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Mixed motives and cycle complexes, II

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Outline:

- Properties of bivariant cycle cohomology
- Morphisms and cycles
- Duality

Recall from Lecture 5:

Theorem (Global PST) *Let F^* be a complex of PSTs on $\mathbf{Sm}/k$: $F \in C^-(PST)$. Suppose that the cohomology presheaves $h^i(F)$ are homotopy invariant. Then*

(1) For $Y \in \mathbf{Sm}/k$, $\mathbb{H}^i(Y_{\text{Nis}}, F^*_{\text{Nis}}) \cong \mathbb{H}^i(Y_{\text{Zar}}, F^*_{\text{Zar}})$

(2) The presheaf $Y \mapsto \mathbb{H}^i(Y_{\text{Nis}}, F^*_{\text{Nis}})$ is homotopy invariant

(1) and (2) follows from the PST theorem using the spectral sequence:

$$E_2^{p,q} = H^p(Y_\tau, h^q(F)_\tau) \Longrightarrow \mathbb{H}^{p+q}(Y_\tau, F_\tau), \tau = \text{Nis}, \text{Zar}.$$

Definition Take $X, Y \in \mathbf{Sch}_k$. The *bivariant cycle cohomology* of Y with coefficients in cycles on X are

$$A_{r,i}(Y,X) := \mathbb{H}^{-i}(Y_{\mathrm{cdh}}, C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{cdh}}).$$

$A_{r,i}(Y,X)$ is contravariant in Y and covariant in X (for proper maps).

We have the natural map

$$h_i(z_r^{\mathrm{equi}}(X))(Y) := H_i(C_*(z_r^{\mathrm{equi}}(X))(Y)) \to A_{r,i}(Y,X).$$

The bivariant cycle cohomology $A_{r,i}(Y, X)$ has long exact Mayer-Vietoris sequence and a blow-up sequence with respect to Y.

Additional properties of $A_{r,i}$ require some fundamental results on the behavior of homotopy invariant PST's with respect to cdh-sheafification. Additionally, we will need some essentially algebro-geometric results comparing different cycle complexes. These two types of results are:

1. Acyclicity theorems. We have already seen the Nisnevich acyclicity theorem:

Theorem *Let F be a PST with $F_{\text{Nis}} = 0$. Then the Suslin complex $C_*(F)_{\text{Zar}}$ is acyclic.*

We will also need the cdh version:

Theorem (cdh-acyclity) *Assume that k admits resolution of singularities. For F a PST with $F_{\mathrm{cdh}} = 0$, the Suslin complex $C_*(F)_{\mathrm{cdh}}$ is acyclic as on $\mathbf{Sch}_k$.*

The cdh-acyclicity theorem follows from the Nisnevich version by a hypercovering argument.

Using a hypercovering argument again, and Voevodsky's PST theorem, these results also show that cdh, Nis and Zar cohomology of a homotopy invariant PST all agree on smooth varieties:

Theorem (cdh-Nis-Zar) *Assume that k admits resolution of singularities. For $U \in \mathbf{Sm}/k$, $F^* \in C^-(PST)$ such that the cohomology presheaves of F are homotopy invariant,*

$$\mathbb{H}^n(U_{\text{Zar}}, F^*_{\text{Zar}}) \cong \mathbb{H}^n(U_{\text{Nis}}, F^*_{\text{Nis}}) \cong \mathbb{H}^n(Y_{\text{cdh}}, F^*_{\text{cdh}})$$

2. Moving lemmas. The bivariant cohomology $A_{r,i}$ is defined using cdh-hypercohomology of z_r^{equi}, so comparing z_r^{equi} with other complexes leads to identification of $A_{r,i}$ with cdh-hypercohomology of the other complexes. These comparisions of z_r^{equi} with other complexes is based partly on a number of very interesting geometric constructions, due to Friedlander-Lawson and Suslin. We will not discuss these results here, except to mention where they come in.

Homotopy Bivariant cycle homology is homotopy invariant:

Proposition *Suppose k admits resolution of singularities. Then the pull-back map*

$$p^* : A_{r,i}(Y, X) \to A_{r,i}(Y \times \mathbb{A}^1, X)$$

is an isomorphism.

Proof. Using hypercovers and resolution of singularities, we reduce to the case of smooth Y.

The cdh-Nis-Zar theorem changes the cdh hypercohomology defining $A_{r,i}$ to Nisnevich hypercohomology:

$$A_{r,i}(Y, X) = \mathbb{H}^i(Y_{\mathrm{Nis}}, C_*(z_r^{\mathrm{equi}}(X)_{\mathrm{Nis}}).$$

By the global PST theorem, the hypercohomology presheaves

$$Y \mapsto \mathbb{H}^i(Y_{\mathsf{Nis}}, C_*(z_r^{\mathsf{equi}}(X)_{\mathsf{Nis}}))$$

are homotopy invariant.

The geometric comparison theorem

Theorem (Geometric comparison) *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then the natural map $z^{\text{equi}}(X,*) \to z_r(X,*)$ is a quasi-isomorphism.*

This is based on *Suslin's moving lemma*, a purely algebro-geometric construction, in case X is affine. In addition, one needs to use the cdh techniques to prove a Meyer-Vietoris property for the complexes $z^{\text{equi}}(X,*)$ (we'll see how this works a bit later).

The geometric duality theorem

Let $z_r^{\text{equi}}(Z, X) := \mathcal{H}om(L(Z), z_r^{\text{equi}}(X))$. Explicitly:

$$z_r^{\text{equi}}(Z, X)(U) = z_r^{\text{equi}}(X)(Z \times U).$$

We have the inclusion $z_r^{\text{equi}}(Z, X) \to z_{r+\dim Z}^{\text{equi}}(X \times Z)$.

Theorem (Geometric duality) *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$, quasi-projective of dimension n.*
The inclusion $z_r^{\text{equi}}(U, X) \to z_{r+n}^{\text{equi}}(X \times U)$ induces a quasi-isomorphism of complexes on $\mathbf{Sm}/k_{\text{Zar}}$:

$$C_*(z_r^{\text{equi}}(U, X))_{\text{Zar}} \to C_*(z_{r+n}^{\text{equi}}(X \times U))_{\text{Zar}}$$

The proof for U and X smooth and projective uses the *Friedlander-Lawson moving lemma* for "moving cycles in a family". The extension to U smooth quasi-projective, and X general uses the cdh-acyclicity theorem.

The cdh comparison and duality theorems

Theorem (cdh **comparison)** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then for U smooth and quasi-projective, the natural map*

$$h_i(z_r^{\text{equi}}(X))(U) \to A_{r,i}(U, X)$$

is an isomorphism.

Theorem (cdh **duality)** *Suppose k admits resolution of singularities. Take $X, Y \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$ of dimension n. There is a canonical isomorphism*

$$A_{r,i}(Y \times U, X) \to A_{r+n,i}(Y, X \times U).$$

To prove the cdh comparison theorem, first use the cdh-Nis-Zar theorem to identify

$$\mathbb{H}^{-i}_{\mathsf{Zar}}(U, C_*(z_r^{\mathsf{equi}}(X))) \xrightarrow{\sim} \mathbb{H}^{-i}_{\mathsf{cdh}}(U, C_*(z_r^{\mathsf{equi}}(X))) =: A_{r,i}(U, X)$$

Next, if V_1, V_2 are Zariski open in U, use the geometric duality theorem to identify the Mayer-Vietoris sequence

$$C_*(z_r^{\mathsf{equi}}(X))(V_1 \cup V_2) \to C_*(z_r^{\mathsf{equi}}(X))(V_1) \oplus C_*(z_r^{\mathsf{equi}}(X))(V_2) \to C_*(z_r^{\mathsf{equi}}(X))(V_1 \cap V_2)$$

with what you get by applying $C_*(-)(\mathsf{Spec}\, k)$ to

$$0 \to z_{r+d}^{\mathsf{equi}}(X \times (V_1 \cup V_2)) \to z_{r+d}^{\mathsf{equi}}(X \times V_1) \oplus z_{r+d}^{\mathsf{equi}}(X \times V_2) \to z_{r+d}^{\mathsf{equi}}(X \times (V_1 \cap V_2))$$

$d = \dim U$.

But this presheaf sequence is exact, and $coker_{\mathsf{cdh}} = 0$. The cdh-acyclicity theorem thus gives us the distinguished triangle

$$
\begin{aligned}
&C_*(z_{r+d}^{\mathsf{equi}}(X \times (V_1 \cup V_2)))_{\mathsf{Zar}} \\
&\qquad \to C_*(z_{r+d}^{\mathsf{equi}}(X \times V_1))_{\mathsf{Zar}} \oplus C_*(z_{r+d}^{\mathsf{equi}}(X \times V_2))_{\mathsf{Zar}} \\
&\qquad\qquad \to C_*(z_{r+d}^{\mathsf{equi}}(X \times (V_1 \cap V_2)))_{\mathsf{Zar}} \to
\end{aligned}
$$

Evaluating at Spec k, we find that our original Mayer-Vietoris sequence for $C_*(z_r^{\mathsf{equi}}(X))$ was in fact a distinguished triangle.

The Mayer-Vietoris property for $C_*(z_r^{\mathsf{equi}}(X))$ then formally implies that

$$h_i(C_*(z_r^{\mathsf{equi}}(X)))(U) \to \mathbb{H}^{-i}_{\mathsf{Zar}}(U, C_*(z_r^{\mathsf{equi}}(X))) = A_{r,i}(U, X)$$

is an isomorphism.

The proof of the cdh-duality theorem is similar, using the geometric duality theorem.

cdh-descent theorem

Theorem (cdh-descent) *Suppose k admits resolution of singularities. Take $Y \in \mathbf{Sch}_k$.*
(1) Let $U \cup V = X$ be a Zariski open cover of $X \in \mathbf{Sch}_k$. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, U \cap V) \to A_{r,i}(Y,U) \oplus A_{r,i}(Y,V) \\ \to A_{r,i}(Y,X) \to A_{r,i-1}(Y, U \cap V) \to \ldots$$

(2) Let $Z \subset X$ be a closed subset. There is a long exact sequence

$$\ldots \to A_{r,i}(Y,Z) \to A_{r,i}(Y,X) \to A_{r,i}(Y, X \setminus Z) \to A_{r,i-1}(Y,Z) \to \ldots$$

(3) Let $p \amalg i : X' \amalg F \to X$ be an abstract blow-up. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, p^{-1}(F)) \to A_{r,i}(Y,X') \oplus A_{r,i}(Y,F) \\ \to A_{r,i}(Y,X) \to A_{r,i-1}(Y, p^{-1}(F)) \to \ldots$$

Proof. For (1) and (3), the analogous properties are obvious in the "first variable", so the theorem follows from duality.

For (2), the presheaf sequence

$$0 \to z_r^{\mathsf{equi}}(Z) \to z_r^{\mathsf{equi}}(X) \to z_r^{\mathsf{equi}}(X \setminus U)$$

is exact and $coker_{\mathsf{cdh}} = 0$. The cdh-acyclicity theorem says that applying $C_*(-)_{\mathsf{cdh}}$ to the above sequence yields a distinguished triangle.

Localization for M^c_{gm}

Continuing the argument for (2), the cdh-Nis-Zar theorem shows that the sequence

$$0 \to C_*(z_r^{\mathrm{equi}}(Z))_{\mathrm{Nis}} \to C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{Nis}} \to C_*(z_r^{\mathrm{equi}}(X \setminus U))_{\mathrm{Nis}}$$

canonically defines a distinguished triangle in $DM^{\mathrm{eff}}_-(k)$. Taking $r = 0$ gives

Theorem (Localization) *Suppose k admits resolution of singularities. Let $i : Z \to X$ be a closed immersion in $\mathbf{Sch}_k$ with complement $j : U \to X$. Then there is a canonical distinguished triangle in $DM^{\mathrm{eff}}_-(k)$*

$$M^c_{\mathrm{gm}}(Z) \xrightarrow{i_*} M^c_{\mathrm{gm}}(X) \xrightarrow{j^*} M^c_{\mathrm{gm}}(U) \to M^c_{\mathrm{gm}}(Z)[1]$$

Corollary *Suppose k admits resolution of singularities. For each $X \in \mathbf{Sch}_k$, $M^c_{\mathrm{gm}}(X)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \subset DM^{\mathrm{eff}}_{-}(k)$.*

Proof. We proceed by induction on $\dim X$. First assume $X \in \mathbf{Sm}/k$. By resolution of singularities, we can find a smooth projective $\bar{X}$ containing X as a dense open subscheme. Since the complement $D := \bar{X} \setminus X$ has $\dim D <$ dim, $M^c_{\mathrm{gm}}(D)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

$M^c_{\mathrm{gm}}(\bar{X}) = M_{\mathrm{gm}}(\bar{X})$ since $\bar{X}$ is for proper. The localization distinguished triangle shows $M^c_{\mathrm{gm}}(X)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

For arbitrary X, take a stratification X_* of X by closed subschemes with $X_i \setminus X_{i-1}$. The localization triangle and the case of smooth X gives the result.

A computation

Proposition $M^c_{\mathrm{gm}}(\mathbb{A}^n) \cong \mathbb{Z}(n)[2n]$

Proof. For Z projective $M^c_{\mathrm{gm}}(Z) = M_{\mathrm{gm}}(Z)$. The localization sequence gives the distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{P}^{n-1}) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M^c_{\mathrm{gm}}(\mathbb{A}^n) \to M_{\mathrm{gm}}(\mathbb{P}^{n-1})[1]$$

Then use the projective bundle formula:

$$M_{\mathrm{gm}}(\mathbb{P}^n) = \oplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$
$$M_{\mathrm{gm}}(\mathbb{P}^{n-1}) = \oplus_{i=0}^{n-1} \mathbb{Z}(i)[2i].$$

Corollary (Duality) *For* $X, Y \in \mathbf{Sch}_k$, $n = \dim Y$ *we have a canonical isomorphism*

$$\mathrm{CH}_{r+n}(X \times Y, i) \cong A_{r,i}(Y, X)$$

Proof. For $U \in \mathbf{Sm}/k$, quasi-projective, we have the quasi-isomorphisms

$$C_*(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k) = z^{\mathrm{equi}}_{r+n}(X \times U, *) \to z_{r+n}(X \times U, *)$$

$$C_*(z^{\mathrm{equi}}_{r}(U, X))(\mathrm{Spec}\, k) \to C_*(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k)$$

and the isomorphisms

$$A_{r,i}(U, X) \to A_{r+n,i}(\mathrm{Spec}\, k, X \times U) \leftarrow h_i(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k)$$

This gives the isomorphism

$$\mathrm{CH}_{r+n}(X \times U, i) \to A_{r,i}(U, X).$$

One checks this map is natural with respect to the localization sequences for $\mathrm{CH}_{r+n}(X \times -, i)$ and $A_{r,i}(-, X)$.

Given $Y \in \mathbf{Sch}_k$, there is a filtration by closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset \ldots \subset Y_m = Y$$

with $Y_i \setminus Y_{i-1} \in \mathbf{Sm}/k$ and quasi-projective (k is perfect), so this extends the result from $U \in \mathbf{Sm}/k$, quasi-projective, to $Y \in \mathbf{Sch}_k$.

Corollary *Suppose k admits resolution of singularities. For $X, Y \in \mathbf{Sch}_k$ we have*
(1) (homotopy) *The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p^* : A_{r,i}(Y, X) \to A_{r+1,i}(Y, X \times \mathbb{A}^1)$.*

(2) (suspension) *The maps $i_0 : X \to X \times \mathbb{P}^1$, $p : X \times \mathbb{P}^1 \to X$ induce an isomorphism*

$$A_{r,i}(Y, X) \oplus A_{r-1,i}(Y, X) \xrightarrow{i_* + p^*} A_{r,i}(Y, X \times \mathbb{P}^1)$$

(3)(cosuspension) *There is a canonical isomorphism*

$$A_{r,i}(Y \times \mathbb{P}^1, X) \cong A_{r,i}(Y, X) \oplus A_{r+1,i}(Y, X)$$

(4) (localization) *Let $i : Z \to U$ be a codimension n closed embedding in $\mathbf{Sm}/k$. Then there is a long exact sequence*

$$\ldots \to A_{r+n,i}(Z, X) \to A_{r,i}(U, X) \xrightarrow{j^*} A_{r,i}(U \setminus Z, X) \to A_{r+n,i-1}(Z, X) \to \ldots$$

Proof. These all follow from the corresponding properties of $\mathrm{CH}^*(-, *)$ and the duality corollary:

(1) from homotopy

(2) and (3) from the projective bundle formula

(4) from the localization sequence.

Morphisms and cycles

We describe how morphisms in $DM^{\text{eff}}_{\text{gm}}(k)$ can be realized as algebraic cycles.

We assume throughout that k admits resolution of singularities.

Bivariant cycle cohomology reappears The cdh-acyclicity theorem relates the bivariant cycle cohomology (and hence higher Chow groups) with the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Theorem *For* $X, Y \in \mathbf{Sch}_k$ $r \geq 0$, $i \in \mathbb{Z}$, *there is a canonical isomorphism*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{-}(k)}(M_{\mathrm{gm}}(Y)(r)[2r+i], M^c_{\mathrm{gm}}(X)) \cong A_{r,i}(Y, X).$$

Proof. First use cdh hypercovers to reduce to $Y \in \mathbf{Sm}/k$.

For $r = 0$, the embedding theorem and localization theorem, together with the cdh-Nis-Zar theorem gives an isomorphism

$$\mathrm{Hom}_{DM_{-}^{\mathrm{eff}}(k)}(C_*(Y)[i], C_*^c(X)) \cong \mathbb{H}^{-i}(Y_{\mathrm{Nis}}, C_*(z_0^{\mathrm{equi}}(X))_{\mathrm{Nis}})$$
$$\cong \mathbb{H}^{-i}(Y_{\mathrm{cdh}}, C_*(z_0^{\mathrm{equi}}(X))_{\mathrm{cdh}}) = A_{0,i}(Y, X).$$

To go to $r > 0$, use the case $r = 0$ for $Y \times (\mathbb{P}^1)^r$:

$$\mathrm{Hom}_{DM_{-}^{\mathrm{eff}}(k)}(C_*(Y \times (\mathbb{P}^1)^r)[i], C_*^c(X)) \cong A_{0,i}(Y \times (\mathbb{P}^1)^r, X).$$

By the cosuspension isomorphism $A_{r,i}(Y, X)$ is a summand of $A_{0,i}(Y \times (\mathbb{P}^1)^r, X)$; by the definition of $\mathbb{Z}(1)$, $M_{\mathrm{gm}}(Y)(r)[2r]$ is a summand of $M_{\mathrm{gm}}(Y \times (\mathbb{P}^1)^r)$. One checks the two summands match up.

Effective Chow motives

Corollary *Sending a smooth projective variety* X *of dimension* n *to* $M_{\mathrm{gm}}(X)$ *extends to a full embedding* $i : CHM^{\mathrm{eff}}(k)^{\mathrm{op}} \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, $CHM^{\mathrm{eff}}(k) :=$ *effective Chow motives,*

$$i(\mathfrak{h}(X)(-r)) = M_{\mathrm{gm}}(X)(r)$$

Proof. For X and Y smooth and projective

$$\begin{aligned}
\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)) &= A_{0,0}(Y, X) \\
&\cong A_{\dim Y,0}(\mathrm{Spec}\, k, Y \times X) \\
&\cong \mathrm{CH}_{\dim Y}(Y \times X) \\
&\cong \mathrm{CH}^{\dim X}(X \times Y) \\
&= \mathrm{Hom}_{CHM^{\mathrm{eff}}(k)}(X, Y).
\end{aligned}$$

One checks that sending $a \in \mathsf{CH}^{\dim X}(X \times Y)$ to the corresponding map

$$[{}^t a] : M_{\mathsf{gm}}(Y) \to M_{\mathsf{gm}}(X)$$

satisfies $[{}^t(b \circ a)] = [{}^t a] \circ [{}^t b]$.

The Chow ring reappears

Corollary *For* $Y \in \mathbf{Sch}_k$, *equi-dimensional over* k, $i \geq 0$, $j \in \mathbb{Z}$, $\mathrm{CH}^i(Y, j) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), \mathbb{Z}(i)[2i-j])$. *That is*

$$\mathrm{CH}^i(Y, j) \cong H^{2i-j}(Y, \mathbb{Z}(i)).$$

Take $i \geq 0$. Then $M^c_{\mathrm{gm}}(\mathbb{A}^i) \cong \mathbb{Z}(i)[2i]$ and

$$\begin{aligned} H^{2i-j}(Y, \mathbb{Z}(i)) &= \mathrm{Hom}_{DM^{\mathrm{eff}}_{-}(k)}(M_{\mathrm{gm}}(Y)[j], M^c_{\mathrm{gm}}(\mathbb{A}^i)) \\ &\cong A_{0,j}(Y, \mathbb{A}^i) \\ &\cong A_{\dim Y, j}(\mathrm{Spec}\, k, Y \times \mathbb{A}^i) \\ &= \mathrm{CH}_{\dim Y}(Y \times \mathbb{A}^i, j) \\ &= \mathrm{CH}^i(Y \times \mathbb{A}^i, j) \\ &\cong \mathrm{CH}^i(Y, j) \end{aligned}$$

Remark Combining the Chern character isomorphism

$$ch : K_j(Y)^{(i)} \cong \mathsf{CH}^i(Y,j)_{\mathbb{Q}}$$

(for $Y \in \mathbf{Sm}/k$) with our isomorphism $\mathsf{CH}^i(Y,j) \cong H^{2i-j}(Y,\mathbb{Z}(i))$ identifies rational motivic cohomology with weight-graded K-theory:

$$H^{2i-j}(Y,\mathbb{Q}(i)) \cong K_j(Y)^{(i)}.$$

Thus motivic cohomology gives an integral version of weight-graded K-theory, in accordance with conjectures of Beilinson on mixed motives.

Corollary (cancellation) *For $A, B \in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ the map*

$$- \otimes \mathrm{id} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A(1), B(1))$$

is an isomorphism. Thus

$$DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_{\mathrm{gm}}(k)$$

is a full embedding.

Corollary *For $Y \in \mathbf{Sch}_k$, $n, i \in \mathbb{Z}$, set*

$$H^n(Y, \mathbb{Z}(i)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), \mathbb{Z}(i)[n]).$$

Then $H^n(Y, \mathbb{Z}(i)) = 0$ for $i < 0$ and for $n > 2i$.

Corollary *The full embedding* $CHM^{\mathsf{eff}}(k)^{\mathsf{op}} \to DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$ *extends to a full embedding*

$$M_{\mathsf{gm}} : CHM(k)^{\mathsf{op}} \to DM_{\mathsf{gm}}(k).$$

Proof of the cancellation theorem.

The Gysin distinguished triangle for for M_{gm} shows that $DM^{\mathsf{eff}}_{\mathsf{gm}}(k)$ is generated by $M_{\mathsf{gm}}(X)$, X smooth and projective. So, we may assume $A = M_{\mathsf{gm}}(Y)[i]$, $B = M_{\mathsf{gm}}(X)$, X and Y smooth and projective, $i \in \mathbb{Z}$.

Then $M_{\mathsf{gm}}(X) = M^c_{\mathsf{gm}}(X)$ and $M_{\mathsf{gm}}(X)(1)[2] = M^c_{\mathsf{gm}}(X \times \mathbb{A}^1)$. Thus:

$$\begin{aligned}
\mathrm{Hom}(M_{\mathsf{gm}}(Y)(1)[i], M_{\mathsf{gm}}(X)(1)) &\cong A_{1,i}(Y, X \times \mathbb{A}^1) \\
&\cong A_{0,i}(Y, X) \\
&\cong \mathrm{Hom}(M_{\mathsf{gm}}(Y)[i], M_{\mathsf{gm}}(X))
\end{aligned}$$

For the second corollary, supposes $i < 0$. Cancellation implies

$$\begin{aligned} H^{2i-j}(Y, \mathbb{Z}(i)) &= \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y)(-i)[j-2i], \mathbb{Z}) \\ &= A_{-i,j}(Y, \mathrm{Spec}\, k) \\ &= A_{\dim Y - i, j}(\mathrm{Spec}\, k, Y) \\ &= H^{-j}(C_*(z^{\mathrm{equi}}_{\dim Y - i}(Y))(\mathrm{Spec}\, k)). \end{aligned}$$

Since $\dim Y - i > \dim Y$, $z^{\mathrm{equi}}_{\dim Y - i}(Y) = 0$.

If $i \geq 0$ but $n > 2i$, then $H^n(Y, \mathbb{Z}(i)) = \mathrm{CH}^i(Y, 2i - n) = 0$.

Duality

We describe the duality involution

$$* : DM_{\mathsf{gm}}(k) \to DM_{\mathsf{gm}}(k)^{\mathsf{op}},$$

assuming k admits resolution of singularities.

A reduction

Proposition *Let $\mathcal{D}$ be a tensor triangulated category, $\mathcal{S}$ a subset of the objects of $\mathcal{D}$. Suppose*

1. Each $M \in \mathcal{S}$ has a dual M^*.

2. $\mathcal{D}$ is equal to the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$ and closed under isomorphisms in $\mathcal{D}$.

Then each object in $\mathcal{D}$ has a dual, i.e. $\mathcal{D}$ is a rigid tensor triangulated category.

Idea of proof For $M \in \mathcal{S}$, we have the unit and trace

$$\delta_M : \mathbb{1} \to M^* \otimes M,\ \epsilon_M : M \otimes M^* \to \mathbb{1}$$

satisfying

$$(\epsilon \otimes \mathrm{id}_M) \circ (\mathrm{id}_M \otimes \delta) = \mathrm{id}_M,\ (\mathrm{id}_{M^*} \otimes \epsilon) \circ (\delta \otimes \mathrm{id}_{M^*}) = \mathrm{id}_{M^*}$$

Show that, if you have such δ, ϵ for M_1, M_2 in a distinguished triangle

$$M_1 \xrightarrow{a} M_2 \to M_3 \to M_1[1]$$

you can construct δ_3, ϵ_3 with M_3^* fitting in a distinguished triangle

$$M_3^* \to M_2^* \xrightarrow{a^*} M_1^* \to M_3^*[1]$$

Duality for X projective

Proposition *For $X \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$, $M_{\mathrm{gm}}(X)(r) \in DM_{\mathrm{gm}}(k)$ has a dual $(M_{\mathrm{gm}}(X)(r))^*$.*

We use the full embedding $CHM(k)^{\mathrm{op}} \hookrightarrow DM_{\mathrm{gm}}(k)$ sending $\mathfrak{h}(X)(-r)$ to $M_{\mathrm{gm}}(X)(r)$, and the fact that $\mathfrak{h}(X)(-r)$ has a dual in $CHM(k)$.

Proposition *Suppose* k *admits resolution of singularities. Then* $DM_{\mathsf{gm}}(k)$ *is the smallest full triangulated subcategory of* $DM_{\mathsf{gm}}(k)$ *containing the* $M_{\mathsf{gm}}(Y)(r)$ *for* $Y \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$ *and closed under isomorphisms in* $DM_{\mathsf{gm}}(k)$.

Proof. Take $X \in \mathbf{Sm}/k$. By resolution of singularities, there is a smooth projective $\bar{X}$ containing X as a dense open subscheme, such that $D := \bar{X} - X$ is a strict normal crossing divisor:

$$D = \cup_{i=1}^{m} D_i$$

with each D_i smooth codimension one on $\bar{X}$ and each intersection: $I = \{i_1, \ldots, i_r\}$

$$D_I := D_{i_1} \cap \ldots \cap D_{i_r}$$

is smooth of codimension r.

Then $\bar{X}$ and each $D_{i_1} \cap \ldots \cap D_{i_r}$ is in $\mathbf{SmProj}/k$. So $M_{\mathsf{gm}} = M^c_{\mathsf{gm}}$ for all these.

The Gysin triangle for $W \subset Y$ both smooth, $n = \mathrm{codim}_Y W$,

$$M_{\mathsf{gm}}(Y \setminus W) \to M_{\mathsf{gm}}(Y) \to M_{\mathsf{gm}}(W)(n)[2n] \to M_{\mathsf{gm}}(Y \setminus W)[1],$$

and induction on $\dim X$ and descending induction on r shows that

$$M_{\mathrm{gm}}(\bar{X} \setminus \cup_{|I|=r} D_I)$$

is in the category generated by the $M_{\mathrm{gm}}(Y)(r)$, $Y \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$.

Theorem *Suppose* k *admits resolution of singularities. Then* $DM_{\mathsf{gm}}(k)$ *is a rigid tensor triangulated category.*

Note. In fact, one can show that (after embedding in $DM_{-}^{\mathsf{eff}}(k)$)

$$M_{\mathsf{gm}}(X)^* = M_{\mathsf{gm}}^c(X)(-d_X)[-2d_X]$$

The End

Thank you!