcal{X}$ is affine. In particular, every separated scheme is semiseparated.

Semiseparated noncommutative (in particular, commutative) schemes and locally affine 'spaces' over an affine scheme are particularly convenient, because the category of quasi-coherent sheaves on them is described by a linear algebra data provided by *flat descent*.

5.1. Semiseparated schemes. Flat descent. We shall consider semiseparated schemes and more general locally affine 'spaces' over an affine scheme, $\mathcal{S} = \mathbf{Sp}(R)$. These are pairs (X, f), where X is a 'space' and f a continuous morphism $X \longrightarrow \mathcal{S}$ for which there exists a finite affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that every morphism u_i is flat and affine. In this case, the corresponding morphism

$$U_J = \coprod_{i\in J} U_i \xrightarrow{\mathfrak{u}} X$$

is fflat (=faithfully flat) and affine. By the (dual version of) Beck's theorem (Lecture 1, 2.3.1), there is a commutative diagram

$$\begin{array}{ccc} C_X & \longrightarrow & \mathcal{G}_{\mathfrak{u}} - Comod \\ \mathfrak{u}^* \searrow & & \swarrow \widehat{\mathfrak{u}}^* \\ & R_{f\mathfrak{u}} - mod & \end{array}$$

where the horizontal arrow is a category equivalence. Here we identify the category C_{U_J} with the category of $R_{f\mathfrak{u}}$-modules for a ring $R_{f\mathfrak{u}}$ over R corresponding (by Beck's theorem) to the affine morphism $U_J \xrightarrow{f\mathfrak{u}} \mathbf{Sp}(R)$ – the monad $\mathcal{F}_{f\mathfrak{u}}$ on $R - mod$ is isomorphic to the monad $R_{f\mathfrak{u}} \otimes_R -$ (see Lecture 1). Since the morphism $\mathfrak{u}$ is affine, the associated comonad $\mathcal{G}_{\mathfrak{u}} = (G_{\mathfrak{u}}, \delta_{\mathfrak{u}})$, that is the functor $G_{\mathfrak{u}} = \mathfrak{u}^*\mathfrak{u}_*$, is continuous: the composition $\mathfrak{u}^!\mathfrak{u}_*$ is its right adjoint. Therefore, $G_{\mathfrak{u}}$ is isomorphic to the tensoring $\mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} -$ by an $R_{f\mathfrak{u}}$-bimodule $\mathcal{M}_{\mathfrak{u}}$ determined uniquely up to isomorphism. The comonad structure $\delta_{\mathfrak{u}}$ induces a map $\mathcal{M} \xrightarrow{\widetilde{\delta}_{\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}}$ which turnes $\mathcal{M}_{\mathfrak{u}}$ into a coalgebra in the monoidal category of $R_{f\mathfrak{u}}$-bimodules. Thus, the category C_X is naturally equivalent to the category $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}}) - Comod$ of $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}})$. Its objects are pairs $(V, V \xrightarrow{\zeta} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} V)$, where V is a left $R_{f\mathfrak{u}}$-module, which satisfy the usual comodule conditions. The structure morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ is encoded in the structure object $\mathcal{O} = f^*(R)$, or, what is the same, a comodule structure $R_{f\mathfrak{u}} \xrightarrow{\zeta_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} R_{f\mathfrak{u}}$ on the left module $R_{f\mathfrak{u}}$, which we can replace, thanks to an isomorphism $\mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} R_{f\mathfrak{u}} \simeq \mathcal{M}_{\mathfrak{u}}$, by a morphism $R_{f\mathfrak{u}} \xrightarrow{\zeta_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}}$ satisfying the natural associativity condition and whose composition with counit $\mathcal{M} \xrightarrow{\varepsilon_{\mathfrak{u}}} R_{f\mathfrak{u}}$ of the coalgebra $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}})$ is the identical morphism.

Thus, Beck's theorem provides a description of the category of quasi-coherent sheaves on a semiseparated noncommutative (that is not necessarily commutative) scheme in terms of linear algebra.

5.2. The category of vector bundles. Fix a locally affine 'space' (X, f). We call an object $\mathcal{M}$ of the category C_X a *vector bundle* if its inverse image, $u_J^*(\mathcal{M})$ is a projective ΓU_J-module of finite type, or, equivalently, $u_i^*(\mathcal{M})$ is a projective ΓU_i-module of finite type for each $i \in J$. We denote by $\mathcal{P}(X)$ the full subcategory of the category C_X whose objects are vector bundles on X.

5.3. The category of coherent objects. We call an object $\mathcal{M}$ of the category C_X *coherent* if $u_i^*(\mathcal{M})$ is coherent for each $i \in J$. We denote by $Coh(X)$ the full subcategory of C_X generated by coherent objects.

5.3.1. Proposition. *(a) The notions of a projective and coherent objects are well defined.*

(b) $Coh(X)$ is a thick subcategory of C_X. In particular, it is an abelian category.

(c) $\mathcal{P}(X)$ an fully exact (i.e. closed under extensions) subcategory of C_X. In particular, $\mathcal{P}(X)$ is an exact category.

Proof. (a) Semiseparated finite covers form a filtered system: if $U_J \xrightarrow{u_J} X \xleftarrow{u_I} \widetilde{\mathcal{U}}_I$ are fflat and affine, then all arrows in the cartesian square

$$\begin{array}{ccc} U_J \times_X \widetilde{\mathcal{U}}_I & \longrightarrow & \widetilde{\mathcal{U}}_I \\ \downarrow & cart & \downarrow \\ U_J & \longrightarrow & X \end{array}$$

are fflat and affine. This follows from the categorical description of the cartesian product corresponding to direct image functors of $U_J \longrightarrow X$ and $\widetilde{\mathcal{U}} \longrightarrow X$.

(b) & (c). An exercise for the reader. ■

5.4. The category of locally affine semiseparated 'spaces'. Let $\mathfrak{Laff}_{\mathcal{S}}$ denote the subdiagram of the category $|Cat|^o_{\mathcal{S}}$ of $\mathcal{S}$-'spaces' whose objects are locally affine quasi-compact semiseparated $\mathcal{S}$-'spaces' and morphisms are those morphisms $X \xrightarrow{f} Y$ of $\mathcal{S}$-'spaces' which can be lifted to a morphism of semiseparated covers. More precisely, for any morphism $X \xrightarrow{f} Y$ of $\mathfrak{Laff}_{\mathcal{S}}$ and any affine cover $U_Y \xrightarrow{\pi_Y} Y$, there is a commutative diagram

$$\begin{array}{ccc} U_X & \xrightarrow{\widetilde{f}} & U_Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where the left vertical arrow is an affine cover of X.

One can see that $\mathfrak{Laff}_{\mathcal{S}}$ is a subcategory of $|Cat|^o_{\mathcal{S}}$.

For each object (X,f) of $\mathfrak{Laff}_{\mathcal{S}}$, let $X_{\mathcal{P}}$ denote the 'space' defined by $C_{X_{\mathcal{P}}} = \mathcal{P}(X)$ and $X_{\mathfrak{C}}$ the 'space' defined by $C_{X_{\mathfrak{C}}} = Coh(X,f)$.

5.5. Proposition. *The map $(X,f) \longmapsto X_{\mathcal{P}}$ is a functor from $\mathfrak{Laff}_{\mathcal{S}}$ to the category $\mathfrak{Esp}_{\mathfrak{r}}$ whose objects are 'spaces' represented by exact categories and whose morphisms have 'exact' inverse image functors.*

Proof. In fact, restricted to the affine schemes, the functor takes values in the category $\mathfrak{Ex}$, because an inverse image of (automatically affine) morphism between affine $\mathcal{S}$-'spaces' maps conflations to conflations. The general case follows from the affine case via affine covers, because the inverse image functors of the covers are fflat. ■

5.6. The functor $\mathcal{K}_\bullet$. We define the K-theory functor $\mathcal{K}_\bullet$ as the universal ∂-functor from the category $\mathfrak{Laff}_{\mathcal{S}}$ semiseparated locally affine 'spaces' endowed with left exact structure induced by the functor from $\mathfrak{Laff}_{\mathcal{S}}$ to the category of right exact 'spaces' which assigns to every locally affine semiseparated 'space' the right exact 'space' represented by the category of vector bundles.

5.7. The category $\mathfrak{Laff}^{\mathfrak{fl}}_{\mathcal{S}}$. We denote this way the subcategory of the category $\mathfrak{Laff}_{\mathcal{S}}$ of locally affine 'spaces' formed by flat morphisms.

5.7.1. Proposition. *The map $(X,f) \longmapsto X_{\mathcal{P}}$ is a functor from $\mathfrak{Laff}^{\mathfrak{fl}}_{\mathcal{S}}$ to the category $\mathfrak{Esp}_{\mathfrak{a}}$ whose objects are 'spaces' represented by abelian categories and whose morphisms have exact inverse image functors.*

Proof. An exercise for the reader. ∎

5.8. The functor $G_\bullet$. We endow the category $\mathfrak{Laff}_{\mathcal{S}}^{\mathfrak{fl}}$ with the left exact structure $\widetilde{\mathfrak{I}}_{\mathfrak{S}}$) induced by the standard left exact structure on $\mathfrak{Esp}_{\mathfrak{a}}$ (inverse image functors of inflations are exact localizations) via the functor of 5.7.1. We define the ∂-functor $G_\bullet$ as the universal ∂-functor from the left exact category $(\mathfrak{Laff}_{\mathcal{S}}^{\mathfrak{fl}}, \widetilde{\mathfrak{I}}_{\mathfrak{S}})$, whose zero component assigns to every locally affine semiseparated 'space' (X, f) the K_0-group of the 'space' represented by the category of coherent sheaves on (X, f).

5.9. Proposition. *Let $i \longmapsto (X_i, f_i)$ be a filtered projective system of locally affine $\mathcal{S}$-'spaces' such that the transition morphisms $(X_i, f_i) \longrightarrow (X_j, f_j)$ are affine, and let $(X, f) = \lim(X_i, f_i)$. Then*

$$\mathcal{K}_\bullet(X, f) \simeq \operatorname{colim}(\mathcal{K}_\bullet(X_i, f_i)). \tag{2}$$

If in addition the transition morphisms are flat, then

$$G_\bullet(X, f) \simeq \operatorname{colim}(G_\bullet(X_i, f_i)). \tag{2'}$$

Proof. It follows from the assumptions that a filtered projective system of locally affine $\mathcal{S}$-'spaces' and affine morphisms induces a filtered inductive system of the exact categories $\mathcal{P}(X_i, f_i)$ of vector-bundles. Its colimit, $\mathcal{P}(X, f)$ is an exact category whose conflations are images of conflations of $\mathcal{P}(X_i, f_i)$. Whence the isomorphism (2).

If, in addition, the transition morphisms are flat, then the inverse image functors of the transition functors induce exact functors between categories of coherent objects. This implies the isomorphism (2'). ∎

5.10. Regular locally affine 'spaces'. For a locally affine $\mathcal{S}$-'space' (X, f), we denote by $\mathbb{H}(X, f)$ the full subcategory of the category $Coh(X, f)$ which have a $\mathcal{P}(X)$-resolution.

5.10.1. Proposition. *(a) $\mathbb{H}(X, f)$ is a fully exact subcategory of the category $Coh(X, f)$. In particular, it is an exact category.*

(b) Set $\mathbb{H}(X, f) = C_{X_{\mathbb{H}}}$. The embedding of categories $\mathcal{P}(X, f) \hookrightarrow \mathbb{H}(X, f)$ induces an isomorphism $\mathcal{K}_\bullet(X, f) \stackrel{\mathrm{def}}{=} K_\bullet(X_{\mathcal{P}}) \xrightarrow{\sim} K_\bullet(X_{\mathbb{H}})$.

Proof. (a) By a standard argument.

(b) The fact is a consequence of the Resolution Theorem. ∎

5.10.2. Definition. A locally affine 'space' is called *regular* if $\mathbb{H}(X, f) = Coh(X, f)$.

Thus, if (X, f) is a regular locally affine 'space', then $\mathcal{K}_\bullet(X, f) = G_\bullet(X, f)$.

5.10.3. Remark. If (X, f) is an affine $\mathcal{S}$-'space', then the regularity coincides with the usual notion of regularity of rings ($\mathcal{S}$ is assumed to be affine). Similarly, if (X, f) is an $\mathcal{S}$-'space' corresponding to a commutative scheme.

The notion of $\mathbb{H}(X, f)$ is *local* in the following sense:

5.10.4. Proposition. *Let (X, f) be a locally affine $\mathcal{S}$-'space'. The following conditions on an object $\mathcal{M}$ of C_X are equivalent:*

(a) $\mathcal{M}$ belongs to $\mathbb{H}(X, f)$;

(b) $u_i^*(\mathcal{M})$ belongs to $\mathbb{H}(U_i, fu_i)$ for some finite cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of (X, f) and for all $i \in J$;

(c) $u_i^*(\mathcal{M})$ belongs to $\mathbb{H}(U_i, fu_i)$ for some finite cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of (X, f) and for all $i \in J$.

Proof. Obviously, $(c) \Rightarrow (b)$. In the rest of the argument, one can assume that the covers consist of one fflat affine morphism. The assertion follows from the fact that such covers form a filtered system. Details are left as an exercise. ∎

5.10.5. Examples. The quantum flag varieties and the corresponding twisted quantum D-schemes [LR] are examples of regular schemes. Noncommutative Grassmannians [KR1], [KR3] are examples of regular locally affine 'spaces' which are not schemes.

6. Remarks on K-theory and quantized enveloping algebras. In a sense, the standard K-theory based on the category of vector bundles, or G-theory based on the category of all coherent sheaves, do not give much valuable information from the point of view of representation theory. For instance, if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field k, then $K_\bullet(U(\mathfrak{g})) \simeq K_\bullet(k)$ and, similarly, $K_\bullet(A_n(k)) \simeq K_\bullet(k)$, where $A_n(k)$ is the n-th Weyl algebra over k. This indicates that one should study K-theory of other subcategories of the category of $U(\mathfrak{g})$-modules. The subcategory which received the most attention in seventies and the beginning of eighties was the category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ of representations of a semi-simple (or reductive) Lie algebra $\mathfrak{g}$ introduced by I.M. Gelfand and his collaborators. The highlight of its study was Kazhdan-Lusztig conjecture and, the most important, its prove, which led to the reformulation of the representation theory of reductive algebraic groups in terms of D-modules and D-schemes making it a part of (actually noncommutative) algebraic geometry, even before this science emerged.

The main basic fact which allowed to reduce the problems of representation theory to the study of D-modules on flag varieties is the Beilinson-Bernstein localization theorem which says that the global section functor induces an equivalence between the category of D-modules on the flag variety of a reductive Lie algebra $\mathfrak{g}$ over a field of zero characteristic and the category of $U(\mathfrak{g})$-modules with trivial central character (and its twisted version). Harish-Chandra modules and their different generalizations turned out to be *holonomic* D-modules. As a result, holonomic modules on flag varieties became the main object of study of representation theory of reductive algebraic groups.

On the other hand, the notions of quantum flag variety and the appropriate categories of twisted D-modules were introduced in [LR]. And it was established a quantum version of Beilinson-Bernstein localization theorem [LR], [T], which reduces the study of representations of the quantized enveloping algebra $U_q(\mathfrak{g})$ to the study of twisted D-modules on quantum flag variety, like in the classical case. The notion of a holonomic D-module is extended to the setting of noncommutative algebraic geometry [R4]. In particular, there exists a notion of a *quantum* holonomic D-module.

All initial ingredients are present and the area of research is wide open.

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