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Constructing the Rost motive

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6 Lecture 6

In this lecture, we will construct a Rost motive for $\underline{a}$ in the sense of Definition 3.4. As we saw in Lecture 3, this suffices to verify the Bloch-Kato conjecture.

Let X be a Rost variety (Definition 3.1), and write $\mathfrak{X}$ for the simplicial Čech variety associated to X. In 1.5, we produced a nonzero $\delta \in H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$, and used it in Proposition 3.3 to construct a nonzero element μ of $H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$. Now any $z \in H^{2b+1,b}(\mathfrak{X})$ can be interpreted as a map $\mathfrak{X} \to \mathbb{L}^b[1]$ in **DM**; tensoring with $\mathfrak{X}$, and using $\mathfrak{X} \otimes \mathfrak{X} \cong \mathfrak{X}$ yields a map $\mathfrak{X} \to \mathfrak{X} \otimes \mathbb{L}^b[1]$, which we also call z, and fit into a triangle

$$\mathfrak{X} \otimes \mathbb{L}^b \xrightarrow{x} A \xrightarrow{y} \mathfrak{X} \xrightarrow{z} \mathfrak{X} \otimes \mathbb{L}^b[1]. \tag{6.1}$$

Applying $\Sigma_{i-1} \subset \Sigma_i$ to $A^{\otimes i}$, we get a corestriction map $S^{i-1}(A) \otimes A \to S^i(A)$. There is also a transfer map $\mathrm{tr}\colon S^i(A) \to S^{i-1}(A) \otimes A$, induced by the endomorphism

$$a_1 \otimes \cdots \otimes a_i \longmapsto \Sigma(\cdots \otimes \hat{a}_j \otimes \cdots) \otimes a_j$$

of $A^{\otimes i}$. Now $S^i(A) \cong \mathfrak{X} \otimes S^i(A)$. Composing tr with $1 \otimes y$ yields a map $u\colon S^i(A) \to S^{i-1}(A)$; composing $1 \otimes x$ with corestriction yields a map $v\colon S^{i-1}(A) \otimes \mathbb{L}^b \to S^i(A)$.

Lemma 6.2. *If $i < \ell$ and $1/(\ell-1)! \in R$, the maps u and v fit into triangles*

(a) $S^{i-1}(A) \otimes \mathbb{L}^b \xrightarrow{v} S^i(A) \xrightarrow{S^i y} \mathfrak{X} \xrightarrow{s} S^{i-1}(A) \otimes \mathbb{L}^b[1]$.

(b) $\mathfrak{X} \otimes \mathbb{L}^{bi} \xrightarrow{S^i x} S^i(A) \xrightarrow{u} S^{i-1}(A) \xrightarrow{r} \mathfrak{X} \otimes \mathbb{L}^{bi}[1]$.

Proof. This is proven in [MC/l, 3.1] using the slice filtration on $A^{\otimes i}$. □

Setting $D = S^{\ell-2}(A)$ and $M = S^{\ell-1}(A)$, we see that M satisfies property 3.4(c) of a Rost motive. The composition of s and $r \otimes 1$ yields a map (for $i = \ell - 1$)

$$\phi(z)\colon\ \mathfrak{X} \xrightarrow{s} D \otimes \mathbb{L}^b[1] \xrightarrow{r\otimes 1} \mathfrak{X} \otimes \mathbb{L}^{b\ell}[2] \to \mathbb{L}^{b\ell}[2]$$

i.e., an element of $H^{2b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$. Consider the function $z \mapsto \phi(z)$.

Proposition 6.3. (Voevodsky) *The function* $\phi\colon H^{2b+1,b}(\mathfrak{X},\mathbb{Z}) \to H^{2b\ell+2,b\ell}(\mathfrak{X},\mathbb{Z}_{(\ell)})$ *extends to a cohomology operation* $\phi\colon H^{2b+1,b}(-,\mathbb{Z}) \to H^{2b\ell+2,b\ell}(-,\mathbb{Z}_{(\ell)})$ *satisfying*

(a) $\phi(az) = a^{\ell}\phi(z)$ *for* $a \in \mathbb{Z}$*;*

(b) $\phi(\Sigma y) = 0$ *for* $y \in H^{2b,b}(-,\mathbb{Z})$*.*

Proof. This is the content of 3.2, 3.5 and 3.6 of [MC/l]. ☐

Corollary 6.4. *The mod–*ℓ *reduction* $\bar{\phi}$ *of* ϕ*, regarded as a motivic cohomology operation* $H^{2b+1,b}(-,\mathbb{Z}) \to H^{2b\ell+2,b\ell}(-,\mathbb{Z}/\ell)$*, is a multiple of* βP^b*.*

Proof. Combine Theorem 5.1 and Proposition 6.3. ☐

Remark 6.4.1. It is easy to show that $\bar{\phi} \neq 0$, so that $\bar{\phi}(x) = c\beta P^b(\bar{x})$ for a nonzero $c \in \mathbb{Z}/\ell$.

Lemma 6.5. *There are maps* $\lambda\colon \mathbb{Z}_{tr}(X) \to S^{\ell-1}(A)$ *such that the inclusion* $\iota\colon X \to \mathfrak{X}$ *factors in* **DM** *as:*

$$\mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A) \xrightarrow{S^{\ell-1}y} \mathfrak{X}.$$

Proof. (Voevodsky) [MC/l, 5.11] Applying $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}_{\mathrm{tr}}X,-)$ to the triangle (6.1) defining A yields the exact sequence

$$\mathrm{Hom}(X,A) \xrightarrow{y} \mathrm{Hom}(X,\mathfrak{X}) \xrightarrow{z} \mathrm{Hom}(X,\mathbb{L}^b[1]) = 0;$$

the group on the right vanishes since it equals $H^{2b+1,b}(X,\mathbb{Z}) = 0$. Hence ι factors through some $\lambda_1\colon \mathbb{Z}_{\mathrm{tr}}X \to A$. Similarly, triangle 6.2(b) yields exact sequences

$$\mathrm{Hom}(X,S^i(A)) \xrightarrow{u} \mathrm{Hom}(X,S^{i-1}(A)) \to \mathrm{Hom}(X,\mathfrak{X}\otimes\mathbb{L}^{bi}[1]) = 0.$$

The group on the right is $H^{2bi+1,bi}(X,\mathbb{Z}) = 0$. By induction, there are maps $\lambda_i\colon \mathbb{Z}_{\mathrm{tr}}(X) \to S^i(A)$ for $i \le \ell-1$ such that $\lambda_{i-1} = u\lambda_i$. By the construction of u, $yu^i = S^iy\colon S^i(A) \to \mathfrak{X}$. ☐

Recall from 2.3.1 that $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{L}^d, \mathbb{Z}_{\mathrm{tr}}(X)) = H_{2d,d}(X, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$ by duality, so there is a fundamental class $\tau\colon \mathbb{L}^d \to \mathbb{Z}_{\mathrm{tr}}(X)$. Since $\mathfrak{X} \otimes X \cong X$, we may also view τ as a map from $\mathfrak{X} \otimes \mathbb{L}^d$ to $\mathbb{Z}_{\mathrm{tr}}(X)$.

Proposition 6.6. *The composition $\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{\tau} \mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A)$ is not divisible by ℓ.*

Proof. (Voevodsky [MC/l, 5.12]) By the definition of ϕ in terms of the map s of 6.2(a), the restriction $S^{\ell-1}y^*(\phi)$ of ϕ to $S^{\ell-1}(A)$ is zero. By 6.4.1, βP^b also vanishes on $S^{\ell-1}(A)$. Since the Q_i anticommute we have $Q_i(\mu) = 0$ for $i \le n-2$.

Consider the element $\alpha = Q_{n-1}(\mu) \in H^{b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}/\ell)$. By 3.3, $\alpha \neq 0$, and $Q_{n-1}(\alpha) = 0$ as $Q_i^2 = 0$. By the definition of Q_{n-1} we have

$$\alpha = Q_{n-1}(\mu) = Q_{n-2}(P^{\ell^{n-2}}(\mu)) = \cdots = \beta P^b(\mu),$$

so $(S^{\ell-1}y)^*(\alpha) = \beta P^n((S^{\ell-1}y)^*\mu) = 0$ in $H^{b\ell+2,b\ell}(S^{\ell-1}(A), R)$. By the Motivic Degree Theorem of [MC/l, 4.4], applied to the factorization in Lemma 6.5, the existence of $\alpha \neq 0$ implies the mod–ℓ reduction of the map $\lambda\tau : \mathfrak{X} \otimes \mathbb{L}^d \to S^{\ell-1}(A)$ is nonzero. $\square$

Because $\mu\colon \mathfrak{X} \to \mathfrak{X} \otimes \mathbb{L}^b[1]$ is a map between Tate objects, it is self dual ($\mu = \mu^* \otimes \mathbb{L}^b$ under the identification of $\mathfrak{X}$ with $\mathfrak{X}^*$). It follows that $A \cong A^* \otimes \mathbb{L}^b$. Since $S^i(M) \cong (S^iM)^*$ for every M we also have $S^i(A) \cong S^i(A)^* \otimes \mathbb{L}^{bi}$. (See [MC/l, 5.7].) For the map λ of 6.5, we write $D\lambda$ for the dual map

$$D\lambda\colon\ S^{\ell-1}(A) \cong S^{\ell-1}(A)^* \otimes \mathbb{L}^d \xrightarrow{\lambda^*\otimes 1} \mathbb{Z}_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d \cong \mathbb{Z}_{\mathrm{tr}}(X).$$

Theorem 6.7. *The composition $\lambda \circ D\lambda$ is an isomorphism on $S^{\ell-1}(A)$ (with coefficients $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$), and there is an integer $c \not\equiv 0 \pmod{\ell}$ so that the following diagram commutes:*

$$\begin{array}{ccc}
S^{\ell-1}(A) & \xrightarrow{\lambda\circ D\lambda} & S^{\ell-1}(A) \\
{\scriptstyle S^{\ell-1}y}\downarrow & & \downarrow{\scriptstyle S^{\ell-1}y} \\
\mathfrak{X} & \xrightarrow{c} & \mathfrak{X}.
\end{array}$$

In particular, $S^{\ell-1}(A)$ is a direct summand of $R_{tr}(X)$ for $R = \mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$.

Proof. (Voevodsky [MC/l, 5.15]) From triangle 6.2(b) we have an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, \mathfrak{X}\otimes\mathbb{L}^d) & \xrightarrow{S^{\ell-1}x} & \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, S^{\ell-1}A) & \xrightarrow{u} & \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, S^{\ell-2}A) = 0. \\ c & \mapsto & \lambda\tau \not\equiv 0 \pmod{\ell} & & \end{array}$$

The fact that the right side is zero follows from the exact sequences of 6.2(a),

$$\mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, S^{i-1}A\otimes\mathbb{L}^b) \xrightarrow{v} \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, S^iA) \to \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, \mathfrak{X}),$$

because the outer terms vanish — the right because maps between Tate objects cannot decrease weight, and the left by induction on i. Hence the map $\lambda\tau$ of Proposition 6.6 lifts to an element c of $\mathbb{Z} = \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d, \mathfrak{X}\otimes\mathbb{L}^d)$. Since $\lambda\tau \not\equiv 0 \pmod{\ell}$ by 6.6, $c \not\equiv 0 \pmod{\ell}$. Dualizing $\lambda\tau = (S^{\ell-1}x)c$ yields the left square in the following diagram, since $S^{\ell-1}y$ is dual to $S^{\ell-1}x$ and ι is dual to $\tau\colon \mathfrak{X}\otimes\mathbb{L}^d \to \mathbb{Z}_{\mathrm{tr}}(X)$, so $\iota\circ D\lambda$ is dual to $\lambda\tau$.

$$\begin{tikzcd} S^{\ell-1}(A) \arrow[r, "D\lambda"] \arrow[d, "S^{\ell-1}y"'] & \mathbb{Z}_{\mathrm{tr}}(X) \arrow[r, "\lambda"] \arrow[d, "\iota"'] & S^{\ell-1}(A) \arrow[dl, "S^{\ell-1}y"] \\ \mathfrak{X} \arrow[r, "c"] & \mathfrak{X}. & \end{tikzcd}$$

The right triangle commutes by Lemma 6.5. □

Corollary 6.8. *When $R = \mathbb{Z}_{(\ell)}$, the maps λ and $D\lambda$ make $M = S^{\ell-1}(A)$ into a direct summand of $R_{tr}(X)$, and the following composition is an isomorphism:*

$$M \cong M^*\otimes\mathbb{L}^d \xrightarrow{\lambda^*} R_{tr}(X)^*\otimes\mathbb{L}^d \cong R_{tr}(X) \xrightarrow{\lambda} M.$$

Indeed, this is just a restatement of Theorem 6.7 in the form of axioms 3.4(a,b) of Lecture 3. Since axiom 3.4(c) holds by Lemma 6.2, M is a Rost motive. We saw in Lecture 3 that the Bloch-Kato conjecture follows form the existence of a Rost motive, so we are done.