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Pure motives, II

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Pure motives: Part II

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Outline:

- Standard conjectures
- Decompositions of the diagonal
- Filtrations on the Chow ring
- Nilpotence conjecture
- Finite dimensionality

The standard conjectures

We would like to think of our functor

$$\mathfrak{h} : \mathbf{SmProj}/k^{\mathsf{op}} \to M_{\mathsf{hom}}(k)$$

as the “universal Weil cohomology”. What is lacking:

• We have the “total cohomology” $\mathfrak{h}(X)$, we would like the individual cohomologies $\mathfrak{h}^r(X)$.

• Other “higher level” properties of cohomology are missing, e.g., Lefschetz theorems.

• $\sim_{\mathsf{hom}}$ could depend on the choice of Weil cohomology.

• $M_{\mathsf{hom}}(k)$ is not a category of vector spaces, but it is at least pseudo-abelian. It would be nice if it were an abelian category.

Künneth projectors

Fix a Weil cohomology H^* and an $X \in \mathbf{SmProj}/k$. By the Künneth formula, we have

$$H^*(X \times X) = H^*(X) \otimes H^*(X)$$

so

$$H^{2d_X}(X \times X)(d_X) = \oplus_{n=0}^{2d_X} H^n(X) \otimes H^{2d_X-n}(X)(d_X)$$

By Poincaré duality, $H^{2d_X-n}(X)(d_X) = H^n(X)^\vee$, so

$$\begin{aligned} H^{2d_X}(X \times X)(d_X) &= \oplus_{n=0}^{2d_X} H^n(X) \otimes H^n(X)^\vee \\ &= \oplus_{n=0}^{2d_X} \mathrm{Hom}_K(H^n(X), H^n(X)). \end{aligned}$$

$$
\begin{aligned}
H^{2d_X}(X \times X)(d_X) &= \oplus_{n=0}^{2d_X} H^n(X) \otimes H^n(X)^\vee \\
&= \oplus_{n=0}^{2d_X} \mathrm{Hom}_K(H^n(X), H^n(X)).
\end{aligned}
$$

This identifies $H^{2d_X}(X \times X)(d_X)$ with the vector space of graded K-linear maps $f : H^*(X) \to H^*(X)$ and writes

$$
\mathrm{id}_{H^*(X)} = \sum_{n=0}^{2d_X} \pi^n_{X,H}; \quad \pi^n_H \in H^n(X) \otimes H^n(X)^\vee.
$$

The term

$$
\pi^n_{X,H} : H^*(X) \to H^*(X)
$$

is the projection on $H^n(X)$, called the *Künneth projector*

Since $\mathrm{id}_{\mathfrak{h}_{\mathrm{hom}}(X)}$ is represented by the diagonal $\Delta_X \in \mathcal{Z}^{d_X}(X \times X)$, we have

$$\gamma_{X,H}(\Delta_*) = \mathrm{id}_{H^*(X)} = \sum_n \pi^n_{X,H}$$

We can ask: are there correspondences $\pi^n_X \in \mathcal{Z}^{d_X}_{\mathrm{hom}}(X \times X)_{\mathbb{Q}}$ with

$$\gamma_{X,H}(\pi^n_X) = \pi^n_{X,H}.$$

Remarks 1. The $\pi^n_{X,H}$ are idempotent endomorphisms $\Longrightarrow$ (X, π^n_X) defines a summand $\mathfrak{h}^n(X)$ of $\mathfrak{h}(X)$ in $M^{\mathrm{eff}}_{\mathrm{hom}}(k)_{\mathbb{Q}}$.

2. If π^n_X exists, it is unique.

3. π^n_X exists iff $\mathfrak{h}_{\mathrm{hom}}(X) = \mathfrak{h}^n(X) \oplus \mathfrak{h}(X)'$ in $M^{\mathrm{eff}}(k)_{\mathbb{Q}}$ with $H^*(\mathfrak{h}^n(X)) \subset H^*(X)$ equal to $H^n(X)$.

If all the π_X^n exist:

$$\mathfrak{h}_{\mathsf{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}_{\mathsf{hom}}^n(X)$$

X has a *Künneth decomposition*.

Examples

1. The decomposition

$$\mathfrak{h}(\mathbb{P}^n) = \oplus_{r=0}^{n} \mathfrak{h}^{2r}(\mathbb{P}^n)$$

in $CHM^{\mathsf{eff}}(k)$ maps to a Künneth decomposition of $\mathfrak{h}_{\mathsf{hom}}(\mathbb{P}^n)$.

2. For a curve C, the decomposition (depending on a choice of $0 \in C(k)$)

$$\mathfrak{h}(C) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C); \quad \mathfrak{h}^0(C) \cong \mathbb{1}, \mathfrak{h}^2(C) \cong \mathbb{1}(-1),$$

in $CHM^{\text{eff}}(k)$ maps to a Künneth decomposition of $\mathfrak{h}_{\text{hom}}(C)$.

3. For each $X \in \mathbf{SmProj}/k$, a choice of a k-point gives factors

$$\begin{aligned}\mathfrak{h}^0(X) &:= (X, 0 \times X) \cong \mathbb{1} \\ \mathfrak{h}^{2d_X}(X) &:= (X, X \times 0) \cong \mathbb{1}(-d_X).\end{aligned}$$

of $\mathfrak{h}(X)$. Using the Picard and Albanese varieties of X, one can also define factors $\mathfrak{h}^1(X)$ and $\mathfrak{h}^{2d-1}(X)$, so

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \mathfrak{h}(X)' \oplus \mathfrak{h}^{2d_X-1}(X) \oplus \mathfrak{h}^{2d_X}(X)$$

which maps to a partial Künneth decomposition in $M^{\text{eff}}_{\text{hom}}(k)_{\mathbb{Q}}$. For $d_X = 2$, this gives a full Künneth decomposition (Murre).

The Künneth conjecture

Conjecture (C(X)) *The Künneth projectors* $\pi^n_{X,H}$ *are algebraic for all* n*:*

$$\mathfrak{h}_{\mathrm{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n_{\mathrm{hom}}(X)$$

with $H^*(\mathfrak{h}^n_{\mathrm{hom}}(X)) = H^n(X) \subset H^*(X)$*.*

Consequence *Let* $a \in \mathcal{Z}^{d_X}(X \times X)_{\mathbb{Q}}$ *be a correspondence.*

1. The characteristic polynomial of $H^n(a)$ *on* $H^n(X)$ *has* $\mathbb{Q}$*-coefficients.*

2. If $H^n(a) : H^n(X) \to H^n(X)$ *is an automorphism, then* $H^n(a)^{-1} = H^*(b)$ *for some correspondence* $b \in \mathcal{Z}^{d_X}(X \times X)_{\mathbb{Q}}$*.*

Proof. (1) The Lefschetz trace formula gives

$$Tr(a^m)_{|H^n(X)} = (-1)^n \deg({}^t a^m \cdot \pi_X^n) \in \mathbb{Q}.$$

But

$$\det(1 - ta_{|H^n(X)}) = \exp(-\sum_{m=1}^{\infty} \frac{1}{m} Tr(a^m_{H^n(X)}) t^m).$$

(2) By Cayley-Hamilton and (1), there is a $Q_n(t) \in \mathbb{Q}[t]$ with

$$\begin{aligned} H^n(a)^{-1} &= Q_n(H^n(a)) \\ &= H^n(Q_n(a)) \\ &= H^*(Q_n(a)\pi_X^n) \end{aligned}$$

Status: $C(X)$ is known for "geometrically cellular" varieties ($\mathbb{P}^n$, Grassmannians, flag varieties, quadrics, etc.), curves, surfaces and abelian varieties: For an abelian variety A, one has

$$\mathfrak{h}^n_{\mathrm{hom}}(A) = \Lambda^n(\mathfrak{h}^1_{\mathrm{hom}}(A)).$$

$C(X)$ is true for all X if the base-field k is a finite field $\mathbb{F}_q$ and $H^* = H^*_{\text{ét}}(-, \mathbb{Q}_\ell)$:

Use the Weil conjectures to show that the characteristic polynomial $P_n(t)$ of Fr_X on $H^n(X, \mathbb{Q}_\ell)$ has $\mathbb{Q}$-coefficients and that $P_n(t)$ and $P_m(t)$ are relatively prime for $n \neq m$. Cayley-Hamilton and the Chinese remainder theorem yield polynomials $Q_n(t)$ with $\mathbb{Q}$-coefficients and

$$Q_n(Fr^*_X)_{|H^m(X)} = \delta_{n,m}\mathrm{id}_{H^m(X)}.$$

Then $\pi^n_X = Q_n({}^t\Gamma_{Fr_X})$.

The sign conjecture $C^+(X)$

This is a weak version of $C(X)$, saying that $\pi^+_{X,H} := \sum_{n=0}^{d_X} \pi^{2n}_{X,H}$ is algebraic. Equivalently, $\pi^-_{X,H} := \sum_{n=1}^{d_X} \pi^{2n-1}_{X,H}$ is algebraic.

$C^+(X)$ for all X/k says that we can impose a $\mathbb{Z}/2$-grading on $M_{\text{hom}}(k)_{\mathbb{Q}}$:

$$\mathfrak{h}_{\text{hom}}(X) = \mathfrak{h}^+_{\text{hom}}(X) \oplus \mathfrak{h}^-_{\text{hom}}(X)$$

so that $H^* : M_{\text{hom}}(k)_{\mathbb{Q}} \to \text{GrVec}_K$ defines

$$H^{\pm} : M_{\text{hom}}(k)_{\mathbb{Q}} \to s\text{Vec}_K$$

respecting the $\mathbb{Z}/2$ grading, where $s\text{Vec}_K$ the tensor category of finite dimensional $\mathbb{Z}/2$-graded K vector spaces.

Consequence *Suppose* $C^+(X)$ *for all* $X \in \mathbf{SmProj}/k$*. Then*

$$M_{\text{hom}}(k)_{\mathbb{Q}} \to M_{\text{num}}(k)_{\mathbb{Q}}$$

is conservative and essentially surjective.

This follows from:

Lemma $C^+(X) \Longrightarrow$
the kernel of $\mathcal{Z}^{d_X}_{\text{hom}}(X \times X)_{\mathbb{Q}} \to \mathcal{Z}^{d_X}_{\text{num}}(X \times X)_{\mathbb{Q}}$ *is a nil-ideal, hence* ker $\subset \mathcal{R}$.

Proof. For $f \in ker$, $\deg(f^n \cdot \pi^+_X) = \deg(f^n \cdot \pi^-_X) = 0$. By Lefschetz

$$Tr(\gamma(f^n)_{|H^+(X)}) = Tr(\gamma(f^n)_{|H^-(X)}) = 0$$

Thus $\gamma(f)_{|H^*(X)}$ has characteristic polynomial t^N, $N = \dim H^*(X)$.
□

Remark. André and Kahn use the fact that the kernel of $M_{\text{hom}}(k)_{\mathbb{Q}} \to M_{\text{num}}(k)_{\mathbb{Q}}$ is a $\otimes$ nilpotent ideal to define a canonical $\otimes$ functor $M_{\text{num}}(k)_{\mathbb{Q}} \to M_{\text{hom}}(k)_{\mathbb{Q}}$. This allows one to define the "homological realization" for $M_{\text{num}}(k)_{\mathbb{Q}}$.

The Lefschetz theorem

Take a smooth projective X over k with an embedding $X \subset \mathbb{P}^N$. Let $i : Y \hookrightarrow X$ be a smooth hyperplane section.

For a Weil cohomology H^*, this gives the operator

$$L : H^*(X) \to H^{*-2}(X)(-1)$$
$$L(x) := i_*(i^*(x)) = \gamma([Y]) \cup x.$$

L lifts to the correspondence $Y \times X \subset X \times X$.

The strong Leschetz theorem is

Theorem *For H^* a "classical" Weil cohomology and $i \leq d_X$*

$$L^{d_X - i} : H^i(X) \to H^{2d_X - i}(X)(d_X - i)$$

is an isomorphism.

The conjecture of Lefschetz type

Let $*_{L,X}$ be the involution of $\oplus_{i,r} H^i(X)(r)$:

$$*_{L,X} \text{ on } H^i(X)(r) := \begin{cases} L^{d_X - i} & \text{for } 0 \le i \le d_X \\ (L^{i-d_X})^{-1} & \text{for } d_X < i \le 2d_X. \end{cases}$$

Conjecture (B(X)) *The Lefschetz involution $*_{L,X}$ is algebraic: there is a correspondence $\alpha_{L,X} \in \mathcal{Z}^*(X \times X)_{\mathbb{Q}}$ with $\gamma(\alpha) = *_{L.X}$*

Status

$B(X)$ is known for curves, and for abelian varieties (Kleiman-Grothendieck). For abelian varieties Lieberman showed that the operator Λ (related to the inverse of L) is given by Pontryagin product (translation) with a rational multiple of $Y^{(d-1)}$.

Homological and numerical equivalence

Conjecture (D(X)) $\mathcal{Z}^*_{\text{hom}}(X)_{\mathbb{Q}} = \mathcal{Z}^*_{\text{num}}(X)_{\mathbb{Q}}$

Proposition *For $X \in \mathbf{SmProj}/k$, $D(X^2) \Longrightarrow \text{End}_{M_{\text{hom}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X))$ is semi-simple.*

$D(X^2) \Longrightarrow \text{End}_{M_{\text{hom}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X)) = \text{End}_{M_{\text{num}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X))$, which is semi-simple by Jannsen's theorem.

Similarly, Jannsen's theorem shows:

Proposition *If $D(X)$ is true for all $X \in \mathbf{SmProj}/k$, then H^* : $M_{\text{hom}}(k)_F \to \text{GrVec}_K$ is conservative and exact.*

In fact: $D(X^2) \Longrightarrow B(X) \Longrightarrow C(X)$.

Thus, if we know that hom = num (with $\mathbb{Q}$-coefficients) we have our universal cohomology of smooth projective varieties

$$\mathfrak{h} = \oplus_i \mathfrak{h}^i : \mathbf{SmProj}(k)^{\mathrm{op}} \to NM(k)_{\mathbb{Q}}$$

with values in the semi-simple abelian category $NM(k)_{\mathbb{Q}}$.

Also, for H^* = Betti cohomology, $B(X) \Longrightarrow D(X)$, so it would suffice to prove the conjecture of Lefschetz type.

$D(X)$ is known in codimension $0, d_X$ and for codimension 1 (Matsusaka's thm). In characteristic 0, also for codimension $2, d_X - 1$ and for abelian varieties (Lieberman).

Decompositions of the diagonal

We look at analogs of the Künneth projectors for $CHM(k)_{\mathbb{Q}}$.

First look at two basic properties of the Chow groups.

Localization

Theorem *Let* $i : W \to X$ *be a closed immersion,* $j : U \to X$ *the complement. Then*

$$\mathsf{CH}_r(W) \xrightarrow{i_*} \mathsf{CH}_r(X) \xrightarrow{j^*} \mathsf{CH}_r(U) \to 0$$

is exact.

Proof.

$$0 \to \mathcal{Z}_r(W) \xrightarrow{i_*} \mathcal{Z}_r(X) \xrightarrow{j^*} \mathcal{Z}_r(U) \to 0$$

is exact: Look at the basis given by subvarieties. At $\mathcal{Z}_r(U)$ take the closure to lift to $\mathcal{Z}_r(X)$. At $\mathcal{Z}_r(X)$ $j^{-1}(Z) = \emptyset$ means $Z \subset W$.

Do the same for $W \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$ and use the snake lemma. □

Continuity

Proposition *Let $t : \mathrm{Spec}\,(L) \to T$ be a geometric generic point and take $X \in \mathbf{Sch}_k$ equi-dimensional. If $\eta \in \mathrm{CH}^r(X \times T)_{\mathbb{Q}} \mapsto 0 \in \mathrm{CH}^r(X_t)_{\mathbb{Q}}$, then there is a Zariski open subset of T containing the image of t such that $\eta \mapsto 0 \in \mathrm{CH}^r(X \times U)$.*

$\eta_t = 0 \Rightarrow \eta_K = 0$ for some $K/k(T)$ finite, Galois.

But $\mathrm{CH}^r(X_K)_{\mathbb{Q}}^{\mathrm{Gal}} = \mathrm{CH}^r(X_{k(X)})_{\mathbb{Q}} \Rightarrow \eta_{k(X)} = 0 \in \mathrm{CH}^r(X_{k(X)})_{\mathbb{Q}}$.

But $\mathrm{CH}^r(X_{k(X)}) = \lim_{\emptyset \neq U \subset T} \mathrm{CH}^r(X \times U)$.

Note. This result is *false* for other $\sim$, e.g. $\sim_{\mathrm{hom}}$, $\sim_{\mathrm{alg}}$.

The first component

Proposition (Bloch) $X \in \mathbf{SmProj}/k$. *Suppose* $\mathrm{CH}_0(X_{\bar{L}})_{\mathbb{Q}} = \mathbb{Q}$ *(by degree) for all finitely generated field extensions* $L \supset k$. *Then*

$$\Delta_X \sim_{\mathrm{rat}} X \times 0 + \rho$$

with $\rho \in \mathcal{Z}^{d_X}(X \times X)$ *supported in* $D \times X$ *for some divisor* $D \subset X$ *and* $0 \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ *any degree 1 cycle.*

Proof. Let $i : \eta \to X$ be a geometric generic point. Then $i^*(X \times 0)$ and $i^*(\Delta_X)$ are in $\mathrm{CH}_0(X_{k(\eta)})$ and both have degree 1. Thus

$$(i \times \mathrm{id})^*(X \times 0) = (i \times \mathrm{id})^*(\Delta_X) \text{ in } \mathrm{CH}_0(X_{k(\eta)})_{\mathbb{Q}}$$

By continuity, there is a dense open subscheme $j : U \hookrightarrow X$ with

$$(j \times \mathrm{id})^*(X \times 0) = (j \times \mathrm{id})^*(\Delta_X^*) \text{ in } \mathrm{CH}_0(U \times X)_{\mathbb{Q}}$$

By localization there is a $\tau \in \mathcal{Z}_{d_X}(D \times X)$ for $D = X \setminus U$ with

$$\Delta_X - X \times 0 = (i_{D*} \times \mathrm{id})_*(\tau) =: \rho.$$

Mumford's theorem Take $k = \bar{k}$. Each X in $\mathbf{SmProj}/k$ has an associated *Albanese variety* $\mathrm{Alb}(X)$. A choice of $0 \in X(k)$ gives a morphism $\alpha_X : X \to \mathrm{Alb}(X)$ sending 0 to 0, which is universal for pointed morphisms to abelian varieties.

Extending by linearity and noting $\mathrm{Alb}(X \times \mathbb{P}^1) = \mathrm{Alb}(X)$ gives a canonical map

$$\alpha_X : \mathrm{CH}_0(X)_{\deg 0} \to \mathrm{Alb}(X)$$

Theorem (Mumford) *X: smooth projective surface over $\mathbb{C}$. If $H^0(X, \Omega^2) \neq 0$, then the Albanese map $\alpha_X : \mathrm{CH}_0(X)_{\deg 0} \to \mathrm{Alb}(X)$ has "infinite dimensional" kernel.*

Here is Bloch's motivic proof (we simplify: assume $\mathrm{Alb}(X) = 0$, and show only that $\mathrm{CH}_0(X)_{\mathbb{Q}}$ is not $\mathbb{Q}$).

Since $\mathbb{C}$ has infinite transcendence degree over $\mathbb{Q}$, $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$ implies $\mathrm{CH}_0(X_{\bar{L}})_{\mathbb{Q}} = \mathbb{Q}$ for all finitely generated fields $L/\mathbb{C}$.

Apply Bloch's decomposition theorem: $\Delta_X \sim_{\mathrm{rat}} X \times 0 + \rho$. Since

$$H^0(X, \Omega^2) = H^0(X \times \mathbb{P}^1, \Omega^2)$$

$\Delta_{X*} = (X \times 0)_* + \rho_*$ on 2-forms.

If $\omega \in H^0(X, \Omega^2)$ is a two form, then

$$\omega = \Delta_*(\omega) = (X \times 0)_*(\omega) + \rho_*(\omega) = 0 :$$

$(X \times 0)_*(\omega)$ is 0 on $X \setminus \{0\}$. $\rho_*(\omega)$ factors through the restriction $\omega_{|D}$. D is a curve, so $\omega_{|D} = 0$.

Jannsen's surjectivity theorem

Theorem (Jannsen) *Take* $X \in \mathbf{SmProj}/\mathbb{C}$*. Suppose the cycle-class map*

$$\gamma^r : \mathrm{CH}^r(X)_{\mathbb{Q}} \to H^{2r}(X(\mathbb{C}), \mathbb{Q})$$

is injective for all r*. Then* $\gamma^* : \mathrm{CH}^*(X) \to H^*(X, \mathbb{Q})$ *is surjective, in particular* $H^{odd}(X, \mathbb{Q}) = 0$*.*

Corollary *If* $\gamma^* : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^*(X(\mathbb{C}), \mathbb{Q})$ *is injective, then the Hodge spaces* $H^{p,q}(X)$ *vanish for* $p \neq q$*.*

Compare with Mumford's theorem: if X is a surface and $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$, then $H^{2,0}(X) = H^{0,2}(X) = 0$.

Note. The proof shows that the injectivity assumption yields a full decomposition of the diagonal

$$\Delta_X = \sum_{i=0}^{d_X} \sum_{j=1}^{n_i} a^{ij} \times b_{ij} \text{ in } \mathrm{CH}^{d_X}(X \times X)_{\mathbb{Q}}$$

with $a^{ij} \in \mathcal{Z}^i(X)_{\mathbb{Q}}$, $b_{ij} \in \mathcal{Z}_i(X)_{\mathbb{Q}}$. Applying Δ_{X*} to a cohomology class $\eta \in H^r(X, \mathbb{Q})$ gives

$$\eta = \Delta_{X*}(\eta) = \sum_{ij} Tr(\eta \cup \gamma(a^{ij})) \times \gamma(b_{ij})$$

This is 0 if r is odd, and is in the $\mathbb{Q}$-span of the $\gamma(b_{ij})$ for $r = 2d_X - 2i$.

Conversley, a decomposition of Δ_X as above yields

$$\mathfrak{h}(X)_{\mathbb{Q}} \cong \sum_{i=0}^{d_X} \mathbb{1}(-i)^{n_i}_{\mathbb{Q}} \text{ in } CHM(\mathbb{C})_{\mathbb{Q}}$$

which implies $\mathrm{CH}_i(X)_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of the b_{ij} and that γ^* is an isomorphism.

Proof. Show by induction that

$$\Delta_X = \sum_{i=0}^{r} \sum_{j=1}^{n_i} a^{ij} \times b_{ij} + \rho^r \text{ in } \mathrm{CH}^{d_X}(X \times X)_{\mathbb{Q}}$$

with $a^{ij} \in \mathcal{Z}^i(X)_{\mathbb{Q}}$, $b_{ij} \in \mathcal{Z}_i(X)_{\mathbb{Q}}$ and ρ^r supported on $Z^r \times X$, $Z^r \subset X$ a closed subset of codimension $r+1$.

The case $r = 0$ is Bloch's decomposition theorem, since $H^{2d_X}(X, \mathbb{Q}) = \mathbb{Q}$.

To go from r to $r+1$: ρ^r has dimension d_X. Think of $\rho^r \to Z^r$ as a familiy of codimension $d_X - r - 1$ cycles on X, parametrized by Z^r (at least over some dense open subschemeof Z^r):

$$z \mapsto \rho^r(z) \in \mathsf{CH}^{d-r-1}(X)_{\mathbb{Q}} \xrightarrow{\gamma} H^{2d-2r-2}(X, \mathbb{Q})$$

For each component Z_i of Z, fix one point z_i. Then

$$\rho^r - \sum_i Z_i \times \rho^r(z_i)$$

goes to zero in $H^{2d-2r-2}(X, \mathbb{Q})$ at each geometric generic point of Z^r. Thus the cycle goes to zero in $\mathsf{CH}^{d-r-1}(X_{k(\eta_j)})$ for each generic point $\eta_j \in Z^r$.

By continuity, there is a dense open $U \subset Z^r$ with

$$(\rho^r - \sum_i Z_i \times \rho^r(z_i)) \cap U \times X = 0 \text{ in } \mathsf{CH}^{d_X}(U \times X)_{\mathbb{Q}}$$

By localization

$$\rho^r = \sum_i Z_i \times \rho^r(z_i) + \rho^{r+1} \in \mathsf{CH}^*(Z^r \times X)_{\mathbb{Q}}$$

with ρ^{r+1} supported in $Z^{r+1} \times X$, $Z^{r+1} = X \setminus U$.

Combining with the identity for r gives

$$\Delta_X = \sum_{i=0}^{r+1} \sum_{j=1}^{n_i} a^{ij} \times b_{ij} + \rho^r \text{ in } \mathsf{CH}^{d_X}(X \times X)_{\mathbb{Q}}$$

Esnault's theorem

Theorem (Esnault) *Let X be a smooth Fano variety over a finite field $\mathbb{F}_q$. Then X has an $\mathbb{F}_q$-rational point.*

Recall: X is a Fano variety if $-K_X$ is ample.

Proof. Kollár shows that X Fano $\Longrightarrow X_{\bar{k}}$ is rationally connected (each two points are connected by a chain of rational curves).

Thus $\mathrm{CH}_0(X_L)_{\mathbb{Q}} = \mathbb{Q}$ for all $L \supset \bar{\mathbb{F}}_q$. Now use Bloch's decomposition (transposed):

$$\Delta_{\bar{X}} = 0 \times \bar{X} + \rho$$

$0 \in X(\bar{\mathbb{F}}_q)$, ρ supported on $\bar{X} \times D$.

Thus $H^n_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell) \to H^n_{\text{ét}}(\bar{X} \setminus D, \mathbb{Q}_\ell)$ is the zero map for all $n \geq 1$.

Purity of étale cohomology $\Longrightarrow$ EV of Fr_X on $H^n_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ are divisible by q for $n \geq 1$.

Lefschetz fixed point formula $\Longrightarrow$

$$\# X(\mathbb{F}_q) = \sum_{n=0}^{2d_X} (-1)^n Tr(Fr_{X|H^n_{\text{ét}}(\bar{X}, \mathbb{Q})}) \equiv 1 \mod q$$

Bloch's conjecture

Conjecture *Let X be a smooth projective surface over $\mathbb{C}$ with $H^0(X, \Omega^2) = 0$. Then the Albanese map*

$$\alpha_X : \mathrm{CH}_0(X) \to \mathrm{Alb}(X)$$

is an isomorphism.

This is known for surfaces *not* of general type (K_X ample) by Bloch-Kas-Lieberman, and for many examples of surfaces of general type.

Roitman has shown that α_X is an isomorphism on the torsion subgroups for arbitrary smooth projective X over $\mathbb{C}$.

A motivic viewpoint

Since X is a surface, we have Murre's decomposition of $\mathfrak{h}_{\mathrm{rat}}(X)_{\mathbb{Q}}$:

$$\mathfrak{h}(X)_{\mathbb{Q}} = \oplus_{i=0}^{4} \mathfrak{h}^i(X)_{\mathbb{Q}} \cong \mathbb{1} \oplus \mathfrak{h}^1 \oplus \mathfrak{h}^2 \oplus \mathfrak{h}^1(-1) \oplus \mathbb{1}(-2).$$

Murre defined a filtration of $\mathrm{CH}^2(X)_{\mathbb{Q}}$:

$$F^0 := \mathrm{CH}^2(X)_{\mathbb{Q}} \subset F^1 = \mathrm{CH}^2(X)_{\mathbb{Q}\,\deg 0} \supset F^2 := \ker \alpha_X \supset F^3 = 0$$

and showed

$$F^2 = \mathrm{CH}^2(\mathfrak{h}^2(X)), \mathrm{gr}^1_F = \mathrm{CH}^2(\mathfrak{h}^3(X)), \mathrm{gr}^0_F = \mathrm{CH}^2(\mathfrak{h}^4(X))$$

$\mathrm{CH}^2(\mathfrak{h}^i(X)) = 0$ for $i = 0, 1$.

Suppose $p_g = 0$. Choose representatives $z_i \in \mathsf{CH}^1(X)$ for $\mathcal{Z}^1_{\mathsf{num}}(X)_{\mathbb{Q}} = H^2(X, \mathbb{Q})(1)$.

Since $\mathsf{CH}^1(X) = \mathsf{Hom}_{CHM}(\mathbb{1}(-1), \mathfrak{h}(X))$, we can use the z_i to lift $\mathfrak{h}^2_{\mathsf{num}}(X) = \mathbb{1}(-1)^\rho$ to a direct factor of $\mathfrak{h}^2(X)_{\mathbb{Q}}$:

$$\mathfrak{h}^2(X)_{\mathbb{Q}} = \mathbb{1}(-1)^\rho \oplus \mathfrak{t}^2$$

with $\mathfrak{t}^2_{\mathsf{num}}(X) = 0$.

$\mathsf{CH}^2(\mathbb{1}(-1)) := \mathsf{Hom}_{CHM(k)}(\mathbb{1}(-2), \mathbb{1}(-1)) = \mathsf{CH}^1(\mathsf{Spec}\, k) = 0$.

So Bloch's conjecture is:

$$\mathsf{CH}^2(\mathfrak{t}^2(X)) = 0.$$

Filtrations on the Chow ring

We have seen that a lifting of the Künneth decompostion in $\mathcal{Z}^*_{\mathsf{num}}(X^2)_{\mathbb{Q}}$ to a sum of products in $\mathsf{CH}^*(X^2)_{\mathbb{Q}}$ imposes strong restrictions on X. However, one can still ask for a lifting of the Künneth projectors π_X^n (assuming $C(X)$) to a mutually orthogonal decomposition of Δ_X in $\mathsf{CH}^*(X^2)_{\mathbb{Q}}$.

This leads to an interesting filtration on $\mathsf{CH}^*(X)_{\mathbb{Q}}$, generalizing the situation for dimension 2.

Murre's conjecture

Conjecture (Murre) *For all $X \in \mathbf{SmProj}/k$:*

1. the Künneth projectors π_X^n are algebraic.

2. There are lifts Π_X^n of π_X^n to $\mathrm{CH}^{d_X}(X^2)_{\mathbb{Q}}$ such that
i. the Π_X^n are mutually orthogonal idempotents with $\sum_n \Pi_X^n = 1$.
ii. Π_X^n acts by 0 on $\mathrm{CH}^r(X)_{\mathbb{Q}}$ for $n > 2r$
iii. the filtration

$$F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}} := \cap_{n>2r-\nu} \ker \Pi_X^n$$

is independent of the choice of lifting
iv. $F^1\mathrm{CH}^*(X)_{\mathbb{Q}} = \ker(\mathrm{CH}^*(X)_{\mathbb{Q}} \to \mathcal{Z}^r_{\mathrm{hom}}(X)_{\mathbb{Q}})$.

In terms of a motivic decomposition, this is the same as:

1. $\mathfrak{h}_{\mathsf{hom}}(X)$ has a Künneth decomposition in $M_{\mathsf{hom}}(k)_{\mathbb{Q}}$:

$$\mathfrak{h}_{\mathsf{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n_{\mathsf{num}}(X)$$

2. This decomposition lifts to a decomposition in $CHM(k)_{\mathbb{Q}}$:

$$\mathfrak{h}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n(X)$$

such that

ii. $\mathsf{CH}^r(\mathfrak{h}^n(X)) = 0$ for $n > 2r$

iii. the filtration

$$F^\nu \mathsf{CH}^r(X)_{\mathbb{Q}} = \sum_{n \leq 2r-\nu} \mathsf{CH}^r(\mathfrak{h}^n(X))$$

is independent of the lifting.

iv. $\mathsf{CH}^r(\mathfrak{h}^{2r}(X)) = \mathcal{Z}^r_{\mathsf{hom}}(X)_{\mathbb{Q}}$.

The Bloch-Beilinson conjecture

Conjecture *For all* $X \in \mathbf{SmProj}/k$*:*

1. the Künneth projectors π_X^n *are algebraic.*

2. For each $r \geq 0$ *there is a filtration* $F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}}$, $\nu \geq 0$ *such that*

i. $F^0 = \mathrm{CH}^r$, $F^1 = \ker(\mathrm{CH}^r \to \mathcal{Z}^r_{\mathrm{hom}})$
ii. $F^\nu \cdot F^\mu \subset F^{\nu+\mu}$
iii. F^ν *is stable under correspondences*
iv. π_X^n *acts by* id *on* $\mathrm{Gr}_F^\nu \mathrm{CH}^r$ *for* $n = 2r - \nu$, *0 otherwise*
v. $F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}} = 0$ *for* $\nu >> 0$.

Murre's conjecture implies the BB conjecture by taking the filtration given in the statement of Murre's conjecture. In fact

Theorem (Jannsen) *The two conjectures are equivalent, and give the same filtrations.*

Also: Assuming the Lefschetz-type conjectures $B(X)$ for all X, the condition (v) in BB is equivalent to $F^{r+1}\mathrm{CH}^r(X) = 0$ i.e.

$$\mathrm{CH}^r(\mathfrak{h}^n(X)) = 0 \text{ for } n < r.$$

Saito's filtration

Saito has defined a functorial filtration on the Chow groups, without requiring any conjectures. This is done inductively: $F^0\mathrm{CH}^r = \mathrm{CH}^r$, $F^1\mathrm{CH}^r := \ker(\mathrm{CH}^r \to \mathcal{Z}^r_{\mathrm{hom}})_{\mathbb{Q}}$ and

$$F^{\nu+1}\mathrm{CH}^r(X)_{\mathbb{Q}} := \sum_{Y,\rho,s} \mathrm{Im}(\rho_* : F^\nu \mathrm{CH}^{r-s}(Y)_{\mathbb{Q}} \to \mathrm{CH}^r(X)_{\mathbb{Q}})$$

with the sum over all $Y \in \mathbf{SmProj}/k$, $s \in \mathbb{Z}$ and $\rho \in \mathcal{Z}^{d_Y+s}(Y \times X)$ such that the map

$$\pi_X^{2r-\nu} \circ \rho_* : H^*(Y) \to H^{2r-\nu}(X)$$

is 0.

There is also a version with the Y restricted to lie in a subcategory $\mathcal{V}$ closed under products and disjoint union.

The only problem with Saito's filtration is the vanishing property: That $F^{\nu}\mathrm{CH}^{r}(X)$ should be 0 for $\nu >> 0$. The other properties for the filtration in the BB conjecture (2) are satisfied.

Consequences of the BBM conjecture

We assume the BBM conjectures are true for the $X \in \mathcal{V}$, some subset of $\mathbf{SmProj}/k$ closed under products and disjoint union. Let $M_\sim(\mathcal{V})$ denote the full tensor pseudo-abelian subcategory of $M_\sim(k)$ generated by the $\mathfrak{h}(X)(r)$ for $X \in \mathcal{V}$, $r \in \mathbb{Z}$.

Lemma *The kernel of $CHM(\mathcal{V})_\mathbb{Q} \to NM(\mathcal{V})_\mathbb{Q}$ is a nilpotent $\otimes$ ideal.*

The nilpotence comes from

1. $\ker(\mathrm{Hom}_{CHM}(\mathfrak{h}(X)(r), \mathfrak{h}(Y)(s)) \to \mathrm{Hom}_{NM}(\mathfrak{h}(X), \mathfrak{h}(Y)))$
$$= F^1\mathrm{CH}^{d_X - r + s}(X \times Y)$$
2. $F^\nu \cdot F^\mu \subset F^{\nu+\mu}$
3. $F^\nu \mathrm{CH}^r(X^2) = 0$ for $\nu >> 0$.

The $\otimes$ property is valid without using the filtration.

Proposition $CHM(\mathcal{V})_{\mathbb{Q}} \to NM(\mathcal{V})_{\mathbb{Q}}$ *is conservative and essentially surjective.*

Indeed: $\ker \subset \mathcal{R}$

Proposition *Let X be a surface over $\mathbb{C}$ with $p_g = 0$. The BBM conjectures for X^n (all n) imply Bloch's conjecture for X.*

Proof. Recall the decomposition $\mathfrak{h}(X) = \oplus_n \mathfrak{h}^n(X)$ and $\mathfrak{h}^2(X) = \mathbb{1}(-1)^\rho \oplus \mathfrak{t}^2(X)$, $\rho = \dim_{\mathbb{Q}} H^2(X, \mathbb{Q})$. We need to show that $\mathrm{CH}^2(\mathfrak{t}^2(X)) = 0$.

But $\mathfrak{h}^2_{\mathrm{hom}} = \mathfrak{h}^2_{\mathrm{num}} = \mathbb{1}(-1)^\rho$, so $\mathfrak{t}^2_{\mathrm{num}} = 0$. By the proposition $\mathfrak{t}^2 = 0$.

Status

The BBM conjectures are valid for X of dimension ≤ 2. For an abelian variety A, one can decompose $\mathrm{CH}^r(A)_{\mathbb{Q}}$ by the common eigenspaces for the multiplication maps $[m] : A \to A$ This gives

$$\mathrm{CH}^r(X)_{\mathbb{Q}} = \oplus_{i \geq 0} \mathrm{CH}^r_{(i)}(A)$$

with $[m]$ acting by $\times m^i$ on $\mathrm{CH}^r_{(i)}(A)$ for all m.

Beauville conjectures that $\mathrm{CH}^r_{(i)}(A) = 0$ for $i > 2r$, which would give a BBM filtration by

$$F^\nu \mathrm{CH}^r(A)_{\mathbb{Q}} = \oplus_{i=0}^{2r-\nu} \mathrm{CH}^r_{(i)}(A).$$

Nilpotence

We have seen how one can compare the categories of motives for $\sim\succ\approx$ if the kernel of $\mathcal{Z}^*_\sim(X^2) \to \mathcal{Z}^*_\approx(X^2)$ is nilpotent. Voevodsky has formalized this via the adequate equivalence relation $\sim_{\otimes \mathrm{nil}}$.

Definition A correspondence $f \in \mathrm{CH}^*(X \times Y)_F$ is *smash nilpotent* if $f \times \ldots \times f \in \mathrm{CH}^*(X^n \times Y^n)$ is zero for some n.

Lemma *The collection of smash nilpotent elements in* $\mathrm{CH}^*(X \times Y)_F$ *for* $X, Y \in \mathbf{SmProj}/k$ *forms a tensor nil-ideal in* $\mathrm{Cor}^*(k)_F$*.*

Proof. For smash nilpotent f, and correspondences $g_0, \ldots, g_m$, the composition $g_0 \circ f \circ g_1 \circ \ldots \circ f \circ g_m$ is formed from $g_0 \times f \times \ldots \times f \times g_m$ by pulling back by diagonals and projecting. After permuting the factors, we see that $g_0 \times f \times \ldots \times f \times g_m = 0$ for $m >> 0$. □

Note. There is a 1-1 correspondence between tensor ideals in $\mathrm{Cor}_{\mathrm{rat}}(k)_F$ and adequate equivalence relations. Thus smash nilpotence defines an adequate equivalence relation $\sim_{\otimes \mathrm{nil}}$.

Corollary *The functor* $CHM(k)_F \to M_{\otimes\text{nil}}(k)_F$ *is conservative and a bijection on isomorphism classes.*

The kernel $\mathcal{I}_{\otimes\text{nil}}$ of $CHM(k)_F \to M_{\otimes\text{nil}}(k)_F$ is a nil-ideal, hence contained in $\mathcal{R}$.

Lemma $\sim_{\otimes\text{nil}} \succ \sim_{\text{hom}}$

If a is in $H^*(X)$ then $a \times \ldots \times a \in H^*(X^r)$ is just $a^{\otimes r} \in (H^*(X))^{\otimes r}$, by the Künneth formula.

Conjecture (Voevodsky) $\sim_{\otimes\text{nil}} = \sim_{\text{num}}$.

This conjecture thus implies the standard conjecture $\sim_{\text{hom}} = \sim_{\text{num}}$.

As some evidence, Voevodsky proves

Proposition *If $f \sim_{\mathrm{alg}} 0$, then $f \sim_{\otimes\mathrm{nil}} 0$ (with $\mathbb{Q}$-coefs).*

By naturality, one reduces to showing $a^{\times n} = 0$ for $a \in \mathrm{CH}_0(C)_{\deg 0}$, $n >> 0$, C a curve.

Pick a point $0 \in C(k)$, giving the decomposition $\mathfrak{h}(C) = \mathbb{1} \oplus \tilde{\mathfrak{h}}(C)$. Since a has degree 0, this gives a map $a : \mathbb{1}(-1) \to \tilde{\mathfrak{h}}(C)$.

We view $a^{\times n}$ as a map $a^{\times n} : \mathbb{1}(-n) \to \tilde{\mathfrak{h}}(C)^{\otimes n}$, i.e. an element of $\mathrm{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})_{\mathbb{Q}}$.

$a^{\times n}$ is symmetric, so is in $\mathsf{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})^{S_n}_{\mathbb{Q}} \subset \mathsf{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})_{\mathbb{Q}}$

But

$$\mathsf{CH}^n(\tilde{\mathfrak{h}}(C))^{S_n}_{\mathbb{Q}} = \mathsf{CH}_0(\mathsf{Sym}^n C)_{\mathbb{Q}}/\mathsf{CH}_0(\mathsf{Sym}^{n-1} C)_{\mathbb{Q}}.$$

For $n > 2g-1$ $\mathsf{Sym}^n C \to \mathsf{Jac}(C)$ and $\mathsf{Sym}^{n-1} C \to \mathsf{Jac}(C)$ are projective space bundles, so the inclusion $\mathsf{Sym}^{n-1} C \to \mathsf{Sym}^n C$ induces an iso on CH_0.

Nilpotence and other conjectures

For X a surface, the nilpotence conjecture for X^2 implies Bloch's conjecture for X: The nilpotence conjecture implies that $\mathfrak{t}^2_{\otimes \mathrm{nil}}(X) = 0$, but then $\mathfrak{t}^2(X) = 0$.

The BBM conjectures imply the nilpotence conjecture (O'Sullivan).

Finite dimensionality

Kimura and O'Sullivan have introduced a new notion for pure motives, that of finite dimensionality.

Multi-linear algebra in tensor categories

For vector spaces over a field F, one has the operations

$$V \mapsto \Lambda^n V, \; V \mapsto \mathrm{Sym}^n V$$

as well as the other Schur functors.

Define elements of $\mathbb{Q}[S_n]$ by

$$\lambda^n := \frac{1}{n!} \sum_{g \in S_n} \mathrm{sgn}(g) \cdot g$$

$$\mathrm{sym}^n := \frac{1}{n!} \sum_{g \in S_n} g$$

λ^n and sym^n are idempotents in $\mathbb{Q}[S_n]$.

Let S_n act on $V^{\otimes_F n}$ by permuting the tensor factors. This makes $V^{\otimes_F n}$ a $\mathbb{Q}[S_n]$ module (assume F has characteristic 0) and

$$\Lambda^n V = \lambda^n(V^{\otimes n}), \mathrm{Sym}^n V = \mathrm{sym}^n(V^{\otimes n}).$$

These operation extend to the abstract setting.

Let $(\mathcal{C}, \otimes, \tau)$ be a pseudo-abelian tensor category (over $\mathbb{Q}$). For each object V of $\mathcal{C}$, S_n acts on $V^{\otimes n}$ with simple transpositions acting by the symmetry isomorphisms τ.

Since $\mathcal{C}$ is pseudo-abelian, we can define

$$\Lambda^n V := \mathrm{Im}(\lambda^n : V^{\otimes n} \to V^{\otimes n})$$
$$\mathrm{Sym}^n V := \mathrm{Im}(\mathrm{sym}^n : V^{\otimes n} \to V^{\otimes n})$$

Note. 1. Let $\mathcal{C} = \mathsf{GrVec}_F$, and let $f : \mathsf{GrVec}_K \to \mathsf{Vec}_K$ be the functor "forget the grading". If V has purely odd degree, then

$$f(\mathsf{Sym}^n V) \cong \Lambda^n f(V), f(\Lambda^n V) = \mathsf{Sym}^n f(V)$$

If V has purely even degree, then

$$f(\mathsf{Sym}^n V) \cong \mathsf{Sym}^n f(V), f(\Lambda^n V) = \Lambda^n f(V).$$

2. Take $\mathcal{C} = \mathsf{Vec}_K^\infty$. Then $V \in \mathcal{C}$ is finite dimensional $\Leftrightarrow \Lambda^n V = 0$ for some n.

3. Take $\mathcal{C} = \mathsf{GrVec}_K^\infty$. Then $V \in \mathcal{C}$ is finite dimensional $\Leftrightarrow$ $V = V^+ \oplus V^-$ with $\Lambda^n V^+ = 0$ and $\mathsf{Sym}^n V^- = 0$ for some n.

Definition Let $\mathcal{C}$ be a pseudo-abelian tensor category over a field F of characteristic 0. Call $M \in \mathcal{C}$ *finite dimensional* if $M \cong M^+ \oplus M^-$ with

$$\Lambda^n M^+ = 0 = \mathrm{Sym}^m M^-$$

for some integers $n, m > 0$.

Proposition (Kimura, O'Sullivan) *If M, N are finite dimensional, then so are $N \oplus M$ and $N \otimes M$.*

The proof uses the extension of the operations Λ^n, Sym^n to *all* Schur functors.

Theorem (Kimura,O'Sullivan) *Let C be a smooth projective curve over k. Then $\mathfrak{h}(C) \in CHM(k)_{\mathbb{Q}}$ is finite dimensional.*

In fact

$\mathfrak{h}(C)^{+} = \mathfrak{h}^0(C) \oplus \mathfrak{h}^2(C)$, $\mathfrak{h}(C)^{-} = \mathfrak{h}^1(C)$ and

$$\lambda^3(\mathfrak{h}^0(C) \oplus \mathfrak{h}^2(C)) = 0 = \mathrm{Sym}^{2g+1}\mathfrak{h}^1(C).$$

The proof that $\mathrm{Sym}^{2g+1}\mathfrak{h}^1(C) = 0$ is similar to the proof that the nilpotence conjecture holds for algebraic equivalence: One uses the structure of $\mathrm{Sym}^N C \to \mathrm{Jac}(C)$ as a projective space bundle.

Corollary *Let M be in the pseudo-abelian tensor subcategory of $CHM(k)_{\mathbb{Q}}$ generated by the $\mathfrak{h}(C)$, as C runs over smooth projective curves over k. Then M is finite dimensional.*

For example $\mathfrak{h}(A)$ is finite dimensional if A is an abelian variety. $\mathfrak{h}(S)$ is finite dimensional if S is a Kummer surface. $\mathfrak{h}(C_1 \times \ldots \times C_r)$ is also finite dimensional.

It is not known if a general quartic surface $S \subset \mathbb{P}^3$ has finite dimensional motive.

Consequences

Theorem *Suppose M is a finite dimensional Chow motive. Then every $f \in \mathrm{Hom}_{CHM(k)_{\mathbb{Q}}}(M, M)$ with $H^*(f) = 0$ is nilpotent. In particular, if $H^*(M) = 0$ then $M = 0$.*

Corollary *Suppose $\mathfrak{h}(X)$ is finite dimensional for a surface X. Then Bloch's conjecture holds for X.*

Indeed, $\mathfrak{h}(X)$ finite dimensional implies $\mathfrak{h}^2(X) = \mathbb{1}(-1)^{\rho} \oplus \mathfrak{t}^2(X)$ is evenly finite dimensional, so $\mathfrak{t}^2(X)$ is finite dimensional. But $\mathfrak{t}^2_{\mathrm{hom}}(X) = 0$.

Conjecture (Kimura, O'Sullivan) *Each object of $CHM(k)_{\mathbb{Q}}$ is finite dimensional.*

Note. The nilpotence conjecture implies the finite dimensionality conjecture.

In fact, let $\mathcal{I}_{\otimes\text{nil}} \subset \mathcal{I}_{\text{hom}} \subset \mathcal{I}_{\text{num}}$ be the various ideals in $CHM(k)_{\mathbb{Q}}$.

Then $\mathcal{I}_{\otimes\text{nil}} \subset \mathcal{R}$ (f smash nilpotent $\Rightarrow$ f nilpotent). So the nilpotence conjecture implies $\mathcal{R} = \mathcal{I}_{\text{num}}$.

Thus $\phi : CHM(k)_{\mathbb{Q}} \to NM(k)_{\mathbb{Q}} = M_{\text{hom}}(k)_{\mathbb{Q}}$ is conservative and essentially surjective.

Since $\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$, the Künneth projectors are algebraic: we can thus lift the decomposition $\mathfrak{h}_{\mathrm{hom}}(X) = \mathfrak{h}^{+}_{\mathrm{hom}}(X) \oplus \mathfrak{h}^{-}_{\mathrm{hom}}(X)$ to $CHM(k)$.

Since ϕ is conservative, $\mathfrak{h}(X) = \mathfrak{h}^{+}X(X) \oplus \mathfrak{h}^{-}(X)$ is finite dimensional:

$$\Lambda^{b^{+}(X)+1}(\mathfrak{h}^{+}(X)) = 0 = \mathrm{Sym}^{b^{-}(X)+1}(\mathfrak{h}^{-}(X)).$$