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Chow groups and motives

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Lecture 2

Chow groups and motives

We will be working with the *algebraic varieties*, which we always assume to be *quasiprojective*. Quasiprojective variety is an open subvariety in the *projective variety*. And the latter one is just a closed subvariety of the projective space $\mathbb{P}^N$, that is subvariety given by the set of (homogeneous) equations $f_1, \ldots, f_r$. The same quasiprojective variety can be embedded in different projective spaces: $\mathbb{P}^N \supset X \subset \mathbb{P}^M$ (in particular, one can define precisely when such subvarieties are isomorphic).

Algebraic variety can be covered by *affine* open subvarieties. Affine varieties correspond to commutative rings (finitely generated, in our case). This correspondence has the form

$$R - \text{ ring} \quad \leftrightarrow \quad \mathrm{Spec}(R),$$

where $\mathrm{Spec}(R)$ is called the *spectrum* of R, and R, in turn is a ring of regular functions on the algebraic variety $\mathrm{Spec}(R)$. The above correspondence is contravariant:

$$\phi : S \to R \quad \leftrightarrow \quad \phi^\vee : \mathrm{Spec}(R) \to \mathrm{Spec}(S).$$

In our situation, affine varieties are just the closed subvarieties of affine space $\mathbb{A}^n = \mathrm{Spec}(k[x_1, \ldots, x_n])$ which is just the translation into geometric language of the fact that the respective rings are finitely generated and so are the quotient rings of the polynomial ring: $R = k[x_1 \ldots, x_n]/(f_1, \ldots, f_r)$. Of course, the same variety can be embedded into many different affine spaces - just choose another set of k-algebra generators $y_1, \ldots, y_m$ and present R as $k[y_1 \ldots, y_m]/(g_1, \ldots, g_s)$.

Algebraic varieties have *points*. Points of the affine variety $\mathrm{Spec}(R)$ are the *prime ideals* $P \subset R$ (that is, such ideals that for any $x, y \in R$, $x \cdot y \in P$ implies that either x, or y belongs to P). The morphism of affine varieties $\phi^\vee : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ acts on points: $P \mapsto \phi^{-1}(P)$. If X is covered by affine open varieties $X = \cup_i U_i$, then

$$(\text{ points of } X) = \coprod_i (\text{ points of } U_i)/(\text{ident.}),$$

where we identify points of $U_i \cap U_j$ in U_i and U_j.

In contrast to topology and usual geometry, the points have different *dimensions*. It is sufficient to consider the case of affine variety.

$$\dim(P) = \max\{d \mid \exists \text{ chain } P = P_0 \subset P_1 \subset \ldots \subset P_d \text{ of distinct prime ideals}\}.$$

Points of dimension 0 are exactly the *maximal* ideals in R. If R has no zero divisors then the ideal (0) is prime, and the respective point is called the *generic point*. In such a case the *dimension of a variety* is just the dimension of its generic point.

To each point one can assign the *residue field* $k(x)$. Namely, if P is prime, then the subset $T = R \backslash P$ is multiplicative ($T \cdot T \subset T$), and we can localise: RT^{-1} will be a *local ring*, and PT^{-1} is the only maximal ideal in it. $k(P) := RT^{-1}/PT^{-1}$. The dimension of a point is just the transcendence degree $\mathrm{trdeg}(k(P)/k)$ of its residue field over k. Any regular function r on $\mathrm{Spec}(R)$ (that is, an element of R), can be evaluated at P with value in $k(P)$:

$$R \to RT^{-1} \to RT^{-1}/PT^{-1} = k(P).$$

Notice that all these fields $k(P)$ come with the natural embedding $k \subset k(P)$, so if one considers only the case of closed points over algebraically closed field k, then all the residue fields are identified, and the evaluation takes values in the same field k (as one used to).

Example: $X = \mathrm{Spec}(k[x_1, \ldots, x_n]) = \mathbb{A}^n$. Then $\dim(X) = n$, residue field of a generic point is the field of rational functions $k(x_1, \ldots, x_n)$, and as a maximal chain of prime ideals one can choose

$$(x_1, \ldots, x_n) \supset (x_1, \ldots, x_{n-1}) \supset \ldots \supset (x_1) \supset (0).$$

Algebraic variety is called irreducible if all of its open affine subvarieties are, and an affine variety $X = \mathrm{Spec}(R)$ is irreducible, if and only if R has no 0-divizors (only "one" generic point).

Examples:

1) $\mathrm{Spec}(k[x, y]/(xy))$ is reducible (consists of the union of x-axis and y-axis on x, y-plane - two components).

2) $\mathrm{Spec}(k[x, y]/(y - x^3))$ is irreducible (consists of just one component).

If (as in the examples above)our variety is a hyersurface in the affine space (given by just one equation), then one simply needs to check if the respective polynomial is decomposable, but if the variety is defined by several equations it could be quite difficult to check the irreducibility.

There is $1-1$ correspondence

$$\{\,\text{irred. closed subvar. of}\,X\} \;\leftrightarrow\; \{\,\text{points of}\,X\}$$

where each point is a *generic point* of some unique closed irreducible subvariety.

Chow groups

Let X be an algebraic variety, then one can define the *Chow group of d-dimensional cylces on X modulo rational equivalence* as

$$\mathrm{CH}_d(X) := \left(\bigoplus_{V\subset X} \mathbb{Z}\cdot[V]\right) /(\,\text{rational equivalence}\,),$$

where V runs over all closed irreducible subvarieties of X of dimension d (that is, over all points of dimension d of X), $[V]$ is just the formal group generator corresponding to V, and the two combinations are called *rationally equivalent* if there exists a combination $W=\sum_l \nu_l\cdot[W_l]$ of $(d+1)$-dimensional irreducible subvarieties on $X\times\mathbb{P}^1$ such that $W|_{X\times\{0\}} = \sum_i \lambda_i\cdot[V_i]$ and $W|_{X\times\{1\}} = \sum_j \mu_j\cdot[U_j]$ (one can give the precise meaning to the notation $W|_{X\times\{a\}}$).

Have the action of various operations on the Chow groups.

<u>Push-forwards</u>

Let $f: X\to Y$ be a map of algebraic varoeties. It is called *projective*, if it can be decomposed as:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Y\times\mathbb{P}^n \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where j is a closed embedding.

Examples:

1) Closed embedding <u>is</u> a projective map.

2) $\mathbb{A}^1\to\mathrm{Spec}(k)$ is <u>not</u> a projective map.

3) X is projective (a closed subvariety in projective space), then any $f: X\to Y$ is projective.

Roughly speaking, f is projective if all the fibers are projective varieties.

If f is projective we have *push-forward maps*

$$f_* : \mathrm{CH}_d(X) \to \mathrm{CH}_d(Y),$$

where if $V \subset X$ is closed irreducible subvariety of X, and $U \subset Y$ is its image under f, then

$$f_*([V]) := \begin{cases} 0, \text{ if } \dim(U) < \dim(V); \\ \deg(k(V)/k(U)) \cdot [U], \text{ if } \dim(U) = \dim(V). \end{cases}$$

The coefficient $\deg(k(V)/k(U))$ here is just the "number of preimages" of the "sufficiently generic" point of U.

One can prove that in the case of projective map such definition respects the rational equivalence.

Warning: if f is not projective one can try to define f_* by the same formula, but the rational equivalence will not be respected.

Pull-backs

Together with the *dimensional* notations one can use the *codimensional* ones. Namely, $\mathrm{codim}(V \subset X) = \dim(X) - \dim(V)$, and we will denote the same Chow groups in two ways:

$$\mathrm{CH}_d(X) = \mathrm{CH}^{\dim(X)-d}(X).$$

Variety X is called *smooth* if locally it can be defined by $(n - \dim(X))$ equations $f_1, \dots, f_{n-\dim(X)}$ in some $\mathbb{A}^n$, so that the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ has (maximal possible) rank $(n - \dim(X))$ everywhere on X.

Examples:

1) Projective space is smooth.

2) q-nondegenerate quadratic, then the respective projective quadric Q is smooth. If q is degenerate, then Q is not smooth.

3) $\mathrm{Spec}(k[x,y]/(y^2 - x^3))$ is not smooth (singularity at $(0,0)$).

If X is smooth, one has *pull-back maps*

$$f^* : \mathrm{CH}^c(Y) \to \mathrm{CH}^c(X).$$

For arbitrary f it is not easy to see, how f^* acts on classes of subvarieties, but if f is *smooth morphism* (roughly speking, all the fibers are smooth varieties)(or even *flat morphism*), then $f^*([U]) = [f^{-1}(U)]$.

For arbitrary varieties X and Y one has the *external product*

$$\mathrm{CH}^a(X) \times \mathrm{CH}^b(Y) \xrightarrow{\times} \mathrm{CH}^{a+b}(X \times Y),$$

given by $[V] \times [U] \mapsto [V \times U]$. If now X is smooth we can combine this product with the pull-back along the diagonal morphism $\Delta : X \to X \times X$ to get a product structure on $\mathrm{CH}^*(X)$.

$$\begin{tikzcd} \mathrm{CH}^a(X) \times \mathrm{CH}^b(X) \arrow[r, "\times"] \arrow[dr, "\cdot"'] & \mathrm{CH}^{a+b}(X \times X) \arrow[d, "\Delta^*"] \\ & \mathrm{CH}^{a+b}(X). \end{tikzcd}$$

This gives the structure of the associative commutative ring on $\mathrm{CH}^*(X)$ for smooth variety X.

Category of Chow motives

Category of correspondences

Define $\mathcal{C}(k)$ - the *category of correspondences*:

$Ob(\mathcal{C}(k)) = \{$ smooth proj. var.$over$k$\} \ni [X]$ - typical representative.

$Mor_{\mathcal{C}(k)}([X],[Y]) = \mathrm{CH}_{\dim(X)}(X \times Y)$, where we assume X - connected.

composition:

Let $\varphi \in Mor_{\mathcal{C}(k)}([X],[Y]), \psi \in Mor_{\mathcal{C}(k)}([Y],[Z])$, in other words, $\varphi \in \mathrm{CH}_{\dim(X)}(X \times Y)$, $\psi \in \mathrm{CH}_{\dim(Y)}(Y \times Z)$.

Consider the natural projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "\pi_{X,Y}"'] \arrow[d, "\pi_{X,Z}"] \arrow[dr, "\pi_{Y,Z}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

Then the composition is defined as:

$$\psi \circ \varphi := ((\pi_{X,Z})_*((\pi_{X,Y})^*(\varphi) \cdot (\pi_{Y,Z})^*(\psi)).$$

It follows from the standard properties of pull-backs and push-forwards, that this operation is asociative.

In particular, one gets the associative ring structure on $\mathrm{CH}^{\dim(X)}(X \times X)$. Warning: do not mess it with the product ring structure on $\mathrm{CH}^*(X \times X)$ - our new composition product is almost never commutative, while the product structure is.

Have a natural functor

$$Sm.Proj./k \stackrel{C}{\to} \mathcal{C}(k)$$

from the category of smooth quasiprojective varieties over k to $\mathcal{C}(k)$, where $X \mapsto [X]$, and $(f : X \to Y) \mapsto [\Gamma_f]$, where $\Gamma_f \subset X \times Y$ is the *graph* of the map f. It is not difficult to check that this is really a functor (respects the composition).

Category of correspondences has a structure of tenzor additive category, where $[X] \oplus [Y] := [X \coprod Y]$ (the class of the disjoint union), and $[X] \otimes [Y] := [X \times Y]$.

Now, one can define the *category of effective Chow-motives* over k as the *Karoubian envelope* of $\mathcal{C}(k)$:

$$\mathrm{Chow}^{eff}(k) := Kar(\mathcal{C}(k)),$$

where the Karoubian (=*pseudo-abelian*) envelope of an additive category $\mathcal{C}$ is defined as follows. $p \in \mathrm{End}_{\mathcal{C}}(A)$ is called *projector*, if $p \circ p = p$. The $Kar(\mathcal{C})$ is a category such that $Ob(Kar(\mathcal{C})) = \{(A, p), A \in Ob(\mathcal{C}), p \in \mathrm{End}_{\mathcal{C}}(A)$ is a projector $\}$. $Mor_{Kar(\mathcal{C})}((A, p), (B, q)) = q \circ Mor_{\mathcal{C}}(A, B) \circ p \subset Mor_{\mathcal{C}}(A, B)$, and the composition is induced by that in $\mathcal{C}$.

There is natural functor $\mathcal{C}(k) \stackrel{Kar}{\to} \mathrm{Chow}^{eff}(k)$ sending $[X]$ to the pair $([X], id)$, and the structure of tenzor additive category descends from $\mathcal{C}(k)$ to $\mathrm{Chow}^{eff}(k)$. The composition of functors C and Kar gives a *motivic functor* from the category of smooth projective varieties over k to the category of the effective Chow motives.

$$\begin{tikzcd} & \mathcal{C}(k) \arrow[dr, "Kar"] & \\ Sm.Proj./k \arrow[ur, "C"] \arrow[rr, "M"'] & & \mathrm{Chow}^{eff}(k). \end{tikzcd}$$

For the smooth projective variety X we will call its image $M(X)$ - the *motive of* X.