SMR/1840-2

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Classifying Spaces and Higher K-theory

Eric Friedlander
Northwestern University, Evanston, USA

2. Classifying spaces and higher K-theory

2.1. **Recollections of homotopy theory.** Much of our discussions will require some basics of homotopy theory. We emphasize the definition of a *homotopy.*

Definition 2.1. Two continuous maps $f, g : X \to Y$ between topological spaces are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ with $F_{|X\times\{0\}} = f, F_{|X\times\{1\}} = g$ (where I denotes the unit interval $[0, 1]$).

If $x \in X, y \in Y$ are chosen ("base points"), then two ("pointed") maps $f, g : (X, \{x\}) \to (Y, \{y\})$ are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ such that $F_{|X\{\times 0\}} = f, F_{|X\times\{1\}} = g$, and $F_{|\{x\}\times I} = \{y\}$ (i.e., F must project $\{x\} \times I$ to $\{y\}$. We use the notation $[(X, x), (Y, y)]$ to denote the pointed homotopy classes of maps from (X, x) (previously denoted $(X, \{x\})$) to $(Y, \{y\})$.

We shall employ the usual notation, $[X, Y]$ to denote homotopy classes of continuous maps from X to Y.

Another basic definition is that of the homotopy groups of a topological space.

Definition 2.2. For any $n \geq 0$ and any pointed space (X, x),

$$\pi_n(X, x) \equiv [(S^n, \infty), (X, x)].$$

For $n = 0$, $\pi_n(X, x)$ is a pointed set; for $n \geq 1$, a group; for $n \geq 2$, an abelian group. If (X, x) is "nice", then $\pi_n(X, x) \simeq [S^n, X]$; moreover, if X is path connected, then the isomorphism class of $\pi_n(X, x)$ is independent of $x \in X$.

A **relative C.W. complex** is a topological pair (X, A) (i.e., A is a subspace of X) such that there exists a sequence of subspaces $A = X_{-1} \subset X_0 \subset \cdots \subset X_n \subset \cdots$ of X with union equal to X such that X_n is obtained from X_{n-1} by "attaching" n-cells (i.e., possibly infinitely many copies of the closed unit disk in $\mathbb{R}^n$, where "attachment" means that the boundary of the disk is identified with its image under a continous map $S^{n-1} \to X_{n-1}$) and such that a subset $F \subset X$ is closed if and only if $X \cap X_n \subset X_n$ is closed for all n. A space X is a C.W. complex if $(X, \emptyset)$ is a relative C.W. complex. A pointed C.W. complex (X, x) is a relative C.W. complex for $(X, \{x\})$.

C.W. complexes have many good properties, one of which is the following.

Theorem 2.3. *(Whitehead theorem) If $f : X \to Y$ is a continuous map of connected C.W. complexes such that $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence.*

Moreover, C.W. complexes are quite general: If (T, t) is a pointed topological space, then there exists a pointed C.W. complex (X, x) and a continuous map $g : (X, x) \to (T, t)$ such that $g_* : \pi_*(X, x) \to \pi_*(T, t)$ is an isomorphism.

Recall that a continuous map $f : X \to Y$ is said to be a fibration if it has the homotopy lifting property: given any commutative square of continuous maps

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ A \times I \arrow[r] & Y \end{tikzcd}$$

then there exits a map $A \times I \to X$ whose restriction to $A \times \{0\}$ is the upper horizontal map and whose composition with the right vertical map equals the lower horizontal map. A very important property of fibrations is that if $f : X \to Y$ is a fibration, then there is a long exact sequence of homotopy groups for any $x_o \in X, y \in Y$:

$$\cdots \to \pi_n(f^{-1}(y), x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \pi_{n-1}(f^{-1}(y), x_0) \to \cdots$$

If $f : (X, x) \to (Y, y)$ is any pointed map of spaces, we can naturally construct a fibration $\tilde{f} : \tilde{X} \to Y$ together with a homotopy equivalence $X \to \tilde{X}$ over Y. We denote by $htyfib(f)$ the fibre $\tilde{f}^{-1}(y)$ of $\tilde{f}$.

2.2. BG**.**

Definition 2.4. Let G be a topological group and X a topological space. Then a **G-torsor** over X (or principal G-bundle) is a continuous map $p : E \to X$ together with a continuous action of G on E over X such that there exists an open covering $\{U_i\}$ of X homeomorphisms $G \times U_i \to E_{|U_i}$ for each i respecting G-actions (where G acts on $G \times U_i$ by left multiplication on G).

Example 2.5. Assume that G is a discrete group. Then a G-torsor $p : E \to X$ is a normal covering space with covering group G.

Theorem 2.6. *(Milnor) Let G be a topological group with the homotopy type of a C.W. complex. There there exists a connected C.W. complex BG and a G-torsor $\pi : EG \to BG$ such that sending a continuous function $X \to BG$ to the G-torsor $X \times_{BG} EG \to X$ over X determines a 1-1 correspondence*

$$[X, BG] \ \stackrel{\simeq}{\to} \ \{isom\ classes\ of\ G\text{-}torsors\ over\ X\}$$

Moreover, the homotopy type of BG is thereby determined; furthermore, EG is contractible.

The topology on G when considering the classifying space BG is crucial. One interesting construction one can consider is the map on classifying spaces induced by the continuous, bijective function $G^\delta \to G$ where G is a topological group and G^δ is the same group but provided with the discrete topology.

Corollary 2.7. *If G is discrete, then $\pi_1(BG, *) = G$ and $\pi_n(BG, *) = 0$ for all $n > 0$ (where $*$ is some choice of base point). Moreover, these properties characterize the C.W. complex BG up to homotopy type.*

Proof. Sketch of proof If $n > 0$, then the facts that $\pi_1(S^n) = 0$ and EG is contractible imply that $[S^n, BG] = \{0\}$. The fact that $\pi_1(BG, *) = G$ is classical covering space theory. □

The proof of the following proposition is fairly elementary, using a standard projection resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\pi]$-module.

Proposition 2.8. *Let π be a discrete group and let A be a $\mathbb{Z}[\pi]$-module. Then*

$$H^*(B\pi, A) = Ext^*_{\mathbb{Z}[\pi]}(\mathbb{Z}, A) \equiv H^*(\pi, A)$$

$$H_*(B\pi, A) = Tor_*^{\mathbb{Z}[\pi]}(\mathbb{Z}, A) \equiv H_*(\pi, A).$$

Now, vector bundles are not G-torsors but rather fibre bundles for the topological groups $O(n)$ (respectively, $U(n)$) in the case of a real (resp., complex) vector bundle of rank n. Nevertheless, because $O(n)$ (resp., $U(n)$) acts faithfully and transitively on $\mathbb{R}^n$ (resp., $\mathbb{C}^n$), we can readily conclude using Theorem 2.6

$$[X, BO(n)] = \{\text{isom classes of real rank n vector bundles over X}\}$$

$$[X, BU(n)] = \{\text{isom classes of complex rank n vector bundles over X}\}.$$

2.3. **Quillen's plus construction.** Daniel Quillen's original definition of $K_i(R), i > 0$, was in terms of the following "Quillen plus construction".

Theorem 2.9. *(Plus construction) Let G be a discrete group and $H \subset G$ be a perfect normal subgroup. Then there exists a C.W. complex BG^+ and a continuous map*

$$\gamma : BG \to BG^+$$

such that $ker\{\pi_1(BG) \to \pi_1(BG^+)\} = H$ and such that $\tilde{H}_*(htyfib(\gamma), \mathbb{Z}) = 0$. Moreover, γ is unique up to homotopy.

The classical "Whitehead Lemma" implies that the commutator subgroup $[GL(R), GL(R)]$ of $GL(R)$ is perfect. (One verifies that an $n \times n$ elementary matrix is itself a commutator of elementary matrices provided that $n \geq 4$.)

Definition 2.10. For any ring R, let

$$\gamma : BGL(R) \to BGL(R)^+$$

denote the Quillen plus construction with respect to $[GL(R), GL(R)] \subset GL(R)$. We define

$$K_i(R) \equiv \pi_i(BGL(R)^+), \quad i > 0.$$

This construction is closely connected to the group completions of our first lecture. In some sense, $\coprod_n BGL(n, R)$ is "up to homotopy, a commutative topological monoid" and $BGL(R)^+ \times \mathbb{Z}$ is a group completion in an appropriate sense. There are several technologies which have been introduced in part to justify this informal description (e.g., the "$S^{-1}S$ construction" discussed below).

Remark 2.11. Essentially by definition, $K_1(R)$ as defined in the first lecture agrees with that of Definition 2.10. Moreover, for any $K_1(R)$-module A,

$$H^*(BGL(R)^+, A) = H^*(BGL(R), A).$$

Moreover, one can verify that $K_2(R)$ as introduced in the first lecture agrees with that of Definition 2.10 for any ring R by identifying this second homotopy group with the second homology group of the perfect group $[GL(R), GL(R)]$.

When Quillen formulated his definition of $K_*(R)$, he also made the following fundamental computation. Indeed, this computation was a motivating factor for Quillen's definition.

Theorem 2.12. *(Quillen's computation for finite fields) Let $\mathbb{F}_q$ be a finite field. Then the space $BGL(\mathbb{F}_q)^+$ can be described as the homotopy fibre of a computable map. This leads to the following computation for $i > 0$:*

$$K_i(\mathbb{F}_q) = \mathbb{Z}/q^j - 1 \qquad \text{if } i = 2j - 1$$
$$K_i(\mathbb{F}_q) = 0 \qquad \text{if } i = 2j.$$

As you probably know, homotopy groups are notoriously hard to compute. So Quillen has played a nasty trick on us, giving us very interesting invariants with which we struggle to make the most basic calculations. For example, a fundamental problem which is still not fully solved is to compute $K_i(\mathbb{Z})$.

Early computations of higher K-groups of a ring R often proceeded by first computing the group homology groups of $GL(n, R)$ for n large, then relating these homology groups to the homotopy groups of $BGL(R)^+$.

2.4. **Abelian and exact categories.** Much of our discussion in these lectures will require the language and concepts of category theory. Indeed, working with categories will give us a method to consider various kinds of K-theories simultaneously.

I shall assume that you are familiar with the notion of an abelian category. Recall that in an abelian category $\mathcal{A}$, the set of morphisms $Hom_{\mathcal{A}}(B, C)$ for any $A, B \in Obj\ \mathcal{A}$ has the natural structure of an abelian group; moreover, for each $A, B \in Obj\ \mathcal{A}$, there is an object $B \oplus C$ which is both a product and a coproduct; moreover, any $f : A \to B$ in $Hom_{\mathcal{A}}(A, B)$ has both a kernel and a cokernel. In an abelian category, we can work with exact sequences just as we do in the category of abelian groups.

Example 2.13. Here are a few "standard" examples of abelian categories.

- the category $Mod(R)$ of (left) R-modules.
- the category $mod(R)$ of finitely generated R-modules (in which case we must take R to be Noetherian).
- the category $QCoh(X)$ of quasi-coherent sheaves on a variety X.
- the category $Coh(X)$ of coherent sheaves on a Notherian variety X.

Warning. The full subcategory $\mathcal{P}(R) \subset mod(R)$ is not an abelian category. For example, if $R = \mathbb{Z}$, then $n : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism of projective R-modules whose cokernel is not projective and thus is not in $\mathcal{P}(\mathbb{Z})$.

Definition 2.14. An exact category $\mathcal{P}$ is a full additive subcategory of some abelian category $\mathcal{A}$ such that
(a.) There exists some set $S \subset Obj\ \mathcal{A}$ such that every $A \in Obj\ \mathcal{A}$ is isomorphic to some element of S.
(b.) If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence in $\mathcal{A}$ with both $A_1, A_3 \in Obj\ \mathcal{P}$, then $A_2 \in Obj\ \mathcal{P}$.

An admissible monomorphism (respectively, epimorphism) in $\mathcal{P}$ is a monomorphism $A_1 \to A_2$ (resp., $A_2 \to A_3$) in $\mathcal{P}$ which fits in an exact sequence of the form of (b.).

Definition 2.15. If $\mathcal{P}$ is an exact category, we define $K_0(\mathcal{P})$ to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of objects of $\mathcal{P}$ (with respect to $\oplus$) modulo the equivalence relation $[A_2] - [A_1] - [A_3]$ for every exact sequence of the form (I.5.b).

Exercise 2.16. *Show that $K_0(R)$ equals $K_0(\mathcal{P}(R))$, where $\mathcal{P}(R)$ is the exact category of finitely generated projective R-modules.*

More generally, show that $K_0(X)$ equals $K_0(\mathcal{V}ect(X))$, where $\mathcal{V}ect(X)$ is the exact category of algebraic vector bundles on the quasi-projective variety X.

Definition 2.17. Let $\mathcal{P}$ be an exact category in which all exact sequences split. Consider pairs (A, α) where $A \in Obj\ \mathcal{P}$ and α is an automorphism of A. Direct sums and exact sequences of such pairs are defined in the obvious way. Then $K_1(\mathcal{P})$ is defined to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of such pairs modulo the relations given by short exact sequences.

You can find a proof in [**?**] that $K_1(\mathcal{P}(R))$ equals $K_1(R)$.

2.5. **The $S^{-1}S$ construction.** Recall that a *symmetric monoidal category* S is a (small) category with a unit object $e \in S$ and a functor $\Box : S \times S \to S$ which is associative and commutative up to coherent natural isomorphisms. For example, if we consider the category $\mathcal{P}$ of finitely generated projective R-modules, then the direct sum $\oplus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative but only commutative up to natural isomorphism. The symmetric monoidal category relevant for the K-theory of a ring R is the category $Iso(\mathcal{P})$ whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms between projective R-modules.

Quillen's construction of $S^{-1}S$ for a symmetric monoidal category S is appealing, modelling one way we would introduce inverses to form the group completion of an abelian monoid.

Definition 2.18. Let S be a symmetric monoidal category. The category $S^{-1}S$ is the category whose objects are pairs $\{a,b\}$ of objects of S and whose maps from $\{a,b\}$ to $\{c,d\}$ are equivalence classes of compositions of the following form:

$$\{a,b\} \overset{s\Box-}{\to} \{s\Box a, s\Box b) \overset{(f,g)}{\to} \{c,d\}$$

where s is some object of S, f,g are morphisms in S. Another such composition

$$\{a,b\} \overset{s'\Box-}{\to} \{s'\Box a, s'\Box b) \overset{(f',g')}{\to} \{c,d\}$$

is declared to be the same map in $S^{-1}S$ from $\{a,b\}$ to $\{c,d\}$ if and only if there exists some isomorphism $\theta : s \to s'$ such that $f = f' \circ (\theta\Box a), g = g' \circ (\theta \Box b)$.

Heuristically, we view $\{a,b\} \in S^{-1}S$ as representing $a-b$, so that $\{s\Box a, s\Box b\}$ also represents $a-b$. Moreover, we are forcing morphisms in S to be invertible in $S^{-1}S$. If we were to apply this construction to the natural numbers $\mathbb{N}$ viewed as a discrete category with addition as the operation, then we get $\mathbb{N}^{-1}\mathbb{N} = \mathbb{Z}$.

The following theorem of Quillen shows how the $S^{-1}S$ construction can provide a homotopy-theoretic group completion

Theorem 2.19. *(Quillen) Let S be a symmetric monoidal category with the property that for all $s,t \in S$ the map $s\Box- : Aut(t) \to Aut(s\Box t)$ is injective. Then the natural map $BS \to B(S^{-1}S)$ of classifying spaces (see the next section) is a homotopy-theoretic group completion.*

In particular, if S denotes the category whose objects are finite dimensional projective R-modules and whose maps are isomorphisms (so that $BS = \coprod_{[P]} BAut(P)$), then $\mathcal{K}(R)$ is homotopy equivalent to $B(S^{-1}S)$.

2.6. **Simplicial sets and the Nerve of a Category.**

Definition 2.20. The category of standard simplices, Δ, has objects $\underline{\mathrm{n}} = \langle 0,1,\ldots,n\rangle$ indexed by $n \in \mathbb{N}$ and morphisms given by

$$Hom_\Delta(\underline{\mathrm{m}},\underline{\mathrm{n}}) = \{\text{non-decreasing maps } \langle 0,1,\ldots,n\rangle \to \langle 0,1,\ldots,m\rangle\}.$$

The special morphisms

$$\partial_i : \underline{\text{n-1}} \to \underline{\mathrm{n}} \ (\textit{skip } i); \quad \sigma_j : \underline{\text{n+1}} \to \underline{\mathrm{n}} \ (\text{repeat j})$$

in Δ generate (under composition) all the morphisms of Δ and satisfy certain standard relations which many topologists know by heart.

A simplicial set $S_\bullet$ is a functor $\Delta^{op} \to (sets)$.

In other words, $S_\bullet$ consists of a set S_n for each $n \geq 0$ and maps $d_i : S_n \to S_{n-1}, s_j : S_n \to S_{n+1}$ satisfying the relations given by the relations satisfied by $\partial_i, \sigma_j \in \Delta$.

Example 2.21. Let T be a topological space. Then the **singular complex** $Sing_\bullet T$ is a simplicial set. Recall that $Sing_n T$ is the set of continuous maps $\Delta^n \to T$, where $\Delta^n \subset \mathbb{R}^{n+1}$ is the standard n-simplex: the subspace consisting of those points

$\underline{x} = (x_0, \ldots, x_n)$ with each $x_i \geq 0$ and $\sum x_i = 1$. Since any map $\mu : \underline{n} \to \underline{m}$ determines a (linear) map $\Delta^n \to \Delta^m$, it also determines $\mu : Sing_m T \to Sing_n T$, so that we may easily verify that

$$Sing_\bullet : \Delta^{op} \to (sets)$$

is a well defined functor.

Definition 2.22. (Milnor's geometric realization functor) For any simplicial set $X_\bullet$, we define its geometric realization as the topological space $|X_.|$ given as follows:

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is given by $(x, \mu \circ t) \simeq (\mu \circ x, t)$ whenever $x \in X_m, t \in \Delta^n, \mu : \underline{n} \to \underline{m}$ a map of Δ. This quotient is given the quotient topology, where each $X_n \times \Delta^n$ is topologized as a disjoint union indexed by $x \in X_n$ of copies of $\Delta^n \subset \mathbb{R}^{n+1}$.

Now, simplicial sets are a very good combinatorial model for homotopy theory as the next theorem reveals.

Theorem 2.23. *(Homotopy category) The categories of topological spaces and simplicial sets satisfy the following relationships.*

- *Milnor's geometric realization functor is left adjoint to the singular functor; in other words, for every simplicial set $X_\bullet$ and every topological space T,*

$$Hom_{(s.sets)}(X_\bullet, Sing_\bullet T) = Hom_{(spaces)}(|X_\bullet|, T).$$

- *For any simplicial set $X_\bullet$, $|X_\bullet|$ is a C.W. complex; moreover, for any topological space T, $Sing._\bullet(T)$ is a particularly well behaved type of simplicial set called a Kan complex.*
- *For any topological space T and any point $t \in T$, the adjunction morphism*

$$(|Sing_\bullet T|, t) \to (T, t)$$

induces an isomorphism on homotopy groups.
- *The adjunction morphisms of (a.) induces an equivalence of categories*

$$(Kan\ cxes)/\sim hom.equiv \quad \simeq \quad (C.W.\ cxes)/\sim hom.equiv$$

Now we can define the classifying space of a (small) category.

Definition 2.24. Let $\mathcal{C}$ be a small category. We define the **nerve** $N\mathcal{C} \in (s.sets)$ to be the simplicial set whose set of n-simplices is the set of composable n-tuples of morphisms in $\mathcal{C}$:

$$N\mathcal{C}_n = \{C_n \xrightarrow{\gamma_n} C_{n-1} \to \cdots \xrightarrow{\gamma_1} C_0\}.$$

For $\partial_i : \underline{n\text{-}1} \to \underline{n}$, we define $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ to send the n-tuple $C_n \to \cdots \to C_0$ to that $n-1$-tuple given by composing γ_{i+1} and γ_i whenever $0 < i < n$, by dropping

$\xrightarrow{\gamma_1} C_0$ if $i = 0$ and by dropping $C_n \xrightarrow{\gamma_n}$ if $i = n$. For $\sigma_j : \underline{\mathrm{n}} \to \underline{\mathrm{n+1}}$, we define $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ by repeating C_j and inserting the identity map.

We define the **classifying space** $B\mathcal{C}$ of the category $\mathcal{C}$ to be $|N\mathcal{C}|$, the geometric realization of the nerve of $\mathcal{C}$.

Example 2.25. Let G be a (discrete) group and let $\mathcal{G}$ denote the category with a single object (denoted $*$) and with $Hom_{\mathcal{G}}(*,*) = G$. Then $B\mathcal{G}$ is a model for BG (i.e., $B\mathcal{G}$ is a connected C.W. complex with $\pi_1(B\mathcal{G},*) = G$ and all higher homotopy groups 0).

Example 2.26. Let X be a polyhedron and let $\mathcal{S}(X)$ denote the category whose objects are simplices of X and maps are the inclusions of simplices. Then $B\mathcal{S}(X)$ can be identified with the first barycentric subdivision of X.

2.7. **Quillen's Q-construction.** What are the higher K-groups of an exact category? In particular, what are the higher K-groups of a quasi-projective variety X (i.e., of the exact category $\mathcal{V}ect(X)$) or more generally of a scheme?

Quillen defines these in terms of another construction, the "Quillen Q-construction."

Definition 2.27. Let $\mathcal{P}$ be an exact category and let $Q\mathcal{P}$ be the category obtained from $\mathcal{P}$ by applying the Quillen Q-construction (as discussed below). Then

$$K_i(\mathcal{P}) = \pi_{i+1}(BQ\mathcal{P}), \quad i \geq 0,$$

the homotopy groups of the geometric realization of the nerve of $Q\mathcal{P}$.

Theorem 2.28. *Let X be a scheme and let $\mathcal{V}ect(X)$ dnoe the exact category of finitely presented, locally free $\mathcal{O}_X$-modules. Then*

$$K_i(X) \;\equiv\; \pi_i(\mathcal{V}ect(X)) \;\equiv\; \pi_{i+1}(BQ\mathcal{V}ect(X))$$

agrees for $i = 0$ with the Grothendieck group of $\mathcal{V}ect(X)$ and for $X = SpecA$ an affine scheme agrees with $K_i(A) = \pi_i(BLG(A)^+)$ provided that $i > 0$.

Quillen proves this theorem using the $S^{-1}S$ construction as an intermediary.

Here is the formulation of Quillen's Q-construction.

Definition 2.29. Let $\mathcal{P}$ be an exact category. We define the category $Q\mathcal{P}$ as follows. We set $Obj\ Q\mathcal{P}$ equal to $Obj\ \mathcal{P}$. For any $A, B \in Obj\ Q\mathcal{P}$, we define

$$Hom_{Q\mathcal{P}}(A,B) = \{A \overset{p}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B; \text{p (resp i) admis epi (resp. mono)}/\sim\}$$

where the equivalence relation is generated by pairs

$$A \twoheadleftarrow X \rightarrowtail B, A \twoheadleftarrow X' \rightarrowtail B$$

which fit in a commutative diagram

$$\begin{tikzcd}
A \arrow[d, "="] & X \arrow[l, "p"'] \arrow[r, "i"] \arrow[d] & B \arrow[d, "="] \\
A & X \arrow[l, "p\prime"] \arrow[r, "i\prime"] & B
\end{tikzcd}$$

Waldhausen gives a somewhat more elaborate construction of Quillen's Q construction which produces "n-fold deloopings" of $BQ\mathcal{P}$ for every $n \geq 0$: pointed spaces T_n with the property that $\Omega^n(T_n)$ is homotopy equivalent to $BQ\mathcal{P}$.

References

[1] J. Milnor, Algebraic K-theory
[2] D. Quillen, Higher Algebraic K-theory, I
[3] G. Segal, Classifying spaces and spectral sequences
[4] F Waldhausen