

COMPUTATIONS IN NON-COMMUTATIVE IWASAWA THEORY

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ABSTRACT. We study special values of L -functions of elliptic curves over \mathbb{Q} twisted by Artin representations that factor through a false Tate curve extension $\mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})/\mathbb{Q}$. In this setting, we explain how to compute L -functions and the corresponding Iwasawa-theoretic invariants of non-abelian twists of elliptic curves. Our results provide both theoretical and computational evidence to the conjectures of non-commutative Iwasawa theory.

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1. INTRODUCTION

Let E/\mathbb{Q} be an elliptic curve and let $F_\infty = \bigcup_n F_n$ be an infinite Galois extension of \mathbb{Q} . The type of questions that we are concerned with is how do the Mordell-Weil group and the Selmer group of E change in the subfields of F_∞ , and the conjectural relations to the special values of twisted L -functions $L(E, \tau, s)|_{s=1}$ for Artin representations τ that factor through $\text{Gal}(F_\infty/\mathbb{Q})$.

Although for an arbitrary algebraic extension of \mathbb{Q} the questions might be hopelessly hard, a good deal is known for the p -adic extension of \mathbb{Q} obtained by adjoining all p -power roots of unity, $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_n \mathbb{Q}(\mu_{p^n})$. This is the subject of cyclotomic Iwasawa theory of elliptic curves. We will consider the false Tate curve extension of \mathbb{Q} , given by

$$F_n = \mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m}), \quad F_\infty = \bigcup_{n=1}^{\infty} F_n$$

for some fixed odd prime p and p -power free integer m . The motivation is that this is the simplest non-abelian p -adic Lie extension to which the conjectures of non-commutative Iwasawa theory apply.

Assume that E has good ordinary reduction at p . For Artin representations τ of $\text{Gal}(F_\infty/\mathbb{Q})$, the authors of [9, 22] have proposed precise modifications $\mathcal{L}_E(\tau)$ of the L -values $L(E, \tau, 1)$, that are supposed to be interpolated by a non-abelian p -adic

L -function of E . For self-dual Artin representations τ with values in $\mathrm{GL}_n(\mathbb{Q}_p)$, their Main Conjecture asserts that

$$(1.1) \quad \mathrm{ord}_p \mathcal{L}_E(\tau) = \mathrm{ord}_p \chi_{na}(E, \tau),$$

where $\chi_{na}(E, \tau)$ is the non-abelian Euler characteristic of the dual of the p^∞ -Selmer group of E/F_∞ twisted by τ (see §4.1). Moreover, the conjectures imply that if τ is congruent to τ' modulo p , then so are the p -adic numbers $\mathcal{L}_E(\tau)$ and $\mathcal{L}_E(\tau')$.

The purpose of this paper is to provide some evidence for these conjectures. We will be concerned with two particular representations σ and ρ of G which are congruent modulo p . First, under suitable hypotheses (see Theorem 5.2) we show that

$$(1.2) \quad \mathrm{ord}_p \chi_{na}(E, \sigma) = 0 \iff \mathrm{ord}_p \chi_{na}(E, \rho) = 0,$$

as predicted by the conjectures. We also show that the corresponding statement for $\mathrm{ord}_p \mathcal{L}_E(\sigma)$ and $\mathrm{ord}_p \mathcal{L}_E(\rho)$ implied by the Main Conjecture also follows from the conjecture of Birch and Swinnerton-Dyer (see §5.11). Finally, we test the full congruence between $\mathcal{L}_E(\sigma)$ and $\mathcal{L}_E(\rho)$ numerically for varying E , p and m .

Let us begin with a description of the Artin representations of $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$ (see, e.g., [19] for details). For $n \geq 1$, let ρ_n denote the representation of G obtained by inducing any character of exact order p^n of $\mathrm{Gal}(F_n/\mathbb{Q}(\mu_{p^n}))$ to $\mathrm{Gal}(F_n/\mathbb{Q})$. Then ρ_n is irreducible, and every irreducible Artin representation of G is of the form χ or $\rho_n \chi$ for some n , where χ is a 1-dimensional character of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. In particular, let σ_n denote the sum of all 1-dimensional characters of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. For $n \geq 1$, both σ_n and ρ_n are defined over \mathbb{Q} and, moreover, the two representations are congruent modulo p (meaning that their reductions modulo p have isomorphic semi-simplifications). For simplicity, we will denote $\sigma = \sigma_1$ and $\rho = \rho_1$.

Now, for any elliptic curve E/\mathbb{Q} and any Artin representation τ , the twisted L -function $L(E, \tau, s)$ is an analytic function for $\mathrm{Re} s > 3/2$ that is conjectured to have an analytic continuation to \mathbb{C} . In fact, for Artin representations of G , this conjecture follows from modularity of elliptic curves over \mathbb{Q} and cyclic base change from the theory of automorphic forms ([19], Theorem 14).

Deligne's conjecture [16] then states that the quotient of $L(E, \tau, 1)$ by the period $\Omega_+(E)^{d^+(\tau)} \Omega_-(E)^{d^-(\tau)}$ is an algebraic number (see the list of notation §1.6). This is also true for Artin representations of G by the results of [5]. There it is also shown that further modifying the value by the global ϵ -factor of τ ,

$$L^*(E, \tau) = \frac{L(E, \tau, 1)}{\Omega_+(E)^{d^+(\tau)} \Omega_-(E)^{d^-(\tau)}} \epsilon(\tau^{-1})$$

lies in the field of definition of τ . In particular, $L^*(E, \sigma_n)$ and $L^*(E, \rho_n)$ are rational numbers for $n \geq 1$.

Assume that E/\mathbb{Q} has good ordinary reduction at p . For the false Tate curve extension F_∞/\mathbb{Q} , the following modification of the L -value $L(E, \tau, 1)$ is proposed in [9]:

$$(1.3) \quad \mathcal{L}_E(\tau) = \frac{L_{v, \not{m} p}(E, \tau, 1)}{\Omega_+(E)^{d^+(\tau)} \Omega_-(E)^{d^-(\tau)}} c_p(\tau) \cdot \frac{P_p(\tau^*, u^{-1})}{P_p(\tau, w^{-1})} \cdot u^{-n(\tau)}.$$

Here $L_{v, \not{m} p}(E, \tau, s)$ is the L -function $L(E, \tau, s)$ with local factors at primes dividing m and p removed. (These are the primes with infinite ramification in F_∞/\mathbb{Q} .) See §1.6 for the definitions of the other quantities in (1.3). Then $\mathcal{L}_E(\tau) \in \mathbb{Q}_p$ and,

furthermore, one has the following conjectures (see [9], 5.6-5.10 and the penultimate paragraph of the paper). The definition of the non-abelian Euler characteristic $\chi_{na}(E, \tau)$ will be given in §4.1.

Conjecture 1.4. Let E be an elliptic curve over \mathbb{Q} with good ordinary reduction at p and let $\tau : G \rightarrow \mathrm{GL}_n(\mathbb{Q}_p)$ be a self-dual Artin representation. Then $L(E, \tau, 1) \neq 0$ if and only if $\chi_{na}(E, \tau)$ is finite. In this case, equation (1.1) holds.

Another consequence of the conjectures is that

$$(1.5) \quad \mathcal{L}_E(\sigma_n) \equiv \mathcal{L}_E(\rho_n) \pmod{p}.$$

(This follows from the fact that the representations are congruent modulo p and the conjectural integrality of the non-abelian p -adic L -function, see Remark A.20.) We can verify (1.5) numerically in many cases (see the Tables in Appendix B) for $n = 1$ and $p = 3, 5$ and 7 . For $p > 7$ or $n > 1$, our approach is too computationally demanding.

In fact, this type of congruence appears in the unpublished work of Balister [2], which was the original motivation for our study of these L -values. We also note that for σ and ρ (i.e. $n = 1$) and $p = 3$, Bouganis [4] has proved the congruence (1.5) in some cases.

There is a remarkable recent result due to Kato [30] that the conjectured existence and integrality of the p -adic L -function implies congruences to a higher power p^k for certain Artin representations of G . It would be very interesting to test these higher congruences, at least numerically. Unfortunately, the conductors of the relevant twists of E are extremely large.

For $p = 3$, if $\mathcal{L}_E(\sigma)$ and $\mathcal{L}_E(\rho)$ are p -adic units, then the congruence between them follows from the Birch–Swinnerton–Dyer conjecture. The point is that $\mathcal{L}_E(\sigma)$ and $\mathcal{L}_E(\rho)$ can be related to the Birch–Swinnerton–Dyer quotient for E/\mathbb{Q} , $E/\mathbb{Q}(\mu_p)$ and $E/\mathbb{Q}(\sqrt[m]{m})$. This gives a formula for the quotient $\mathcal{L}_E(\sigma)/\mathcal{L}_E(\rho)$, where the only difficult terms are the orders of the Tate–Shafarevich groups. As these are squares, for $p = 3$ they do not affect the congruence modulo 3. For $p \geq 5$ this argument gives the congruence modulo squares in \mathbb{F}_p , but the full congruence appears to be stronger than what can be deduced from the Birch–Swinnerton–Dyer conjecture.

The structure of the paper is as follows.

In §2 and §3 we fix the notation and review some basic results concerning systems of l -adic representations associated to twists of elliptic curves, Selmer groups and the cyclotomic Iwasawa theory of elliptic curves. Also, in §3.21 we prove an auxiliary result concerning characteristic elements that we use later on (Proposition 3.23, Corollary 3.24).

Then we turn to false Tate curve extensions and non-commutative Iwasawa theory of elliptic curves in §4. We prove the relation (1.2) in Theorem 5.2. From here, the Main Conjecture implies the corresponding relation between the values $\mathcal{L}_E(\sigma)$ and $\mathcal{L}_E(\rho)$ of the p -adic L -function, that we show to be compatible with the Birch–Swinnerton–Dyer conjecture (Proposition 5.13).

Next, we turn to numerical verification of the congruence (1.5) for $n = 1$. In §6 we explain how to compute $L(E, \sigma, 1)$, $L(E, \rho, 1)$ and various related arithmetic invariants. We give an example for the elliptic curve 21A4 over $\mathbb{Q}(\mu_5, \sqrt[5]{2})$ illustrating both our results and computations in §7, and tabulate all other examples in Appendix B. In §7.19 we also illustrate how similar computations can be carried

out when the false Tate curve extension is replaced by a GL_2 extension, by taking the curves 11A2 and 11A3 over their 5-division fields.

Appendix A by J. Coates and R. Sujatha deals with theoretical questions related to regularity and Heegner like phenomena for the extension F_∞/\mathbb{Q} . In particular, they generalise (1.2) to a larger class of curves and ρ_n in place of ρ (Corollary A.14).

1.6. Notation. By a number field we will always mean a finite extension of \mathbb{Q} . We fix, once and for all, embeddings of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_l$ for all l . Throughout the paper we keep the following notation:

p	odd prime.
μ_{p^n}	group of p^n -th roots of unity.
μ_{p^∞}	union of μ_{p^n} for $n \geq 1$.
k_v	completion of a field k at a prime v ,
\mathbb{F}_v	residue field at a prime v ,
τ^*	contragredient representation of an Artin representation τ ,
$d^\pm(\tau)$	dimensions of the ± 1 eigenspaces of complex conjugation on τ ,
$\epsilon(\tau)$	the global ϵ -factor of τ ,
$\epsilon_p(\tau)$	the local ϵ -factor of τ at p (see §6.10),
$n(\tau)$	the p -valuation of the conductor $N(\tau)$ (see §2.4),
$ S $	cardinality of S ,
$A[p]$	p -torsion of an abelian group A ,
$A[p^\infty]$	p -primary component of an abelian group A .

Notation for the false Tate curve extension and its representations:

m	an integer ≥ 2 such that $n^p \nmid m$ for $n > 1$,
F	$= F_1 = \mathbb{Q}(\mu_p, \sqrt[p]{m})$,
K	$= \mathbb{Q}(\mu_p)$,
σ	$= \sigma_1$, the regular representation of $\text{Gal}(K/\mathbb{Q})$,
ρ	$= \rho_1$, the unique $(p-1)$ -dimensional irreducible representation of $\text{Gal}(F/\mathbb{Q})$, as above.

Notation relating to elliptic curves and Iwasawa theory:

E	elliptic curve over \mathbb{Q} with good ordinary reduction at p ,
$P_q(-, T)$	local polynomial of the L -series $L(-, s)$ at q (see §2.2),
u, w	p -adic numbers, chosen so that u is a p -adic unit, defined by
	$P_p(E, T) = 1 - a_p T + pT^2 = (1 - uT)(1 - wT)$.

We denote by $\Omega_\pm(E)$ the periods of E , defined by integrating the Néron differential of a global minimal Weierstrass equation over the generators of the ± 1 -eigenspaces $H_1(E(\mathbb{C}), \mathbb{Z})^\pm$ of complex conjugation. As usual, Ω_- is chosen to lie in $i\mathbb{R}$.

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2. TWISTED L -FUNCTIONS

We briefly recall the definition of L -functions of elliptic curves by Artin representations, and the invariants attached to them which we will need. We refer to [39] and [38] for ζ - and L -functions of varieties, and to [16], [15] §§3-4 and [49] §4 for their twists.

2.1. Systems of l -adic representations. Let E/\mathbb{Q} be an arbitrary elliptic curve and $\tau : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_n(\bar{\mathbb{Q}})$ be an Artin representation. Both E and τ determine (a compatible system of) l -adic representations for primes l of \mathbb{Q} . In case of E , the l -adic representation is $M_l(E) = H_{\text{ét}}^1(E, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$ or, equivalently, the dual of the l -adic Tate module $T_l(E)$ with scalars extended to $\bar{\mathbb{Q}}_l$. The l -adic representation that corresponds to τ is $M_l(\tau) = \tau \otimes \bar{\mathbb{Q}}_l$. Now we can construct a system of representations

$$M_l(E, \tau) = M_l(E) \otimes_{\bar{\mathbb{Q}}_l} M_l(\tau) .$$

2.2. L -functions. To a system of l -adic representations $M = \{M_l\}_l$ we can associate an L -function $L(M, s)$ as follows. For a prime q of \mathbb{Q} , the local polynomials of $L(M, s)$ are

$$(2.3) \quad P_q(M, T) = \det(1 - \text{Frob}_q^{-1} T \mid M_l^{I_q})$$

for any prime $l \neq q$. We define the local L -factor

$$L_q(M, s) = P_q(M, q^{-s})^{-1}$$

and the global L -function (or L -series)

$$L(M, s) = \prod_q L_q(M, s) .$$

We write

$$L(E, s) = L(M(E), s), \quad L(\tau, s) = L(M(\tau), s), \quad L(E, \tau, s) = L(M(E, \tau), s) .$$

The L -series $L(\tau, s)$ converges to an analytic function on the half-plane $\text{Re } s > 1$, and it is classical that it has a meromorphic continuation to the entire complex plane. The L -series $L(E, s)$ and $L(E, \tau, s)$ converge to analytic functions on $\text{Re } s > 3/2$. It follows from the modularity of elliptic curves [55, 50, 7] that $L(E, s)$ possesses an analytic continuation to \mathbb{C} . We will be concerned with $L(E, \tau, s)$ when τ factors through a false Tate curve extension, in which case there is also an analytic continuation ([19], Theorem 14).

Finally, recall that L -functions are multiplicative,

$$L(E, \tau_1 \oplus \tau_2, s) = L(E, \tau_1, s) L(E, \tau_2, s) .$$

2.4. Conductors. In the formula (2.3) defining the local polynomials of $L(M, s)$, the inertia group I_q acts trivially on M_l for all but finitely many primes q . Therefore for almost all primes q (the *good* primes of M) we have $\deg P_q(M, T) = \dim M_l$. The primes q with $\deg P_q(M, T) < \dim M_l$ are called *bad* primes of M .

For a prime q the local conductor $N_q(M)$ is given by (see [39], §2)

$$N_q(M) = q^{t_q + \delta_q}, \quad t_q = \text{codim}_{M_l} M_l^{I_q}, \quad \delta_q \geq 0 .$$

The term δ_q is defined in terms of the representation of the wild inertia subgroup on M_l . It is zero if and only if this subgroup acts trivially, i.e. M_l is tamely ramified. Finally, the (global) conductor is $N(M) = \prod_q N_q(M)$. Once again, we write

$$N(E) = N(M(E)), \quad N(\tau) = N(M(\tau)), \quad N(E, \tau) = N(M(E, \tau)) .$$

2.5. Functional equation. The twisted L -functions $L(E, \tau, s)$ conjecturally satisfy a functional equation of the following form (see, e.g, [16] 5.2, [49] 4.5). Let

$$\hat{L}(E, \tau, s) = \left(\frac{N(E, \tau)}{\pi^{2 \dim \tau}} \right)^{s/2} \Gamma\left(\frac{s}{2}\right)^{\dim \tau} \Gamma\left(\frac{s+1}{2}\right)^{\dim \tau} L(E, \tau, s).$$

Then, conjecturally,

$$(2.6) \quad \hat{L}(E, \tau, s) = w(E, \tau) \hat{L}(E, \tau^*, 2-s)$$

with $w(E, \tau)$ an algebraic number of complex absolute value 1. If $\tau \cong \tau^*$, then $w(E, \tau) = \pm 1$ and we call it the sign in the functional equation.

2.7. L -functions of elliptic curves over number fields. Recall that for an elliptic curve E a number field k , the L -function $L(E/k, s)$ is given by the Euler product

$$L(E/k, s) = \prod_v L_v(E/k, s) = \prod_v P_v(E/k, N_{k/\mathbb{Q}}(v)^{-s})^{-1},$$

where v runs over primes of k and

$$P_v(E/k, T) = \det(1 - \text{Frob}_v^{-1} T \mid M_l(E)^{I_v}), \quad v \nmid l.$$

The polynomials $P_v(E/k, T)$ depend on the reduction type of E over the local field k_v and are given explicitly by:

$$(2.8) \quad P_v(E/k, T) = \begin{cases} 1 - a_v T + N_{k/\mathbb{Q}}(v) T^2, & \text{good reduction,} \\ 1 - T, & \text{split multiplicative reduction,} \\ 1 + T, & \text{non-split multiplicative reduction,} \\ 1, & \text{additive reduction.} \end{cases}$$

In the case of good reduction, $P_v(E/k, 1) = |\tilde{E}(\mathbb{F}_v)|$, the number of points on the reduction of E at v .

Suppose the elliptic curve E is defined over \mathbb{Q} and let k/\mathbb{Q} be a finite extension. Define R_k to be the representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ induced from the trivial representation of $\text{Gal}(\bar{\mathbb{Q}}/k)$. In particular, if k/\mathbb{Q} is Galois, R_k is the regular representation of $\text{Gal}(k/\mathbb{Q})$ considered as a representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Also, for k arbitrary, R_k is the permutation representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set of embeddings of k into $\bar{\mathbb{Q}}$. One can show that the local factors satisfy Artin formalism,

$$L_q(E, R_k, s) = \prod_{v|q} L_v(E/k, s).$$

In particular, taking the product over all primes,

$$L(E, R_k, s) = L(E/k, s).$$

This allows us to write $L(E/k, s)$ as a product of twisted L -functions of E/\mathbb{Q} that correspond to irreducible pieces of R_k . If k/\mathbb{Q} is abelian, $L(E/k, s)$ becomes a product of L -functions of E twisted by Dirichlet characters.

3. CYCLOTOMIC IWASAWA THEORY OF ELLIPTIC CURVES

In this section we introduce the main tools from cyclotomic Iwasawa theory of elliptic curves. We use the following notation

- p odd prime.
- k a number field.
- k^{cyc} the maximal pro- p extension of k in $k(\mu_{p^\infty})$.
- Γ_k the Galois group $\text{Gal}(k^{cyc}/k)$.
- Λ_k Iwasawa algebra of Γ_k .

Recall that $\Lambda_k = \varprojlim_H \mathbb{Z}_p[\Gamma_k/H]$ where the limit is taken over the open subgroups H of Γ_k . So it is simply the completion of the group ring $\mathbb{Z}_p[\Gamma_k]$ in the profinite topology. The group Γ_k is isomorphic to \mathbb{Z}_p and there is an isomorphism

$$\Lambda_k \cong \mathbb{Z}_p[[T]]$$

obtained by sending a fixed topological generator of Γ_k to $1 + T$.

We recall the definition of the p^n - and p^∞ -Selmer groups of an elliptic curve E over an arbitrary algebraic extension of \mathbb{Q} . (These definitions do not require that p is odd or that E is good ordinary at p .)

If k is a number field, we have an exact sequence

$$0 \longrightarrow \text{Sel}_{p^n}(E/k) \longrightarrow H^1(k, E[p^n]) \longrightarrow \prod_v H^1(k_v, E(\bar{k}_v))[p^n]$$

where v runs over the places of k ([43], §X.4). For arbitrary $L \subset \bar{\mathbb{Q}}$, let

$$\text{Sel}_{p^n}(E/L) = \varinjlim_{k \subset L} \text{Sel}_{p^n}(E/k)$$

where k runs over finite extensions of \mathbb{Q} and the maps that define the inductive system are induced by the restriction maps on cohomology. Finally, define the p^∞ -Selmer group by

$$\text{Sel}_{p^\infty}(E/L) = \varinjlim_{n \geq 1} \text{Sel}_{p^n}(E/L).$$

Over a number field, the group $\text{Sel}_{p^\infty}(E/k)$ is a part of the fundamental exact sequence

$$(3.1) \quad 0 \longrightarrow E(k) \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \text{Sel}_{p^\infty}(E/k) \longrightarrow \text{III}(E/k)[p^\infty] \longrightarrow 0,$$

where $\text{III}(E/k)$ is the Tate-Shafarevich group of E/k .

The Selmer group $\text{Sel}_{p^\infty}(E/k^{cyc})$ is a discrete Λ_k -module. Let $X(E/k^{cyc})$ be its Pontryagin dual,

$$X(E/k^{cyc}) = \text{Hom}(\text{Sel}_{p^\infty}(E/k^{cyc}), \mathbb{Q}_p/\mathbb{Z}_p),$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is a Λ_k -module with trivial Γ_k -action. This is a compact Λ_k -module and it is an elementary fact in Iwasawa theory that $X(E/k^{cyc})$ is finitely generated over Λ_k (see, e.g., [32] Thm. 4.5.(a) or [12]).

3.2. Torsion modules. A Λ_k -module is said to be *torsion* if for each element of the module there is some non-zero element in Λ_k annihilating it.

We recall an important result due to Mazur and its corollary which we will use later on:

Theorem 3.3. (Mazur's control theorem, [34], [24] Theorem 4.1). *Let k be a number field and E/k an elliptic curve with good ordinary reduction at all primes above p . Assume that $k_\infty = \bigcup_n k_n$ is a Galois extension of k with Galois group \mathbb{Z}_p . Then the natural maps*

$$\mathrm{Sel}_{p^\infty}(E/k_n) \longrightarrow \mathrm{Sel}_{p^\infty}(E/k_\infty)^{\mathrm{Gal}(k_\infty/k_n)}$$

have finite kernels and cokernels. Their orders are bounded as $n \rightarrow \infty$.

Theorem 3.4. ([34], [24] Corollary 4.9) *Let k be a number field and let E/k be an elliptic curve with good ordinary reduction at all primes above p . If $\mathrm{Sel}_{p^\infty}(E/k)$ is finite, then $X(E/k^{cy_c})$ is Λ_k -torsion.*

Mazur has conjectured that $X(E/k^{cy_c})$ is always Λ_k -torsion when E has good ordinary reduction at all primes v of k dividing p . The best result to date in this direction is the following deep theorem due to Kato.

Theorem 3.5. (Kato [29]) *If k/\mathbb{Q} is abelian and E/\mathbb{Q} has good ordinary reduction at p , then $X(E/k^{cy_c})$ is Λ_k -torsion.*

3.6. Structure theory. A map of Λ_k -modules is a *pseudo-isomorphism* (denoted $X \cong_{p_s} Y$) if it has finite kernel and cokernel. There is a well-known structure theory for finitely generated Λ_k -torsion modules up to pseudo-isomorphism (see e.g. [6] Ch. 7 or [54] Theorem 13.12). For such a module X , one has

$$(3.7) \quad X \cong_{p_s} \bigoplus_i \Lambda_k/p^{\mu_i} \oplus \bigoplus_j \Lambda_k/f_j^{m_j},$$

Here the direct sums are finite, $\mu_i, m_j \geq 1$ are integers and f_j are elements of Λ_k . The f_j are defined up to units in Λ_k and they can be chosen to map to irreducible distinguished polynomials under the isomorphism $\Lambda_k \cong \mathbb{Z}_p[[T]]$. Recall that *distinguished* means monic with all other coefficients divisible by p . With such a choice of a canonical form for the f_j , the above decomposition is unique up to order. The product

$$f_X = \prod_i p^{\mu_i} \prod_j f_j^{m_j}$$

is called a characteristic element for X . It is uniquely defined up to a Λ_k -unit and the characteristic ideal $(f_X) \subset \Lambda_k$ is well-defined. Characteristic elements are multiplicative in short exact sequences.

Now let E/\mathbb{Q} be an elliptic curve with good ordinary reduction at p . Provided that $X(E/k^{cy_c})$ is a Λ_k -torsion module (e.g. if k/\mathbb{Q} is abelian), we can decompose $X(E/k^{cy_c})$ as in (3.7) and define the following invariants:

$$\begin{aligned} f_{E/k} &= f_{X(E/k^{cy_c})} = \prod_i p^{\mu_i} \prod_j f_j^{m_j}, \text{ the characteristic element.} \\ \mu_{E/k} &= \sum_i \mu_i, \text{ the } \mu\text{-invariant.} \\ \lambda_{E/k} &= \deg f_{E/k} = \dim_{\mathbb{Q}_p}(X(E/k^{cy_c}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p), \text{ the } \lambda\text{-invariant.} \end{aligned}$$

Remark 3.8. We note that $\mu_{E/k} = 0$ if and only if $X(E/k^{cy_c})$ is a finitely generated \mathbb{Z}_p -module, and $\lambda_{E/k} = \mu_{E/k} = 0$ if and only if $X(E/k^{cy_c})$ is finite.

Remark 3.9. Assuming $X(E/k^{cy_c})$ is a Λ_k -torsion module, it is easy to see that $f_{E/k}$ is divisible by T if and only if the coinvariant space $(X(E/k^{cy_c}))_{\Gamma_k}$ is infinite. By Mazur's control theorem (3.3), this is true if and only if $\mathrm{Sel}_{p^\infty}(E/k)$ is infinite. In particular, $\mathrm{Sel}_{p^\infty}(E/k)$ is finite if and only if $f_{E/k}$ has a non-zero constant term.

We also mention the following result.

Theorem 3.10. (Matsuno [33]) *Let p be an odd prime. Let k be a totally imaginary algebraic number field. Let E/k be an elliptic curve which has good reduction at all primes above p . If $X(E/k^{cy_c})$ is Λ_k -torsion and $\mu_{E/k} = 0$, then $X(E/k^{cy_c})$ is p -torsion-free. In particular, if $\lambda_{E/k}$ is also zero, then $X(E/k^{cy_c}) = 0$.*

3.11. Euler characteristics. Let E be an elliptic curve over a number field k and assume that $X(E/k^{cy_c})$ is Λ_k -torsion. When $\text{Sel}_{p^\infty}(E/k)$ is finite, the homology groups $H_i(\Gamma_k, X(E/k^{cy_c}))$ are finite ($i = 0, 1$), and the Γ_k -Euler characteristic is defined by

$$\chi_{cy_c}(E/k) = \frac{|H_0(\Gamma_k, X(E/k^{cy_c}))|}{|H_1(\Gamma_k, X(E/k^{cy_c}))|}.$$

(When $\text{Sel}_{p^\infty}(E/k)$ is infinite, the groups are no longer finite, but one can nevertheless make sense of the quotient and define a ‘‘generalised Euler characteristic’’, see e.g. [11]).

Of importance to us are the following two formulae. First, $\chi_{cy_c}(E/k)$ is a power of p , and we have the basic Euler characteristic formula [36, 37]

$$(3.12) \quad \text{ord}_p \chi_{cy_c}(E/k) = \text{ord}_p \frac{|\text{III}(E/k)[p^\infty]| \prod_{v|p} |\tilde{E}(\mathbb{F}_v)|^2 \prod_v c_v}{|E(k)|^2}.$$

Second, under the isomorphism $\Lambda \cong \mathbb{Z}_p[[T]]$, the constant term of the characteristic element recovers $\chi_{cy_c}(E/k)$,

$$(3.13) \quad \text{ord}_p \chi_{cy_c}(E/k) = \text{ord}_p f_{E/k}(0).$$

Lemma 3.14. *Let E be an elliptic curve over a number field k such that E has good ordinary reduction at all primes above p . Suppose that $\text{Sel}_{p^\infty}(E/k)$ is finite. Then $X(E/k^{cy_c})$ is finite if and only if $\chi_{cy_c}(E/k) = 1$.*

Proof. By Theorem 3.4, $X(E/k^{cy_c})$ is Λ_k -torsion. By (3.13) we have $f_j \neq T$ for every j . The f_j are distinguished irreducible polynomials, so that every $f_j \neq T$ has a constant term divisible by p . Lemma follows from (3.13). \square

3.15. λ - and μ -invariants in p -extensions. The following theorem, due to Hachimori and Matsuno, concerns the change in the λ -invariant of $X(E/k^{cy_c})$ when k is replaced by a finite p -extension of k . This will be very useful when studying the behaviour of E in a false Tate curve tower.

Theorem 3.16. ([25], Theorem 3.1) *Let k'/k be a Galois extension of number fields of p -power degree. Let E/k be an elliptic curve, good ordinary at primes above p and such that whenever E has additive reduction at a prime v , the reduction stays additive at every prime of k'^{cy_c} above v . If $X(E/k)$ is Λ_k -torsion and $\mu_{E/k} = 0$, then $X(E/k')$ is $\Lambda_{k'}$ -torsion, $\mu_{E/k'} = 0$ and*

$$(3.17) \quad \lambda_{E/k'} = [k'^{cy_c} : k^{cy_c}] \lambda_{E/k} + \sum_{v \in V_1} (e(v) - 1) + 2 \sum_{v \in V_2} (e(v) - 1).$$

Here $e(v)$ is the ramification index of v in k'^{cy_c}/k^{cy_c} , V_1 is the set of primes $v \nmid p$ of k'^{cy_c} where E has split multiplicative reduction and V_2 is the set of primes $v \nmid p$ of k'^{cy_c} where E has good reduction and $E(k'_v)^{cy_c}[p] \neq 0$.

Remark 3.18. The condition that the reduction stays additive is automatically satisfied if $p \geq 5$ ([42], p. 498), and is vacuous if E/k is semistable.

It is useful to reformulate some of the conditions in the theorem as follows:

Lemma 3.19. *Let k'/k be a Galois extension of number fields of p -power degree. Let E/k be an elliptic curve such that whenever E has additive reduction at a prime of k , the reduction stays additive at every prime of k' above it. Let $v \nmid p$ be a prime of k .*

- (1) *v ramifies in k' if and only if any prime above v in k^{cy_c} ramifies in k'^{cy_c} .*
- (2) *E has split multiplicative reduction at v if and only if E has split multiplicative reduction at any prime of k'^{cy_c} above v .*
- (3) *If E/k has good reduction at $v \nmid p$ and w is a prime of k'^{cy_c} above v , then $E(k_w'^{cy_c})[p] = 0$ if and only if $\tilde{E}(\mathbb{F}_v)[p] = 0$.*

Proof. First observe that the condition that the reduction stays additive at every prime of k' above a given prime of k ensures that it also stays additive over every prime of k'^{cy_c} above it. This follows from the fact that p is the only prime that ramifies in the p -cyclotomic extension of a number field.

(1). This also follows from the fact that p is the only prime that ramifies in the p -cyclotomic extension of a number field.

(2). If E/k has additive reduction, it stays additive in k'^{cy_c} by assumption. If the reduction is non-split multiplicative, it stays the same because the degree of the extension is (pro-)odd.

(3). We show the asserted equivalence in two steps. First, for a prime w of k'^{cy_c} above v apply the multiplication by p map to the exact sequence

$$0 \rightarrow \hat{E}(m_w) \rightarrow E(k_w'^{cy_c}) \rightarrow \tilde{E}(\mathbb{F}_w) \rightarrow 0$$

where m_w is the maximal ideal and \mathbb{F}_w is the residue field of the local field $k_w'^{cy_c}$. We get a piece of the kernel-cokernel exact sequence

$$\hat{E}(m_w)[p] \rightarrow E(k_w'^{cy_c})[p] \rightarrow \tilde{E}(\mathbb{F}_w)[p] \rightarrow \hat{E}(m_w)/p\hat{E}(m_w).$$

Since $[p]$ on $\hat{E}(m_w)$ is an isomorphism [43], IV.2.3, it follows that $E(k_w'^{cy_c})[p] \cong \tilde{E}(\mathbb{F}_w)[p]$.

It remains to show that $E(\mathbb{F}_v)[p] = 0$ if and only if $E(\mathbb{F}_w)[p] = 0$. This follows from Nakayama's lemma: $G = \text{Gal}(\mathbb{F}_w/\mathbb{F}_v)$ is pro- p , so for any discrete p -primary G -module $A = 0$ if and only if $A^G = 0$. \square

Corollary 3.20. *Under the assumptions of Theorem 3.16, we have that $\lambda_{E/k'} = \lambda_{E/k}$ if and only if either $k' \subset k^{cy_c}$ or the following conditions are satisfied*

- (1) $\lambda_{E/k} = 0$.
- (2) *There are no primes of split multiplicative reduction of E/k that ramify in k'/k .*
- (3) *There are no primes $v \nmid p$ of good reduction of E/k that ramify in k'/k and such that there is a non-trivial point of order p on the reduced curve \tilde{E}/\mathbb{F}_v .*

Proof. This follows from Theorem 3.16 and Lemma 3.19. \square

3.21. Divisibility of characteristic elements. The main result of this section (Proposition 3.23) concerns the behaviour of the characteristic element $f_{E/k}$ when changing the base field.

Lemma 3.22. *Let k'/k be a finite Galois extension with Galois group Δ and let E be an elliptic curve over k . Then the natural restriction map*

$$H^1(k, E[p^\infty]) \xrightarrow{\text{Res}} H^1(k', E[p^\infty])^\Delta$$

has finite p -power kernel and cokernel. If either $p \nmid |\Delta|$ or E/k' has trivial p -torsion, then Res is an isomorphism.

Proof. Denote $M = E[p^\infty](k')$. The map Res is a part of the inflation-restriction sequence

$$0 \longrightarrow H^1(\Delta, M) \longrightarrow H^1(k, E[p^\infty]) \xrightarrow{\text{Res}} H^1(k', E[p^\infty])^\Delta \longrightarrow H^2(\Delta, M).$$

The groups $H^i(\Delta, M)$ are annihilated by $|\Delta|$. Furthermore, they are cofinitely generated continuous \mathbb{Z}_p -modules (since M is), and are therefore finite. If either $p \nmid |\Delta|$ or $M = 0$, these groups are trivial and Res is an isomorphism. \square

Proposition 3.23. *Let k'/k be a finite Galois extension of number fields such that $k' \cap k^{cy_c} = k$. Let E/k be an elliptic curve such that $X(E/k^{cy_c})$ is Λ_k -torsion and $X(E/k'^{cy_c})$ is $\Lambda_{k'}$ -torsion. Fix an isomorphism $\Lambda_k \cong \mathbb{Z}_p[[T]]$. It induces naturally an isomorphism $\Lambda_{k'} \cong \mathbb{Z}_p[[T]]$ and, with this identification, $f_{E/k}$ divides $f_{E/k'}$.*

Proof. Denote $\Delta = \text{Gal}(k'/k)$. By Lemma 3.22 and the definition of the Selmer group, we have an exact sequence of Γ_k -modules

$$0 \longrightarrow A \longrightarrow \text{Sel}_{p^\infty}(k^{cy_c}) \longrightarrow \text{Sel}_{p^\infty}(k'^{cy_c})^\Delta \longrightarrow B \longrightarrow 0$$

with A finite. As Pontryagin dual is exact,

$$0 \longrightarrow \hat{B} \longrightarrow X(E/k'^{cy_c})/I \longrightarrow X(E/k^{cy_c}) \longrightarrow \hat{A} \longrightarrow 0$$

for some ideal I , so $X(E/k^{cy_c})$ is pseudo-isomorphic to a quotient of $X(E/k'^{cy_c})/I$ as a Λ_k -module.

We have two structures of a Λ_k -module on $X(E/k'^{cy_c})/I$.

First, we have that $\Lambda_{k'} \cong \Lambda_k$ as follows: if $\sigma \in \Gamma_{k'} = \text{Gal}(k'^{cy_c}/k')$, we can consider it as an element of $\text{Gal}(k'^{cy_c}/k)$ and we take it modulo $\Delta = \text{Gal}(k'^{cy_c}/k^{cy_c})$. This gives an element of $\text{Gal}(k^{cy_c}/k) = \Gamma_k$. This gives an isomorphism $\Gamma_{k'} \cong \Gamma_k$ since $\text{Gal}(k'^{cy_c}/k)$ is a direct product of Δ and $\Gamma_{k'}$. Hence $\Lambda_{k'} \cong \Lambda_k$, inducing the asserted isomorphism $\Lambda_{k'} \cong \mathbb{Z}_p[[T]]$. This also allows us to consider $X(E/k'^{cy_c})/I$ as a Λ_k -module.

Second, $\text{Sel}(k')$ is a $\text{Gal}(k'^{cy_c}/k)$ -module and $\text{Sel}(k')^\Delta$ is a $\text{Gal}(k^{cy_c}/k)$ -module. This gives a Λ_k -module structure on its Pontryagin dual $X(E/k'^{cy_c})/I$.

By inspection, these Λ_k -module structures are the same. Hence the Λ_k -characteristic element of $X(E/k'^{cy_c})/I$ can be identified with its $\Lambda_{k'}$ -characteristic element. By multiplicativity in short exact sequences, the former is divisible by $f_{E/k}$ while the latter divides $f_{E/k'}$. This completes the proof. \square

Corollary 3.24. *Let k'/k be a finite Galois extension of number fields such that $k' \cap k^{cy_c} = k$. Let E/k be an elliptic curve such that $X(E/k^{cy_c})$ is Λ_k -torsion and $X(E/k'^{cy_c})$ is $\Lambda_{k'}$ -torsion. Then $\lambda_{E/k'} \geq \lambda_{E/k}$ and $\mu_{E/k'} \geq \mu_{E/k}$. Moreover, assuming that $\text{Sel}_{p^\infty}(E/k)$ is finite, we have*

$$\begin{aligned} \lambda_{E/k'} = \lambda_{E/k}, & \iff |\text{Sel}_{p^\infty}(E/k')| < \infty, \\ \mu_{E/k'} = \mu_{E/k} & \iff \chi_{cy_c}(E/k') = \chi_{cy_c}(E/k). \end{aligned}$$

Proof. The first statement follows directly from the proposition. For the second statement, observe that the proposition implies that $\lambda_{E/k'} = \lambda_{E/k}$ and $\mu_{E/k'} = \mu_{E/k}$ if and only if $f_{E/k'} = f_{E/k}$. Now the implication “ \Leftarrow ” follows from formula (3.13). For the reverse implication, $|\text{Sel}_{p^\infty}(E/k')| < \infty$ is a consequence of Remark 3.9, and the equality of Euler characteristics follows from formula (3.13). \square

4. FALSE TATE CURVE EXTENSIONS

We now turn to non-commutative Iwasawa theory in the setting of a false Tate curve extension F_∞/\mathbb{Q} , with $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$. As always, p is an odd prime, $m \geq 2$ a p -power free integer, and E/\mathbb{Q} is an elliptic curve with good ordinary reduction at p . We write

$$\begin{aligned} G_k & \text{ Gal}(F_\infty/k), \text{ for a number field } k \subset F_\infty, \\ \Lambda(G_k) & \text{ Iwasawa algebra of } G_k, \text{ that is } \varprojlim_{H \leq G_k, \text{ open}} \mathbb{Z}_p[G_k/H], \\ X(E/F_\infty) & \text{ Pontryagin dual of } \text{Sel}_{p^\infty}(E/F_\infty), \text{ as in the cyclotomic case,} \\ P_1^{(k)} & \text{ for } k \subset F_\infty, \text{ this is the set of primes } w \text{ of } k \text{ with } w|m, w \nmid p, \text{ where} \\ & \text{ } E \text{ has split multiplicative reduction,} \\ P_2^{(k)} & \text{ for } k \subset F_\infty, \text{ this is the set of primes } w \text{ of } k \text{ with } w|m, w \nmid p, \text{ where} \\ & \text{ } E \text{ has good reduction and the reduced curve has a point of order } p. \end{aligned}$$

4.1. Non-abelian Euler characteristic. Let $k \subset F_\infty$ be a number field and $G_k = \text{Gal}(F_\infty/k)$. Similarly to the cyclotomic case, one can define a non-abelian G_k -Euler characteristic of a compact G_k -module X by

$$\chi(G_k, X) = \prod_{i=0}^2 |H_i(G_k, X)|^{(-1)^i},$$

provided the above groups are finite. For an elliptic curve E/k we also write

$$\chi_{na}(E/k) = \chi(G_k, X(E/F_\infty))$$

and call it the non-abelian Euler characteristic of E/k . One can also define it using the G_k -homology groups of $\text{Sel}_{p^\infty}(E/F_\infty)$, as for instance in [26]. The equivalence of these definitions is a simple duality argument (see, e.g. [27] §1.1).

For a p -adic representation $\tau : G_k \rightarrow \text{GL}_n(\mathbb{Z}_p)$ with finite image, write

$$\chi_{na}(E, \tau) = \chi_{na}(G_k, X(E/F_\infty) \otimes_{\mathbb{Z}_p} \tau).$$

In fact, one can define the Euler characteristic for all Artin twists and not just those defined over \mathbb{Z}_p (see [9]), but we will not need this.

4.2. A formula of Hachimori–Venjakob. There is an explicit formula for $\chi_{na}(E/k)$ in terms of $\chi_{cyc}(E/k)$ due to Hachimori and Venjakob ([26], thm 4.1 and the note following it).

Theorem 4.3. *Assume $p \geq 5$. Let $k \subset F_\infty$ be a number field that contains $K = \mathbb{Q}(\mu_p)$. Suppose E/k has good ordinary reduction at all primes in k above p , and that $\text{Sel}_{p^\infty}(E/k)$ is finite. Then $\chi_{cyc}(E/k)$ and $\chi_{na}(E/k)$ are finite and satisfy*

$$(4.4) \quad \chi_{na}(E/k) = \chi_{cyc}(E/k) \prod_{v \in P_1^{(k)} \cup P_2^{(k)}} |L_v(E/k, 1)|_p,$$

Remark 4.5. In fact Hachimori and Venjakob have a slightly different definition of $P_2^{(k)}$. For a prime $v \nmid p$ where E has good reduction, their condition is that $E(k_v)[p] \neq 0$, where k_v is the completion of k at v . It is equivalent to our requirement that $\tilde{E}(\mathbb{F}_v)[p] = 0$ for the reduced curve, since multiplication by p is an automorphism of the formal group of E at v (see [43], VII.2.1, VII.2.2 and IV.2.3).

Remark 4.6. Let $v \nmid p$ be a prime of k . Denote the residue field of v by \mathbb{F}_v and its number of elements by Nv . Consider the local polynomial $P_v(p^{-s}) = L_v(E, s)^{-1}$ and let us write down how the valuation of $P_v(1/Nv) = L_v(E, 1)^{-1}$ depends on the reduction type of E at v :

- (1) If E has split multiplicative reduction at v , then $P_v(T) = 1 - T$ and $Nv \equiv 1$ modulo p , so $P_v(1/Nv)$ has positive p -valuation.
- (2) If E has non-split multiplicative reduction at v , then $P_v(T) = 1 + T$ and $Nv \equiv 1$ modulo p , so $P_v(1/Nv)$ is a p -adic unit.
- (3) If E has additive reduction at v , then $P_v(T) = 1$.
- (4) Finally, if E has good reduction at v , we have

$$\begin{aligned} P_v(1/Nv) &= 1 - a_v \frac{1}{Nv} + Nv \frac{1}{(Nv)^2} \\ &= (Nv)^{-1} (Nv - a_v + 1) = (Nv)^{-1} \# \tilde{E}(\mathbb{F}_v). \end{aligned}$$

So $P_v(1/Nv)$ has positive p -valuation precisely when $E(\mathbb{F}_v)$ has non-trivial p -torsion.

Thus $P_1^{(k)}$ and $P_2^{(k)}$ contain precisely those primes not dividing p for which $|L_v(E/k, 1)|_p$ is non-trivial. (Note that this also applies when $p = 3$). In particular, every term in the product in (4.4) gives a non-trivial contribution and $\chi_{na}(E/k) = \chi_{cyc}(E/k)$ if and only if both $P_1^{(k)}$ and $P_2^{(k)}$ are empty.

4.7. Artin Formalism. Euler characteristics satisfy the ‘‘Artin formalism’’. Recall that we write $F = \mathbb{Q}(\mu_p, \sqrt[m]{m})$ and $K = \mathbb{Q}(\mu_p)$, and that we have defined in §1 two representations, σ and ρ , of $\text{Gal}(F/\mathbb{Q})$. Both σ and ρ can be realised over the integers, and the regular representation of $\text{Gal}(F/\mathbb{Q})$ is $\sigma \oplus \rho^{p-1}$. The following result is a direct consequence of [9] Theorem 3.10, and the multiplicative properties of Euler characteristics.

Proposition 4.8. *Let E/\mathbb{Q} be an elliptic curve with good ordinary reduction at p . Then we have the following equalities*

$$(4.9) \quad \chi_{na}(E/K) = \chi_{na}(E, \sigma),$$

$$(4.10) \quad \chi_{na}(E/F) = \chi_{na}(E, \rho)^{p-1} \cdot \chi_{na}(E, \sigma).$$

In particular, in the above equations the left-hand side is defined if and only if the right-hand side is.

5. COMPATIBILITY OF THE CONJECTURES

In this section we establish the relation (1.2) between $\chi_{na}(E, \rho)$ and $\chi_{na}(E, \sigma)$. This is the content of our main theorem (Theorem 5.2); see Theorem A.13 (Appendix) for a more general argument.

Theorem 5.2 allows us to formulate explicit consequences of Conjecture 1.4 for the L -values $L(E, \sigma, 1)$ and $L(E, \rho, 1)$, namely Conjectures 5.9 and 5.10. These are consistent with the congruence (1.5), and we also show that they follow from the Birch–Swinnerton-Dyer conjecture.

5.1. Main theorem. Recall that p is an odd prime, $m \geq 2$ a p -power free integer, $K = \mathbb{Q}(\mu_p)$, and σ and ρ are the $(p-1)$ -dimensional representations of $\text{Gal}(F/\mathbb{Q})$ defined in §1.

Theorem 5.2. *Let E be an elliptic curve over \mathbb{Q} with good ordinary reduction at $p \geq 5$. Assume that $\mu_{E/K} = 0$ and that $\text{Sel}_{p^\infty}(E/K)$ is finite. Then $\chi_{na}(E, \sigma)$ is defined and, moreover,*

$$\chi_{na}(E, \sigma) = 1,$$

if and only if

$$|\text{Sel}_{p^\infty}(E/F)| < \infty \quad \text{and} \quad \chi_{na}(E, \rho) = 1.$$

Proof. The theorem follows from the four lemmas below. To be able to use Iwasawa theory, we first show that $X(E/K^{cyc})$ and $X(E/F^{cyc})$ are torsion and that other relevant technical conditions are satisfied (Lemma 5.3). Then we prove the asserted equivalence in a sequence of steps (Lemmas 5.4, 5.5, 5.6):

$$\begin{aligned} \chi_{na}(E, \sigma) = 1 & \\ \iff \lambda_{E/F} = \lambda_{E/K} & \\ \iff |\text{Sel}_{p^\infty}(E/F)| < \infty \text{ and } \text{ord}_p \chi_{cyc}(E/F) = \text{ord}_p \chi_{cyc}(E/K) & \\ \iff |\text{Sel}_{p^\infty}(E/F)| < \infty \text{ and } \chi_{na}(E, \rho) = 1. & \end{aligned}$$

□

Lemma 5.3. *Under the assumptions of the theorem, $X(E/K^{cyc})$ is Λ_K -torsion, $X(E/F^{cyc})$ is Λ_F -torsion, $\mu_{E/F} = 0$, $\chi_{na}(E, \sigma)$ is defined and $\chi_{na}(E, \rho)$ is defined provided that $|\text{Sel}_{p^\infty}(E/F)| < \infty$.*

Proof. By Theorem 3.4, $X(E/K^{cyc})$ is Λ_K -torsion. By Hachimori–Matsuno (Theorem 3.16), $X(E/F^{cyc})$ is Λ_F -torsion and $\mu_{E/F} = 0$. By Hachimori–Venjakob (Theorem 4.3), $\chi_{na}(E/K)$ is defined and, provided that $|\text{Sel}_{p^\infty}(E/F)|$ is finite, $\chi_{na}(E/F)$ is also defined. Now, Artin formalism for non-abelian Euler characteristics applied to σ and ρ (equations (4.9) and (4.10)) proves the claim for $\chi_{na}(E, \sigma)$ and $\chi_{na}(E, \rho)$. □

Lemma 5.4. *Under the assumptions of the theorem,*

$$\chi_{na}(E, \sigma) = 1$$

if and only if

$$\lambda_{E/F} = \lambda_{E/K}.$$

Proof. By assumption, $X(E/K^{cyc})$ is Λ_K -torsion and, by Lemma 5.3, $X(E/F^{cyc})$ is Λ_F -torsion. Thus $\lambda_{E/K}$ and $\lambda_{E/F}$ are defined.

Artin formalism (Proposition 4.8, formula (4.9)) together with the theorem of Hachimori and Venjakob (Theorem 4.3), imply that

$$\chi_{na}(E, \sigma) = \chi_{na}(E/K) = \chi_{cyc}(E/K) \prod_{v \in P_1^{(K)} \cup P_2^{(K)}} |L_v(E/K, 1)|_p.$$

Now, each term $|L_v(E/K, 1)|_p$ in the above product is divisible by p (Remark 4.6). Moreover, $\text{ord}_p \chi_{cyc}(E/K) \geq 0$ (see formula (3.13)), and, as $\mu_{E/K} = 0$ by hypothesis, $\text{ord}_p \chi_{cyc}(E/K) = 0$ if and only if $\lambda_{E/K} = 0$ (Lemma 3.14).

Therefore, $\text{ord}_p \chi_{na}(E, \sigma) = 0$ if and only if $\lambda_{E/K} = 0$ and the sets $P_1^{(K)}$ and $P_2^{(K)}$ are both empty. By the theorem of Hachimori and Matsuno (Theorem 3.16 and Lemma 3.19), this is equivalent to $\lambda_{E/K} = \lambda_{E/F}$. □

Lemma 5.5. *Under the assumptions of the theorem,*

$$\lambda_{E/F} = \lambda_{E/K}$$

if and only if

$$|\mathrm{Sel}_{p^\infty}(E/F)| < \infty \quad \text{and} \quad \mathrm{ord}_p \chi_{\mathrm{cyc}}(E/F) = \mathrm{ord}_p \chi_{\mathrm{cyc}}(E/K).$$

Proof. This is a direct consequence of Lemma 5.3 and Corollary 3.24. \square

Lemma 5.6. *Under the assumptions of the theorem and supposing, furthermore, that $\mathrm{Sel}_{p^\infty}(E/F)$ is finite, we have*

$$\mathrm{ord}_p \chi_{\mathrm{cyc}}(E/F) = \mathrm{ord}_p \chi_{\mathrm{cyc}}(E/K).$$

if and only if

$$\chi_{na}(E, \rho) = 1.$$

Proof. By the Artin formalism for non-abelian Euler characteristics (Proposition 4.8, formula (4.10)),

$$\chi_{na}(E, \rho)^{p-1} = \frac{\chi_{na}(E/F)}{\chi_{na}(E/K)}.$$

Applying the formula of Hachimori and Venjakob for non-abelian Euler characteristics (Theorem 4.3) for both E/F and E/K gives

$$\chi_{na}(E, \rho)^{p-1} = \frac{\chi_{\mathrm{cyc}}(E/F)}{\chi_{\mathrm{cyc}}(E/K)} \cdot \frac{\prod_{w \in P_1^{(F)} \cup P_2^{(F)}} |L_w(E/F, 1)|_p}{\prod_{v \in P_1^{(K)} \cup P_2^{(K)}} |L_v(E/K, 1)|_p}.$$

The primes v in $P_1^{(K)}$ (respectively $P_2^{(K)}$) are in one-to-one correspondence with the primes w in $P_1^{(F)}$ (respectively $P_2^{(F)}$), since all such primes ramify totally in the extension F/K . Moreover, if $w|v$ is such a pair, both the reduction type of E and the residue fields are the same at v and at w . So $|L_v(E/K, 1)|_p = |L_w(E/F, 1)|_p$ and the products in the above formula cancel out,

$$\chi_{na}(E, \rho)^{p-1} = \frac{\chi_{\mathrm{cyc}}(E/F)}{\chi_{\mathrm{cyc}}(E/K)}.$$

The lemma follows. \square

The proof of Theorem 5.2 is now complete.

The difficult part of the theorem is the implication $\chi_{na}(E, \rho) = 1 \implies \chi_{na}(E, \sigma) = 1$. For the converse, if $\chi_{na}(E, \sigma) = 1$, then it is not hard to show that the whole module $X(E/F_\infty)$ is trivial, so that $\chi_{na} = 1$ for any twist:

Theorem 5.7. *Let E be an elliptic curve over \mathbb{Q} with good ordinary reduction at $p \geq 5$. Assume that $\mathrm{Sel}_{p^\infty}(E/K)$ is finite and that $\chi_{na}(E, \sigma)$ is defined and equals 1. Then $X(E/F_\infty) = 0$. In particular, $\chi_{na}(E, \tau) = 1$ for every Artin representation τ that factors through F_∞ .*

Proof. By Theorem 3.4, $X(E/K^{\mathrm{cyc}})$ is Λ_K -torsion. The formula of Hachimori and Venjakob (see 4.3) for $\chi_{na}(E/K) = \chi_{na}(E, \sigma)$ shows that $P_1^{(K)}$ and $P_2^{(K)}$ are both empty and that $\chi_{\mathrm{cyc}}(E/K) = 1$. By Lemma 3.14, $\lambda_{E/K} = \mu_{E/K} = 0$.

Now suppose that $k \subset F_\infty$ is a finite Galois extension of K . Note that $P_1^{(k)}$ and $P_2^{(k)}$ remain empty. The theorem of Hachimori and Matsuno (Theorem 3.16) shows that $X(E/k^{\mathrm{cyc}})$ is Λ_k -torsion and that $\mu_{E/k} = \lambda_{E/k} = 0$. It follows that $X(E/k^{\mathrm{cyc}})$

is finite and hence, by the theorem of Matsuno (Theorem 3.10), $X(E/k^{cyc}) = 0$. Taking the limit over intermediate fields $K \subset k \subset F_\infty$, we get $X(E/F_\infty) = 0$. \square

5.8. Conjectures for the L -values. In view of our Theorem 5.2, the Main Conjecture predicts the following behaviour of L -values. Note that the conjectures below are a special case of the congruence (1.5).

Conjecture 5.9. (consequence of Conjecture 1.4) Let E be an elliptic curve over \mathbb{Q} with good ordinary reduction at $p \geq 5$. Assume that $\mu_{E/K} = 0$ and that $\text{Sel}_{p^\infty}(E/K)$ is finite. Then

$$\text{ord}_p(\mathcal{L}_E(\rho)) = 0 \iff \text{ord}_p(\mathcal{L}_E(\sigma)) = 0.$$

Theorem 5.2 and the above conjecture are stated under the assumption that $p \geq 5$. The primary obstacle in generalising the theorem to $p = 3$ is that the result of Hachimori–Venjakob (Theorem 4.3), which gives an explicit formula for χ_{na} , assumes that $p \geq 5$.

We believe that Theorem 4.3 also holds for $p = 3$ under the additional assumption that whenever E has additive reduction at a prime of K , the reduction stays additive in F (as in 3.16). If that were the case, it would imply that our Theorem 5.2 is also valid for $p = 3$ under the same hypothesis. Then the Main Conjecture would imply the following:

Conjecture 5.10. (Version of 5.9 for $p = 3$.) Let E be an elliptic curve over \mathbb{Q} with good ordinary reduction at $p = 3$. Assume that whenever E has additive reduction at a prime of K , the reduction stays additive in F . Suppose furthermore that $\mu_{E/K} = 0$ and that $\text{Sel}_{p^\infty}(E/K)$ is finite. Then

$$\text{ord}_p(\mathcal{L}_E(\rho)) = 0 \iff \text{ord}_p(\mathcal{L}_E(\sigma)) = 0.$$

5.11. Relation with the Birch–Swinnerton–Dyer conjecture. Let E/\mathbb{Q} be an elliptic curve and k/\mathbb{Q} a number field. Recall that a global Weierstrass minimal model of E over \mathbb{Q} does not necessarily stay minimal over k . To correctly state the Birch–Swinnerton–Dyer formula for E/k using the periods of E/\mathbb{Q} , we define the fractional ideal $\mathcal{A}_{E/k}$ of k by

$$H^0(\mathcal{E}_k, \Omega^1) = \mathcal{A}_{E/k} \omega_E$$

where Ω^1 is the sheaf of invariant differentials on the Néron model \mathcal{E}_k of E over k and ω_E is the Néron differential of E over \mathbb{Q} . The ideal $\mathcal{A}_{E/k}$ is v -adically trivial whenever the Néron model of E over \mathbb{Q}_l (with $v|l$) remains a Néron model of E over k_v . In particular this holds if E/\mathbb{Q}_l has good or multiplicative reduction, so $\mathcal{A}_{E/k}$ is the unit ideal if E/\mathbb{Q} is semistable.

Conjecture 5.12. (Consequence of the Birch–Swinnerton–Dyer conjecture for E/K and E/F , [47]). Let p be an odd prime and E/\mathbb{Q} an elliptic curve with $E(K)$ finite. Then $\text{III}(E/K)$ and $\text{III}(E/F)$ are finite,

$$\frac{L(E, \sigma, 1) \epsilon(\sigma)}{\Omega_+(E)^{(p-1)/2} (2\Omega_-(E))^{(p-1)/2}} = \frac{N_{K/\mathbb{Q}}(\mathcal{A}_{E/K}) \text{III}(E/K) \prod_v c_v}{|E(K)|^2}$$

and

$$\frac{L(E, \rho, 1) \epsilon(\rho)}{\Omega_+(E)^{(p-1)/2} (2\Omega_-(E))^{(p-1)/2}} = \sqrt[p-1]{\frac{|E(K)|^2 N_{F/\mathbb{Q}}(\mathcal{A}_{E/F}) \text{III}(E/F) \prod_w c_w}{|E(F)|^2 N_{K/\mathbb{Q}}(\mathcal{A}_{E/K}) \text{III}(E/K) \prod_v c_v}}.$$

The second formula may read $0 = 0$, in which case it states that $L(E, \rho, 1) = 0$ if and only if $E(F)$ is infinite.

Proposition 5.13. *Let p be an odd prime and E/\mathbb{Q} an elliptic curve with good ordinary reduction at p and $E(K)$ finite.*

(a) *Assume the Birch–Swinnerton-Dyer conjecture for E/K . Then*

$$\text{ord}_p \mathcal{L}_E(\sigma) = \text{ord}_p \left(\chi_{cyc}(E/K) \prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1)^{-1} \right).$$

(b) *Assume the Birch–Swinnerton-Dyer conjecture for E/K and E/F . Then $\mathcal{L}_E(\rho) = 0$ if $\text{Sel}_{p^\infty}(E/F)$ is infinite, and otherwise*

$$\text{ord}_p \mathcal{L}_E(\rho) = \frac{1}{p-1} \text{ord}_p \left(\frac{\chi_{cyc}(E/F)}{\chi_{cyc}(E/K)} \cdot \frac{\prod_{w \in P_1^{(F)} \cup P_2^{(F)}} L_w(E/F, 1)^{-1}}{\prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1)^{-1}} \right).$$

(c) *Consequently, the conjectures 5.9 and 5.10 follow from the Birch–Swinnerton-Dyer conjecture.*

Proof. (a) First we substitute the formula for $L(E, \sigma, 1)$ from 5.12 into the formula (1.3) defining $\mathcal{L}_E(\sigma)$,

$$\mathcal{L}_E(\sigma) = \frac{2^{\frac{p-1}{2}} N_{K/\mathbb{Q}}(\mathcal{A}_{E/K}) \text{III}(E/K) \prod_v c_v}{|E(K)|^2} \cdot \prod_{l|mp} L_l(E, \sigma, 1)^{-1} \cdot \frac{\epsilon_p(\sigma)}{\epsilon(\sigma)} \cdot \frac{P_p(\sigma, u^{-1})}{P_p(\sigma, w^{-1})} \cdot u^{-n(\sigma)}$$

The terms $2^{\frac{p-1}{2}}$, $\frac{\epsilon_p(\sigma)}{\epsilon(\sigma)}$, $u^{-n(\sigma)}$ and $N_{K/\mathbb{Q}}(\mathcal{A}_{E/K})$ are p -adic units. The formula (3.12) yields

$$\text{ord}_p \mathcal{L}_E(\sigma) = \text{ord}_p \left(\frac{\chi_{cyc}(E/K)}{|\tilde{E}(\mathbb{F}_p)|^2} \cdot \prod_{l|mp} L_l(E, \sigma, 1)^{-1} \cdot \frac{P_p(\sigma, u^{-1})}{P_p(\sigma, w^{-1})} \right).$$

By Remark 4.6,

$$\text{ord}_p \prod_{l|m, l \neq p} L_l(E, \sigma, 1) = \text{ord}_p \prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1).$$

Decomposing σ into 1-dimensionals, all of which except $\mathbf{1}$ are ramified at p , we see that $P_p(\sigma, T) = 1 - T$. Hence, by Lemma 5.14,

$$\text{ord}_p \left(\frac{P_p(\sigma, u^{-1})}{P_p(\sigma, w^{-1})} \cdot \frac{1}{|\tilde{E}(\mathbb{F}_p)|^2} \cdot L_p(E, \sigma, 1)^{-1} \right) = 0.$$

The assertion follows.

(b) To begin with, if $\text{Sel}_{p^\infty}(E/F)$ is infinite, then finiteness of $\text{III}(E/F)$ (5.12) implies that $E(F)$ is infinite, which in turn (5.12 again) implies $L(E, \rho, 1) = 0$.

Now we proceed as in the proof of (a), first using the Birch–Swinnerton-Dyer formula from Conjecture 5.12 to eliminate $L(E, \rho, 1) = \sqrt[p-1]{L(E/F, 1)/L(E/K, 1)}$ from (1.3). Then we substitute χ_{cyc} from (3.12) and deduce that

$$\text{ord}_p \mathcal{L}_E(\rho) = \text{ord}_p \left(\sqrt[p-1]{\frac{\chi_{cyc}(E/F)}{\chi_{cyc}(E/K)}} \cdot |E(\mathbb{F}_p)|^{-2\delta} \cdot \sqrt[p-1]{\frac{\prod_{w|mp} L_w(E/F, 1)^{-1}}{\prod_{v|mp} L_v(E/K, 1)^{-1}}} \right).$$

$$\cdot p^{-1} \sqrt{\frac{N_{F/\mathbb{Q}}(\mathcal{A}_{E/F})}{N_{K/\mathbb{Q}}(\mathcal{A}_{E/K})}} \cdot \frac{\epsilon_p(\rho)}{p^{-1} \sqrt{\Delta_F^{1/2}/\Delta_K^{1/2}}} \cdot \frac{P_p(\rho, u^{-1})}{P_p(\rho, w^{-1})} \cdot u^{-n(\rho)} \cdot 2^{\frac{p-1}{2}}$$

where $\delta = 0$ or 1 depending on whether the prime above p ramifies or splits in F/K .

Again $2^{\frac{p-1}{2}}$, $u^{-n(\rho)}$, $N_{F/\mathbb{Q}}(\mathcal{A}_{E/F})$ and $N_{K/\mathbb{Q}}(\mathcal{A}_{E/K})$ are p -adic units. By the conductor-discriminant formula and the fact that the conductor of ρ is $\epsilon_p(\rho)^2$ up to a unit, so is the term $\epsilon_p(\rho)/p^{-1}\sqrt{\dots}$. By Remark 4.6,

$$\text{ord}_p \frac{\prod_{w|m, w \nmid p} L_w(E/F, 1)}{\prod_{v|m, v \nmid p} L_v(E/K, 1)} = \text{ord}_p \frac{\prod_{w \in P_1^{(F)} \cup P_2^{(F)}} L_w(E/F, 1)}{\prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1)}$$

It is not hard to see that $P_p(\rho, T) = (1 - T)^\delta$ with δ as above. Then, by Lemma 5.14,

$$\text{ord}_p \left(\frac{P_p(\rho, u^{-1})}{P_p(\rho, w^{-1})} \cdot \frac{1}{|\tilde{E}(\mathbb{F}_p)|^{2\delta}} \cdot L_p(E, \rho, 1)^{-1} \right) = 0.$$

The assertion follows.

(c) It remains to prove that Conjectures 5.9, 5.10 follow from (a) and (b). For $p \geq 5$ we know (Theorem 4.3) that the right-hand sides in the formulae in (a) and (b) are the p -adic valuations of $\chi_{na}(E, \sigma)$ and $\chi_{na}(E, \rho)$. Then our main theorem (Theorem 5.2) implies Conjecture 5.9.

The reason that this does not apply when $p = 3$ is that we do not have Hachimori-Venjakob's formula (Theorem 4.3) in this case. However, except for this formula and Artin formalism, our proof of main theorem (Theorem 5.2) does not actually use anything else about χ_{na} . Thus, if we define

$$\chi'_{na}(E, \sigma) = \chi_{cyc}(E/K) \prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1)^{-1}$$

and

$$\chi'_{na}(E, \rho) = \left(\frac{\chi_{cyc}(E/F)}{\chi_{cyc}(E/K)} \cdot \frac{\prod_{w \in P_1^{(F)} \cup P_2^{(F)}} L_w(E/F, 1)^{-1}}{\prod_{v \in P_1^{(K)} \cup P_2^{(K)}} L_v(E/K, 1)^{-1}} \right)^{\frac{1}{p-1}},$$

the proof goes through word for word with χ'_{na} in place of χ_{na} . The condition on additive reduction in Conjecture 5.10 is needed for the formula of Hachimori-Matsuno (Theorem 3.16) when $p = 3$. \square

Lemma 5.14. *Let p be an odd prime and E/\mathbb{Q} an elliptic curve with good ordinary reduction at p . Let τ be an Artin representation with $P_p(\tau, T) = (1 - T)^\delta$. Then*

$$|\tilde{E}(\mathbb{F}_p)|^{2\delta} \equiv \frac{P_p(\tau, u^{-1})}{P_p(\tau, w^{-1})} P_p(E, \tau, 1/p) \pmod{\mathbb{Z}_p^*}.$$

Proof. As E has good reduction at p , we have $P_p(E, \tau, T) = P_p(E, T)^\delta$. By multiplicativity, it suffices to prove the formula when $\delta = 1$. Thus $P_p(\tau, T) = 1 - T$,

$$\begin{aligned} P_p(E, \tau, 1/p) &= P_p(E, 1/p) = (1 - a_p T + pT^2) \Big|_{T=1/p} = |\tilde{E}(\mathbb{F}_p)|/p, \\ P_p(\tau, w^{-1}) &= 1 - w^{-1} = 1 - u/p \equiv 1/p \pmod{\mathbb{Z}_p^*}, \\ P_p(\tau, u^{-1}) &= 1 - u^{-1} \equiv 1 - u \equiv 1 - (u + w) \equiv 1 - a_p + p \equiv |\tilde{E}(\mathbb{F}_p)| \pmod{\mathbb{Z}_p^*}. \end{aligned}$$

\square

6. COMPUTATIONS

The main purpose of the remainder of the paper is to provide numerical evidence for the congruence (1.5). In view of Theorem 5.2, this also supports Conjectures 5.9 and 5.10, and thus the main conjecture of [9].

To do this, we pick a small odd prime p and an elliptic curve E/\mathbb{Q} with good ordinary reduction at p , finite $\text{Sel}_{p^\infty}(E/K)$ and $\mu_{E/K} = 0$. Recall that we are working with $K = \mathbb{Q}(\mu_p)$, $F = \mathbb{Q}(\mu_p, \sqrt[p]{m})$ and two $(p-1)$ -dimensional Artin representations σ and ρ which factor through $\text{Gal}(F/\mathbb{Q})$. The congruence reads

$$(6.1) \quad \mathcal{L}_E(\sigma) \equiv \mathcal{L}_E(\rho) \pmod{p}.$$

For varying m , we compute $L(E, \sigma, 1)$ and $L(E, \rho, 1)$ numerically, and deduce the values $\mathcal{L}_E(\sigma)$ and $\mathcal{L}_E(\rho)$. The tables with our results can be found in Appendix B. In all cases, the computations agree with the congruence.

In this section, we first explain how to compute the two L -values $L(E, \sigma, 1)$ and $L(E, \rho, 1)$. Next, we show how we determine the local epsilon factors of σ and ρ at p . The other modifying factors in the expressions for \mathcal{L}_E are fairly straightforward to determine. Finally, we make some remarks concerning the data in our tables.

6.2. Computing L -values: analytic side. To begin with, the two L -functions $L(E, \sigma, s)$ and $L(E, \rho, s)$ have an analytic continuation and satisfy the functional equation (2.6). This can be shown using cyclic base change from the theory of automorphic forms (see [19], Theorem 14).

There is a standard procedure for computing values of so-called motivic L -functions. This method applies to L -functions of twists of elliptic curves, and we will briefly outline it in our case. See [51], [13] 10.3 and [18] for details.

Let E be an elliptic curve over \mathbb{Q} , and let τ be a d -dimensional Artin representation, such that $\tau \cong \tau^*$. For brevity, denote

$$N = N(E, \tau).$$

Assume furthermore that the twisted L -function $L(E, \tau, s)$ has an analytic continuation to \mathbb{C} and satisfies the functional equation

$$(6.3) \quad \hat{L}(E, \tau, s) = \pm \hat{L}(E, \tau, 2-s),$$

where

$$\hat{L}(E, \tau, s) = \left(\frac{N}{\pi^{2d}}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^d L(E, \tau, s).$$

Expanding the Euler product defining the L -function, we see that it can be written as a Dirichlet series,

$$L(E, \tau, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{Re}(s) > 3/2.$$

Consider the inverse Mellin transform $\phi(t)$ of the full Γ -factor, defined by

$$\Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^d = \int_0^{\infty} \phi(t) t^{s-1} dt.$$

This is a real-valued function for $t > 0$, which decays exponentially as $t \rightarrow \infty$. One can show that the functional equation (6.3) is equivalent to

$$(6.4) \quad t^2 \sum_{n=1}^{\infty} a_n \phi\left(n \frac{t\pi^d}{\sqrt{N}}\right) = \pm \sum_{n=1}^{\infty} a_n \phi\left(n \frac{\pi^d}{t\sqrt{N}}\right).$$

We remark that the above formula allows one to test the functional equation numerically. Provided that $\phi(t)$ can be computed, evaluate the two sides of (6.4), say for $t = 1$, and check that the two (exponentially convergent) sums yield the same result.

By considering the Mellin transform of $\sum_{n=1}^{\infty} a_n \phi(nt\pi^d/\sqrt{N})$, splitting the integral into two, and using (6.4), one arrives at a formula for $\hat{L}(E, \tau, s)$,

$$(6.5) \quad \hat{L}(E, \tau, s) = \sum_{n=1}^{\infty} a_n G_s\left(\frac{n\pi^d}{\sqrt{N}}\right) \pm \sum_{n=1}^{\infty} a_n G_{2-s}\left(\frac{n\pi^d}{\sqrt{N}}\right),$$

where

$$G_s(t) = t^{-s} \int_t^{\infty} \phi(x) x^s \frac{dx}{x}, \quad t > 0$$

is the incomplete Mellin transform of $\phi(t)$. Again, for fixed s , the function $G_s(t)$ decays exponentially with t , so that (6.5) gives an exponentially converging series for $L(E, \tau, s)$ for arbitrary $s \in \mathbb{C}$. Note that, in particular, unlike the initial Dirichlet series, this expression is valid not only when $\operatorname{Re}(s) > 3/2$.

The remaining issue is that of being able to efficiently compute the function $G_s(t)$ for $s \in \mathbb{C}$ and $t > 0$. (In fact, we are only interested in the critical value $L(E, \tau, 1)$, i.e. only when $s = 1$.) There are various ways to compute $G_s(t)$. In principle, there is a method due to Tollis [51], based on earlier work of Lavrik [31], which is applicable in our situation and gives precise and explicit error bounds. Let us note here that $G_s(t)$ is totally independent of the elliptic curve E , and depends only on the dimension d of τ . Thus this is purely a problem in transcendental function theory.

However, although $G_s(t)$ depends only on d , the conductor N influences the rate of convergence of the series (6.5). To obtain an approximation to the value $L(E, \tau, 1)$ we need roughly \sqrt{N} terms in the series. In our examples, we often deal with very large N (e.g. for $E = 19A3$, $p = 7$ and $m = 2$, we have $N(E, \rho) = 2^{12}7^{14}19^6 \approx 10^{23}$). Thus the issue of computing $G_s(t)$ efficiently becomes significant. We used empirical algorithms described in [18] and implemented as a PARI package ComputeL [17] to compute the L -values.

Remark 6.6. If τ decomposes as $\tau \cong \bigoplus_i \tau_i$, then

$$L(E, \tau, s) = \prod_i L(E, \tau_i, s).$$

It is more efficient to compute the individual terms in the right-hand side of the equation, as these have smaller conductors. For instance, this applies to $\tau = \sigma$, which decomposes as a sum of 1-dimensional representations. We should mention that these need not satisfy $\tau_i \cong \tau_i^*$. However, it is easy to modify the above method to deal with such representations as well (see [18], Remark 2.7).

6.7. Computing L -values: arithmetic side. We now specialise to our setting. As usual, we write $K = \mathbb{Q}(\mu_p)$. Recall that the regular representation of $\operatorname{Gal}(F/\mathbb{Q})$ decomposes as $\sigma \oplus \rho^{p-1}$, where σ is a sum of 1-dimensional representations, and ρ is irreducible. Both σ and ρ have dimension $p-1$, and $\sigma^* \cong \sigma$ and $\rho^* \cong \rho$.

We aim to compute $L(E, \sigma, s)$ and $L(E, \rho, s)$ at $s = 1$. To apply the methods described in §6.2, we need to determine arithmetic invariants of the two L -functions: the Dirichlet coefficients, the conductor and the sign in the functional equation. We

refer the reader to §2 for the definitions of these quantities and the basic results and notation.

6.7.1. *Dirichlet coefficients.* In order to compute the Dirichlet coefficients, we determine the local factors of $L(E, \sigma, s)$ and $L(E, \rho, s)$ at all primes. The two L -functions are related to L -functions of E over number fields as follows:

$$L(E, \sigma, s) = L(E/K, s), \quad L(E, \rho, s) = L(E/\mathbb{Q}(\sqrt[m]{m}), s)/L(E/\mathbb{Q}, s).$$

Similar formulae hold for the local L -factors. If q is a prime where E/\mathbb{Q} has good reduction, write the local polynomial of $L(E/\mathbb{Q}, s)$ at q as

$$P_q(E, T) = 1 - a_q T + qT^2 = (1 - \alpha_q T)(1 - \beta_q T).$$

For any number field k ,

$$L_q(E, R_k, s) = \prod_{v|q} L_v(E/k, s) = \prod_{v|q} (1 - (\alpha_q^{n_v} + \beta_q^{n_v})q^{-s} + q^{2-n_v s})^{-1},$$

where $n_v = [\mathbb{F}_v : \mathbb{F}_q]$. Hence we can compute these local factors if we know how to determine a_q for rational primes q , and how primes decompose in k . In particular, this gives us a method for computing the local factors of $L(E, \sigma, s)$ and $L(E, \rho, s)$ for all primes of good reduction of E .

It remains to deal with the finitely many factors $L_v(E/k, s)$, for primes v of k dividing $N(E)$. Tate's algorithm ([48],[44] §IV.9) determines the reduction type of E at v and a local minimal Weierstrass equation. The local factor is then given by (2.8). If bad reduction of E/\mathbb{Q}_q becomes good for E/k_v , we can compute a_v by brute force, counting points on E over the residue field of v .

6.7.2. *Conductors.* Next we consider the conductors $N(E, \sigma)$ and $N(E, \rho)$. For every prime q we need to determine the local conductor of the l -adic representations $M_l(E) \otimes M_l(\sigma)$ and $M_l(E) \otimes M_l(\rho)$. The conductors of E , σ and ρ themselves are computed using Tate's algorithm for E , and the conductor-discriminant formula,

$$N(\sigma) = |\Delta_K|, \quad N(\rho) = \frac{|\Delta_{\mathbb{Q}(\sqrt[m]{m})}|}{|\Delta_{\mathbb{Q}}|} = |\Delta_{\mathbb{Q}(\sqrt[m]{m})}|.$$

We now describe the local conductors of $M_l(E) \otimes M_l(\sigma)$ and $M_l(E) \otimes M_l(\rho)$. As the local conductor at q depends only on the action of the inertia subgroup I_q , it follows that for a prime q where either $M_l(E)$ or $M_l(\rho)$ is good,

$$(6.8) \quad N_q(E, \rho) = \begin{cases} N_q(E)^{\dim \rho}, & \rho \text{ good at } q, \\ N_q(\rho)^2, & E \text{ has good reduction at } q. \end{cases}$$

The same formula holds for σ .

It remains to deal with the case when q is a bad prime for both E and ρ . (This does not happen with σ , as the only bad prime for σ is p , and E has good reduction at p .) The local conductor has a tame and a wild part, $N_q(E, \rho) = q^{t_q(E, \rho) + \delta_q(E, \rho)}$. The tame contribution is, by definition,

$$t_q(E, \rho) = 2 \dim \rho - \deg(P_q(E, \rho, T)).$$

To compute the wild contribution, first observe that the only prime where ρ can be wildly ramified is $q = p$. Indeed, primes $q \neq p$ are unramified in K/\mathbb{Q} , so the action of the inertia subgroup I_q on ρ factors through $\text{Gal}(F/K)$. The latter group has no elements of order q , so wild inertia acts trivially.

As E has good reduction at p , we now see that there are no primes q where both $M_i(E)$ and $M_i(\rho)$ are wildly ramified. We can thus use an analogue to formula (6.8) for the wild inertia subgroup,

$$\delta_q(E, \rho) = \begin{cases} (\dim \rho) \delta_q(E), & M_i(\rho) \text{ tamely ramified at } q, \\ 2\delta_q(\rho), & M_i(E) \text{ tamely ramified at } q. \end{cases}$$

6.8.3. *Sign.* Finally, to determine the signs in the functional equations of $L(E, \sigma, s)$ and $L(E, \rho, s)$, we use the following result:

Theorem 6.9. ([19], Theorem 1) *Let E be an elliptic curve over \mathbb{Q} . Let τ be an Artin representation with $\tau \cong \tau^*$. Set P_M (respectively, P_A) to be the set of primes where E has multiplicative (respectively, additive) reduction. Suppose that no prime in P_A is bad for τ .*

If the sign in the functional equation for $L(E, s)$ is w_E , then the sign in the functional equation for $L(E, \tau, s)$ is

$$w_E^{\dim \tau} (-1)^{d^-(\tau)} \prod_{p \in P_M} s_p^{\dim \tau - \dim \tau^{I_p}} \det(\text{Frob}_p^{-1} | \tau^{I_p}) \prod_{p \in P_A} \det(\text{Frob}_p^{-1} | \tau)^{N_p(E)},$$

where s_p is -1 if E has split multiplicative reduction at p , and $+1$ if the reduction is non-split.

The theorem allows to determine the sign for $L(E, \sigma, s)$. It also applies to $L(E, \rho, s)$, except for the case when E has additive reduction at a prime dividing m , since such a prime is also bad for ρ . In such cases, we compute the sign numerically from (6.4).

6.10. **The local ϵ -factors.** In this section we briefly sketch how the local epsilon factors at p for ρ and σ can be computed. For the precise definitions and properties we refer the reader to [49]. The necessary results in class field theory are contained in the articles of Serre and Tate in [8].

To define the local epsilon factor of an Artin representation over \mathbb{Q} , it is first necessary to pick a Haar measure μ on \mathbb{Q}_p and an additive character s given by a homomorphism from $(\mathbb{Q}_p, +)$ to \mathbb{C}^* . We take μ to be the canonical measure determined by $\mu(\mathbb{Z}_p) = 1$ and

$$s(ap^{-n}) = e^{2\pi ia/p^n}, \quad a \in \mathbb{Z}_p.$$

In particular $s(x) = 1$ iff $x \in \mathbb{Z}_p$. We then write

$$\epsilon_p(\tau) = \epsilon_p(\tau, \mu, s).$$

The local epsilon factor $\epsilon_p(\sigma)$ can be computed by writing $\sigma = \bigoplus_i \chi_i$ as the sum of all 1-dimensional representations of $\text{Gal}(K/\mathbb{Q})$,

$$\epsilon_p(\sigma) = \prod_{i=1}^{p-1} \epsilon_p(\chi_i).$$

The local epsilon factor of the trivial representation $\mathbf{1}$ is 1 and, for $\chi_i \neq \mathbf{1}$, it is given by

$$\epsilon_p(\chi_i) = \sum_{j=1}^{p-1} (\chi_i \circ \theta_p)(j/p) \cdot e^{2\pi ij/p}.$$

Here θ_p is the local reciprocity map at p given by class field theory.

Next, we compute the local epsilon factor $\epsilon_p(\rho)$. Both σ and ρ are induced from 1-dimensional representations of $\text{Gal}(F/K)$,

$$\sigma = \text{Ind } \mathbf{1}, \quad \rho = \text{Ind } \psi,$$

with any non-trivial 1-dimensional ψ . The inductive property of local epsilon factors yields

$$\frac{\epsilon_p(\rho)}{\epsilon_p(\sigma)} = \frac{\epsilon_v(\chi)}{\epsilon_v(\mathbf{1})}.$$

Here ϵ_v is the local epsilon factor at the prime v of K above p defined with any measure and the additive character $s \circ \text{Tr}_{K_v/\mathbb{Q}_p}$. Choose a uniformiser π of K_v . Denote $n(\psi) = \text{ord}_v N(\psi)$, which can be computed from the conductor-discriminant formula in F/K . Then it is not hard to see that

$$\frac{\epsilon_v(\chi)}{\epsilon_v(\mathbf{1})} = \sum_{x \in T} (\psi \circ \theta_v)(x) \cdot e^{2\pi i \text{Tr}_{K_v/\mathbb{Q}_p}(x)},$$

where θ_v is the local reciprocity map at v and T is a set of additive representatives of $\pi^{2-p-n(\psi)}\mathcal{O}_{K_v}^*$ modulo $\pi^{2-p}\mathcal{O}_{K_v}$, for instance

$$T = \left\{ \sum_{i=2-p-n(\psi)}^{1-p} a_i \pi^i \mid 0 \leq a_i \leq p-1, a_{2-p-n(\psi)} \neq 0 \right\}.$$

It remains to explain how compute $\psi \circ \theta_v(x)$ for $x \in T$. First, we can approximate x by $x' \in K$ such that $\psi \circ \theta_v(x) = \psi \circ \theta_v(x')$ because $\psi \circ \theta$ is trivial on $1 + \pi^{n(\psi)}\mathcal{O}_{K_v}$. Now, by the product formula in global class field theory,

$$\prod_w \theta_w(x') = 1,$$

where the product is taken over all (Archimedean and non-Archimedean) places of K . Whenever w is unramified in F/K ,

$$\theta_w(x') = \text{Frob}_w^{-\text{ord}_w(x')},$$

which is easy to compute from the explicit action of the Galois group on the p -th roots of m . To make sure that the ramified primes do not contribute to the product, use the Chinese remainder theorem for ideals to choose x' with

$$x' \equiv x \pmod{v^{n(\psi)}}, \quad x' \equiv 1 \pmod{\frac{n(\psi)}{v^{n(\psi)}}}.$$

The above congruences are taken modulo ideals, so v is treated as a prime ideal of \mathcal{O}_K . Note that the conditions at the infinite places are automatically satisfied as K has no real embeddings. Now $\theta_v(x) = \theta_v(x') = \prod_{w \neq v} \theta_w(x')^{-1}$, which can be explicitly computed.

6.11. The finiteness of $\text{Sel}_{p^\infty}(E/K)$ and the μ -invariant. Conjectures 5.9 and 5.10 require that the curve in question has finite $\text{Sel}_{p^\infty}(E/K)$ and $\mu_{E/K} = 0$. In view of Theorem A.13 (Appendix), this restriction should in fact not be necessary. We now indicate what we can say about these conditions for the curves in our tables in Appendix B.

For $p = 3$ we perform 3-descent for E/\mathbb{Q} and for the quadratic twist of E by -3 , using Magma [3]. For those curves in our tables for which 3-descent is implemented, we find that $\text{Sel}_3(E/K)$ is generated by the 3-torsion points, so the Mordell-Weil rank of E/K is zero and $\text{III}(E/K)[3] = 0$. It follows from (3.1) that

$\text{Sel}_{3^\infty}(E/K) = 0$. For $p = 5$ and the curve $X_1(11)$ (Table 5-11A3), Fisher [20] has done 5-descent over $\mathbb{Q}(\mu_5)$, and he has shown that $\text{Sel}_{5^\infty}(E/K) = 0$. For all other curves, we appeal to the Birch–Swinnerton-Dyer conjecture and deduce from the non-vanishing of the L -functions that $\text{Sel}_{p^\infty}(E/K)$ is finite.

The check whether $\mu_{E/K} = 0$ we first compute the Euler characteristic $\chi_{cyc}(E/K)$. For this we need the order of $\text{III}(E/K)[p^\infty]$, that we either know to be 1 or compute from $L(E/K, 1)$ using the second part of the Birch–Swinnerton-Dyer conjecture,

$$\chi_{cyc}(E/K) = L^*(E/K) \cdot |\tilde{E}(\mathbb{F}_p)|^2 \pmod{\mathbb{Z}_p^*}.$$

(Note that $|\tilde{E}(\mathbb{F}_p)| = P_p(E, 1)$ can be read off from the tables.)

If $\chi_{cyc}(E/K) = 1$, then by Lemma 3.14 the module $X(E/K^{cyc})$ is finite and so $\lambda_{E/K} = \mu_{E/K} = 0$. When $\chi_{cyc}(E/K)$ is non-trivial, the question is whether the power of p comes from the λ -invariant. If we can find a point of infinite order of E in $\mathbb{Q}(\mu_{p^\infty}) = K^{cyc}$, it guarantees that the λ -invariant is non-trivial. Thus if $\chi_{cyc}(E/K) = p$, this ensures that $\mu_{E/K} = 0$. We can do this for $p = 3$ in case of the curves 20A1, 92A1 and 116C2 where we have found explicit points of infinite order over $\mathbb{Q}(\mu_9)$. For the curve 128B2 we also found such a point but, as $\chi_{cyc}(E/K) = p^2$, we only know that $\lambda_{E/K} > 0$ and $\mu_{E/K} \leq 1$.

6.12. Reliability of computations. As explained earlier, the most computationally demanding task is that of determining the value of $L(E, \rho, 1)$. The time to compute it numerically to a given precision is roughly proportional to the square root of the conductor $N(E, \rho)$. For $p = 3$, we restrict ourselves to those $m < 2000$ for which $N(E, \rho)$ is small enough that we can evaluate $L(E, \rho, 1)$ to at least 6 digits precision with $2 \cdot 10^6$ Dirichlet coefficients of $L(E, \rho, s)$. For $p = 5$ we require at least 2 digits precision with $1 \cdot 10^8$ coefficients. In many cases the precision is much higher, up to 40 decimal digits. The error bounds for the method [18] that we use have not been proved in general, but the ones for the computations of $L(E, \sigma, 1)$ for all p and of $L(E, \rho, 1)$ for $p = 3$ can be established.

Recall that the L -functions $L(E, \sigma, s)$ and $L(E, \rho, s)$ are defined on all of \mathbb{C} (and, in particular, at $s = 1$) and the modified L -values $L^*(E/\mathbb{Q})$, $L^*(E/K)$ and $L^*(E/\mathbb{Q}(\sqrt[m]{m}))$ are rational (see §1). If we had an upper bound on their denominators, we could then use our numerical approximations to find the actual values of $L^*(E/\mathbb{Q})$, $L^*(E/K)$ and $L^*(E/\mathbb{Q}(\sqrt[m]{m}))$ (or, equivalently, of $L(E, 1)$, $L(E, \sigma, 1)$ and $L(E, \rho, 1)$). We do not have such upper bounds, so we used the conjectural ones predicted by the Birch–Swinnerton-Dyer conjecture: the denominator of $L^*(E/k)$ is at most $|E(k)_{tors}|^2$.

As a sanity check, we compute the analytic order of III of E/\mathbb{Q} , E/K and $E/\mathbb{Q}(\sqrt[m]{m})$. To begin with, this number is always an integer to the correct precision. Next, this integer (possibly 0) is always a square. Finally, we use 2-descent over the three fields to compute the order of the corresponding 2-Selmer groups and check that they are consistent with our conjectural orders of III . Moreover, over \mathbb{Q} and, for $p = 3$, over K we use 3-descent for a similar comparison between the 3-Selmer groups and $\text{III}[3]$. The descents are carried out using Magma [3].

We have also computed 4 examples for $p = 7$ (Tables 7-17A1, 7-19A3), but the conductors in these cases are too large to make the computations reliable. Nevertheless, the L -values lead to plausible orders of III .

6.13. Observations. For all curves in our tables, the congruence (6.1) holds. Note that for some curves, the congruence reads $0 \equiv 0 \pmod p$ for most m . In view of Conjecture 1.4, this corresponds to $\chi_{na}(E, \sigma)$ being non-unit for such m . This can happen for various reasons (cf. Theorem 4.3):

- (1) If $E(K)[p] \neq 0$, then every prime v of K with $v \nmid p$, $v|m$ and where E has good reduction lies in the set $P_2^{(K)}$ of Theorem 4.3 and thus contributes to $\chi_{na}(E, \sigma)$. See, for instance, Tables 3-20A3, 5-11A3.
- (2) If $E(K)[p] = 0$ but $\tilde{E}(\mathbb{F}_p)[p] \neq 0$, then $\chi_{na}(E, \sigma)$ is non-unit for every m . See, for instance, Tables 3-128B2, 3-275B1.
- (3) If $E(K)[p] = 0$ and p divides one of the Tamagawa numbers c_v for a prime v of K , then again $\chi_{na}(E, \sigma)$ is non-unit for every m . See, for instance, Table 3-116C2, 3-260A1.

It is interesting to note that in cases (2) and (3) there appear to be congruences modulo a higher power p^{b+1} . In all our examples, this extra power b is exactly the power of p in the product of $\prod_v c_v$ and $|\tilde{E}(\mathbb{F}_p)[p]|$. Note, however, that (at least in case (3)) the congruence only holds when both terms have valuation exactly b . It would be interesting to have an explanation of this.

7. NUMERICAL EXAMPLES

To illustrate the theory in the previous sections, let us describe an example in detail. We take the elliptic curve 21A4, in the notation of Cremona [14], of conductor 21. We will twist it by Artin representations coming from $\mathbb{Q}(\mu_5, \sqrt[5]{2})/\mathbb{Q}$. Thus

$$\begin{aligned} E_{21} : y^2 + xy &= x^3 + x & (21A4) \\ p &= 5 \\ m &= 2 \\ K &= \mathbb{Q}(\mu_5), & F &= \mathbb{Q}(\mu_5, \sqrt[5]{2}). \end{aligned}$$

In this section, we set $L = \mathbb{Q}(\sqrt[5]{2})$. This example is the first row in Table 5-21A4.

7.1. Reduction types. First, consider E_{21} over \mathbb{Q} . Its standard invariants are

$$j(E_{21}) = \frac{47^3}{3^2 \cdot 7}, \quad \Delta(E_{21}) = -63 = -3^2 \cdot 7 \quad N(E_{21}) = 21 = 3 \cdot 7.$$

The curve has multiplicative reduction at $q = 3$ and $q = 7$ and good reduction otherwise. The Kodaira symbols for the reduction types and the local Tamagawa numbers are

$$\begin{aligned} 3 : & \quad I_2 \text{ split} & c_3 &= 2, \\ 7 : & \quad I_1 \text{ non-split} & c_7 &= 1. \end{aligned}$$

We will now look at E_{21} over the number fields K , L and F . The corresponding local information can be computed using Tate's algorithm. However, for semistable curves (like E_{21}), it can also be determined from the reduction behaviour over \mathbb{Q} as follows.

Recall that split multiplicative reduction stays split multiplicative in any extension. Non-split multiplicative reduction becomes either split or non-split, depending on the parity of the degree of the residue field extension. Recall also from Tate's

algorithm that in case of multiplicative reduction,

$$c_q = \begin{cases} \text{ord}_q(\Delta(E)), & \text{split,} \\ 2, & \text{non-split, } \text{ord}_q(\Delta(E)) \text{ even,} \\ 1, & \text{non-split, } \text{ord}_q(\Delta(E)) \text{ odd.} \end{cases}$$

Now we look at E_{21}/K . The curve has good reduction at all primes of K not dividing 3 or 7. Both 3 and 7 are inert in K/\mathbb{Q} , so there are unique primes $v_3|3$ and $v_7|7$ with residue field extensions $[\mathbb{F}_{v_3} : \mathbb{F}_3] = 4$ and $[\mathbb{F}_{v_7} : \mathbb{F}_7] = 4$. So for E_{21}/K we have

$$\begin{aligned} v_3 : & \quad I_2 \text{ split} & c_{v_3} = 2, \\ v_7 : & \quad I_1 \text{ split} & c_{v_7} = 1. \end{aligned}$$

Next, consider E_{21}/L . The primes 3 and 7 decompose in L as $3 = z_3^{(1)}z_3^{(2)}$ and $7 = z_7^{(1)}z_7^{(2)}$ with $[\mathbb{F}_{z_3^{(1)}} : \mathbb{F}_3] = 1$, $[\mathbb{F}_{z_7^{(2)}} : \mathbb{F}_7] = 4$ and $[\mathbb{F}_{z_3^{(1)}} : \mathbb{F}_3] = 1$, $[\mathbb{F}_{z_7^{(2)}} : \mathbb{F}_7] = 4$. This can be seen by factoring $x^5 - 2$ modulo 3 and 7. So for E_{21}/L we have

$$\begin{aligned} z_3^{(1)}, z_3^{(2)} : & \quad I_2 \text{ split} & c_{z_3^{(i)}} = 2, \\ z_7^{(1)} : & \quad I_1 \text{ non-split} & c_{z_7^{(1)}} = 1, \\ z_7^{(2)} : & \quad I_1 \text{ split} & c_{z_7^{(2)}} = 1. \end{aligned}$$

Finally, take E_{21}/F . The primes v_3 and v_7 of K split in F/K as can be seen from the decomposition of 3 and 7 in L . Hence, in F/\mathbb{Q} ,

$$3 = w_3^{(1)}w_3^{(2)}w_3^{(3)}w_3^{(4)}w_3^{(5)}, \quad 7 = w_7^{(1)}w_7^{(2)}w_7^{(3)}w_7^{(4)}w_7^{(5)},$$

with all residue degrees equal to 4. For E_{21}/F we have

$$\begin{aligned} w_3^{(i)} : & \quad I_2 \text{ split} & c_{w_3^{(i)}} = 2, \\ w_7^{(i)} : & \quad I_1 \text{ split} & c_{w_7^{(i)}} = 1. \end{aligned}$$

7.2. Groups of Mordell, Weil, Selmer, Tate and Shafarevich. The curve E_{21}/\mathbb{Q} has Mordell-Weil rank 0, and $E_{21}(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, with the point $(1, 1)$ as a generator. Using 2-descent over \mathbb{Q} , K and L (e.g. using Magma [3]), we find

$$\begin{aligned} \text{Sel}_2(E_{21}/\mathbb{Q}) & \cong \mathbb{Z}/2\mathbb{Z}, \\ \text{Sel}_2(E_{21}/K) & \cong \mathbb{Z}/2\mathbb{Z}, \\ \text{Sel}_2(E_{21}/L) & \cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Thus, $E_{21}(k)/2E_{21}(k) \rightarrow \text{Sel}_2(E_{21}/k)$ is an isomorphism for $k = \mathbb{Q}, K, L$, mapping the 4-torsion point $(1, 1)$ to the generator. In particular, E_{21} has no points of infinite order and $\text{III}(E_{21})[2]$ is trivial over the three fields. Note that this implies that $E_{21}(F)$ has Mordell-Weil rank 0 as well.

7.3. Artin representations. Recall that we are interested in the two representations σ and ρ of the Galois group $\text{Gal}(F/\mathbb{Q})$. Write $\zeta = e^{2\pi i/5}$.

The group $\text{Gal}(F/\mathbb{Q})$ has order 20 and acts faithfully on the 5 roots of $x^5 - 2$. The subgroup $\text{Gal}(F/L)$ fixes $\sqrt[5]{2}$ and $\text{Gal}(F/K)$ acts as a 5-cycle. We fix their generators by the requirement

$$\begin{aligned} g(\sqrt[5]{2}) & = \sqrt[5]{2}, & g(\zeta\sqrt[5]{2}) & = \zeta^2\sqrt[5]{2}, \\ h(\sqrt[5]{2}) & = \zeta\sqrt[5]{2}, & h(\zeta\sqrt[5]{2}) & = \zeta^2\sqrt[5]{2}. \end{aligned}$$

As an abstract group, $\text{Gal}(F/\mathbb{Q})$ is determined by the relations $g^4 = h^5 = 1$ and $ghg^{-1} = h^2$, and it is usually called G_{20} .

The regular representation of $\text{Gal}(F/\mathbb{Q})$ decomposes as $\sigma \oplus \rho^4$ where $\sigma \cong \mathbf{1} \oplus \chi_1 \oplus \chi_2 \oplus \chi_3$ is the sum of distinct one-dimensional representations and ρ is irreducible. Both σ and ρ are 4-dimensional and σ factors through $\text{Gal}(K/\mathbb{Q})$.

The character table of $\text{Gal}(F/\mathbb{Q})$ (including the character of σ) is

$$(7.4) \quad \begin{array}{c|ccccc} & 1 & C_h & C_{gh} & C_{g^2h} & C_{g^3h} \\ \hline \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \chi_1 & 1 & 1 & i & -1 & -i \\ \chi_2 & 1 & 1 & -1 & 1 & -1 \\ \chi_3 & 1 & 1 & -i & -1 & i \\ \rho & 4 & -1 & 0 & 0 & 0 \\ \hline \sigma & 4 & 4 & 0 & 0 & 0 \end{array}$$

We also record the actual characteristic polynomials of the elements of the Galois group in σ and ρ ,

$$(7.5) \quad \begin{array}{c|ccccc} & 1 & C_h & C_{gh} & C_{g^2h} & C_{g^3h} \\ \hline \sigma & (T-1)^4 & (T-1)^4 & T^4-1 & (T^2-1)^2 & T^4-1 \\ \rho & (T-1)^4 & T^4+T^3+T^2+T+1 & T^4-1 & (T^2-1)^2 & T^4-1 \end{array}$$

Note that the characters and, moreover, the characteristic polynomials of ρ and σ are congruent modulo 5. In fact, the representations themselves are congruent. (They can be realised over \mathbb{Z} and the semi-simplifications of their reductions modulo 5 are isomorphic.)

The conductors of σ and ρ are given by the conductor-discriminant formula,

$$N(\sigma) = |\Delta_K| = 5^3, \quad N(\rho) = \frac{|\Delta_L|}{|\Delta_{\mathbb{Q}}|} = 2^4 5^5.$$

Next, we determine the inertia subgroups and Frobenius elements of $\text{Gal}(F/\mathbb{Q})$ at all primes.

The primes 2 and 5 ramify in the extension F/\mathbb{Q} . The prime 5 is totally ramified (as it ramifies both in K/\mathbb{Q} and in L/\mathbb{Q}), so the inertia subgroup I_5 is the whole of $\text{Gal}(F/\mathbb{Q})$ and the Frobenius element is trivial. The inertia invariant subspaces of σ and ρ are, respectively, 1 and 0-dimensional.

The prime 2 is inert in K/\mathbb{Q} and ramifies in F/K . The inertia subgroup $I_2 \subset \text{Gal}(F/\mathbb{Q})$ is $\text{Gal}(F/K)$ and the Frobenius element is g (modulo inertia). The inertia invariant subspaces of σ and ρ are, respectively, the whole of σ and 0.

If $q \neq 2, 5$, then the inertia subgroup $I_q \subset \text{Gal}(F/\mathbb{Q})$ is trivial, so the representations σ and ρ are unramified at q . Primes $q \equiv \pm 2 \pmod{5}$ are inert in K/\mathbb{Q} , so that $\text{Frob}_q \in \text{Gal}(F/\mathbb{Q})$ (defined up to conjugation) has order either 4 or 20. Since G_{20} only has elements of orders 1, 2, 4 and 5, this implies that q splits in F/K and Frob_q has exact order 4. Similarly, for primes $q \equiv -1 \pmod{5}$, the Frobenius $\text{Frob}_q \in \text{Gal}(F/\mathbb{Q})$ has order 2.

It remains to deal with the primes $q \equiv 1 \pmod{5}$. These split in K/\mathbb{Q} and may either be totally split in F/\mathbb{Q} or remain inert in F/K . In the first case $\text{Frob}_q = 1$, and in the second case Frob_q has order 5. To see which category q falls into, it suffices to determine whether q is totally split or inert in L/\mathbb{Q} . This can be done by counting roots of $x^5 - 2$ modulo q . For instance, $x^5 - 2$ is irreducible modulo 11, so $q = 11$ is inert in L/\mathbb{Q} . On the other hand,

$$x^5 - 2 = (x - 22)(x - 25)(x - 49)(x - 90)(x - 116) \pmod{151},$$

so $q = 151$ splits completely in L/\mathbb{Q} .

7.6. Twists by Artin representations. Now we turn to the conductors and the Euler products for $L(E_{21}, \sigma, s)$ and $L(E_{21}, \rho, s)$. To begin with, the local L -factors for E_{21} are (see §7.1 and (2.8)),

$$P_q(E_{21}/\mathbb{Q}, T) = \begin{cases} 1 - (q + 1 - |\tilde{E}_{21}(\mathbb{F}_q)|)T + qT^2, & q \neq 3, 7 \\ 1 - T, & q = 3, \\ 1 + T, & q = 7. \end{cases}$$

The local L -factors σ and ρ are determined by the action of Frobenius on the inertia invariant subspace of the representation. From their classification in §7.3, we have

$$P_q(\sigma, T) = \begin{cases} 1 - T, & q = 5, \\ 1 - T^4, & q \equiv \pm 2 \pmod{5}, \\ (1 - T^2)^2, & q \equiv -1 \pmod{5}, \\ (1 - T)^4, & q \equiv 1 \pmod{5}. \end{cases}$$

and

$$P_q(\rho, T) = \begin{cases} 1, & q = 2, \\ 1, & q = 5, \\ 1 - T^4, & q \equiv \pm 2 \pmod{5}, q \neq 2 \\ (1 - T^2)^2, & q \equiv -1 \pmod{5}, \\ 1 + T + T^2 + T^3 + T^4, & q \equiv 1 \pmod{5}, x^5 - 2 \text{ irreducible over } \mathbb{F}_q, \\ (1 - T)^4, & q \equiv 1 \pmod{5}, x^5 - 2 \text{ splits over } \mathbb{F}_q. \end{cases}$$

Now we can determine the local polynomials for the twisted L -functions $P_q(E_{21}, \sigma, T)$ and $P_q(E_{21}, \rho, T)$. We illustrate this for ρ , the case of σ being identical. By definition,

$$P_q(E_{21}, \rho, T) = \det(1 - \text{Frob}_q^{-1} T \mid (M_l(E_{21}) \otimes M_l(\rho))^{I_q})$$

Recall that $M_l(E_{21})$ is essentially the dual of the l -adic Tate module $T_l(E_{21})$ and $M_l(\rho)$ is just ρ with coefficients taken in $\hat{\mathbb{Q}}_l$ rather than \mathbb{C} . If $q \neq 2, 3, 5, 7$, then the inertia group I_q acts trivially on both $M_l(E_{21})$ and $M_l(\rho)$. Then the eigenvalues of Frobenius on $M_l(E_{21}) \otimes M_l(\rho)$ are all pairwise products of eigenvalues on $M_l(E_{21})$ with eigenvalues on $M_l(\rho)$. In other words, the local polynomial $P_q(E_{21}, \rho, T)$ can be constructed from $P_q(E_{21}, T)$ and $P_q(\rho, T)$ that we already know.

The same holds for σ for $q \neq 3, 5, 7$. For instance, if $q = 2$, we have

$$\begin{aligned} P_2(E_{21}, T) &= 1 + T + 2T^2 = (1 - \alpha_1 T)(1 - \alpha_2 T), & \alpha_{1,2} &= \frac{-1 \pm \sqrt{-7}}{2}, \\ P_2(\sigma, T) &= 1 - T^4 = (1 - T)(1 + T)(1 - iT)(1 + iT). \end{aligned}$$

Thus,

$$\begin{aligned} P_2(E_{21}, T) &= \\ &= (1 - \alpha_1 T)(1 + \alpha_1 T)(1 - i\alpha_1 T)(1 + i\alpha_1 T)(1 - \alpha_2 T)(1 + \alpha_2 T)(1 - i\alpha_2 T)(1 + i\alpha_2 T) \\ &= 1 - T^4 + 16T^8. \end{aligned}$$

For primes $q = 2, 3, 5$ and 7 it is still true in our case that

$$(7.7) \quad (M_l(E_{21}) \otimes M_l(\rho))^{I_q} \cong M_l(E_{21})^{I_q} \otimes M_l(\rho)^{I_q}$$

since one of the constituents is always unramified. So the same process of constructing the local polynomials $P_q(E_{21}, \rho, T)$ from $P_q(E_{21}, T)$ and $P_q(\rho, T)$ works. Again, the same holds for σ in place of ρ . (In fact, (7.7) holds for any semistable curve and any Artin representation.)

Here is the summary of local polynomials for $q = 2, 3, 5$ and 7 :

q	$P_q(E_{21}, T)$	$P_q(\sigma, T)$	$P_q(\rho, T)$	$P_q(E_{21}, \sigma, T)$	$P_q(E_{21}, \rho, T)$
2	$1 + T + 2T^2$	$1 - T^4$	1	$1 - T^4 + 16T^8$	1
3	$1 - T$	$1 - T^4$	$1 - T^4$	$1 - T^4$	$1 - T^4$
5	$1 + 2T + 5T^2$	$1 - T$	1	$1 + 2T + 5T^2$	1
7	$1 + T$	$1 - T^4$	$1 - T^4$	$1 - T^4$	$1 - T^4$

This completes the description of all the local factors.

Finally, the formula (6.8) gives us the conductors for the twisted L -functions,

$$N(E_{21}, \sigma) = 3^4 5^6 7^4, \quad N(E_{21}, \rho) = 2^8 3^4 5^{10} 7^4.$$

Note that, except for 5^{10} in case of $N(E_{21}, \rho)$, all other exponents are 8 (which is $\dim M_l(E_{21}) \otimes M_l(\rho) = \dim M_l(E_{21}) \otimes M_l(\sigma)$) less the degree of the corresponding local polynomial. In other words, except for $q = 5$ for ρ , there is no wild ramification.

7.8. L -functions. Now that we have the local polynomials for the twisted L -functions, we can compute their Dirichlet expansions,

$$\begin{aligned} L(E_{21}, \sigma, s) &= \prod_q P_q(E_{21}, \sigma, q^{-s})^{-1} = \frac{1}{1^s} - \frac{2}{5^s} + \frac{16}{11^s} + \frac{1}{16^s} - \frac{1}{25^s} + \frac{8}{41^s} + \dots, \\ L(E_{21}, \rho, s) &= \prod_q P_q(E_{21}, \rho, q^{-s})^{-1} = \frac{1}{1^s} - \frac{4}{11^s} + \frac{1}{16^s} - \frac{2}{41^s} + \frac{2}{61^s} + \frac{1}{81^s} + \dots. \end{aligned}$$

We recall that these functions can be extended to entire functions on \mathbb{C} and satisfy functional equations (2.6). We have already computed the conductors. Theorem 6.9 allows us to compute the signs in the functional equations,

$$\begin{aligned} w_{E_{21}, \sigma} &= w_{E_{21}}^{\dim \sigma} (-1)^{\dim \sigma^-} \prod_{q \in \{3, 7\}} s_q^{\dim \sigma - \dim \sigma^{Iq}} \det(\text{Frob}_q^{-1} | \sigma^{Iq}), \\ w_{E_{21}, \rho} &= w_{E_{21}}^{\dim \rho} (-1)^{\dim \rho^-} \prod_{q \in \{3, 7\}} s_q^{\dim \rho - \dim \rho^{Iq}} \det(\text{Frob}_q^{-1} | \rho^{Iq}). \end{aligned}$$

In the notation of the theorem, $w_{E_{21}} = 1$ is the sign for $L(E_{21}/\mathbb{Q}, s)$. Next, $\dim \sigma^- = \dim \rho^- = 2$ since complex conjugation, as an element of $\text{Gal}(F/\mathbb{Q})$, lies in the conjugacy class of C_{g^2h} (see (7.4) and (7.5)). The curve has split multiplicative reduction at $q = 3$ and non-split multiplicative reduction at $q = 7$, so $s_3 = -1$ and $s_7 = 1$. These primes are good for σ and ρ , so I_3 and I_7 act trivially. Finally, Frob_3 and Frob_7 both have characteristic polynomial $T^4 - 1$ on both σ and ρ (see (7.5)) and, in particular, have determinant 1. We get

$$w_{E_{21}, \sigma} = +1, \quad w_{E_{21}, \rho} = +1.$$

The signs $+1$ are consistent with the Birch–Swinnerton-Dyer conjecture for E_{21}/K and E_{21}/L as the Mordell-Weil rank is 0 over both fields.

7.9. L -values. We have now collected all the necessary information to compute the values $L(E_{21}, \sigma, 1)$ and $L(E_{21}, \rho, 1)$ as in §6.2. In fact, for efficiency reasons, we may compute $L(E_{21}, \sigma, 1)$ as the product of 1-dimensional twists (see Remark 6.6). We find

$$\begin{aligned} L(E_{21}/\mathbb{Q}, 1) &\approx 0.451115405388, \\ L(E_{21}, \sigma, 1) &\approx 2.12709564136, \\ L(E_{21}, \rho, 1) &\approx 1.70167651313. \end{aligned}$$

Note that the L -values are non-zero as predicted by the Birch–Swinnerton-Dyer conjecture for E_{21}/\mathbb{Q} , E_{21}/K and E_{21}/L , given that E_{21} has no points of infinite order over these fields.

The periods of E_{21} are

$$\Omega_+ \approx 3.60892324311, \quad \Omega_- \approx 1.91098978075 i.$$

Now we can compute the orders of $\text{III}(E_{21}/\mathbb{Q})$, $\text{III}(E_{21}/K)$ and $\text{III}(E_{21}/L)$ predicted by the Birch–Swinnerton-Dyer formula 5.12,

$$(7.10) \quad \text{III}_{an}(E_{21}/\mathbb{Q}) = \frac{L(E_{21}, 1) |E_{21}(\mathbb{Q})|^2}{\Omega_+(E_{21}) c_3 c_7} \approx 1.0000000000,$$

$$(7.11) \quad \text{III}_{an}(E_{21}/K) = \left| \frac{L(E_{21}, \sigma, 1) \Delta_K}{\Omega_+(E_{21})^2 (2\Omega_-(E_{21}))^2} \right| \frac{|E_{21}(K)|^2}{c_{v_3} c_{v_7}} \approx 1.0000000000,$$

$$(7.12) \quad \text{III}_{an}(E_{21}/L) = \left| \frac{L(E_{21}, \rho, 1) L(E_{21}/\mathbb{Q}, 1) \Delta_L}{\Omega_+(E_{21})^3 (2\Omega_-(E_{21}))^2} \right| \frac{|E_{21}(L)|^2}{c_{z_3^{(1)}} c_{z_3^{(2)}} c_{z_7^{(1)}} c_{z_7^{(2)}}} \approx 1.0000000000,$$

in agreement with $\text{III}(E_{21}/\mathbb{Q})[2] = 1$, $\text{III}(E_{21}/K)[2] = 1$ and $\text{III}(E_{21}/L)[2] = 1$.

Remark 7.13. The L -functions of the Artin representations, $L(\sigma, s)$ and $L(\rho, s)$, also have a meromorphic continuation to \mathbb{C} and satisfy a functional equation. In fact, they can be expressed in terms of Dedekind zeta functions,

$$L(\sigma, s) = \zeta_K(s), \quad L(\rho, s) = \frac{\zeta_L(s)}{\zeta(s)}.$$

Thus their values at $s = 1$ have a similar arithmetic interpretation. Recall that by the class number formula, for a number field k/\mathbb{Q} ,

$$\text{Res}_{s=1} \zeta_k(s) = \frac{2^{r_{k,1}} (2\pi)^{r_{k,2}} \text{Reg}_k |\text{Cl}(\mathcal{O}_k)|}{|\mu_k| \sqrt{\Delta_k}},$$

where $r_{k,1}$ (resp. $r_{k,2}$) is the number of real (resp. pairs of complex) embeddings of k , Reg_k is the regulator, $\text{Cl}(\mathcal{O}_k)$ is the class group of k and μ_k is the set of all roots of unity in k .

7.14. Iwasawa theory. To apply results of Iwasawa theory to E_{21} , we first need to know that E_{21} has good ordinary reduction at $p = 5$. This is easily verified, as $P_5(E/\mathbb{Q}) = 1 + 2T + 5T^2$ has a non-zero linear term modulo 5.

Next, we need the μ -invariant of E_{21}/K to be zero for $p = 5$. This condition is genuinely hard. At present, this requires knowing $\text{Sel}_{5^\infty}(E_{21}/K)$. For some curves with rational 5-torsion subgroups, Fisher [20, 21] carried out 5-descent to determine the structure of their 5-Selmer groups. Our curve has

$$\text{Gal}(\mathbb{Q}(E_{21}[5])/\mathbb{Q}) \cong \text{GL}_2(\mathbb{F}_5) \quad (\text{order } 480),$$

so doing 5-descent is probably unrealistic.

We can numerically determine the conjectural order of $\text{III}(E_{21}/K)$ from the L -value $L(E_{21}, \sigma, 1)$ (see §7.9). We find

$$\text{III}(E_{21}/K) \stackrel{?}{=} 1,$$

where the equality would hold if we knew the Birch–Swinnerton-Dyer conjecture for E_{21}/K and precise estimates for our L -value computations. As we already know that the Mordell-Weil rank of E_{21}/K is zero, this would imply that $\text{Sel}_{5^\infty}(E_{21}/K)$ is trivial.

We can compute the cyclotomic Euler characteristic of E_{21}/K , given by (3.12),

$$\chi_{cyc}(E_{21}/K) = \frac{|\tilde{E}_{21}(\mathbb{F}_5)|^2 c_{v_3} c_{v_7} |\text{III}(E/K)[5^\infty]|}{|E(K)_{tors}|^2} \pmod{\mathbb{Z}_5^*}.$$

As all the terms are coprime to 5, we get

$$\chi_{cyc}(E_{21}/K) = 1.$$

Now, by Lemma 3.14, we get

$$\mu_{E_{21}/K} = \lambda_{E_{21}/K} = 0.$$

In other words, the characteristic element $f_{E_{21}/K} = 1$, so that $\text{Sel}_{5^\infty}(E_{21}/\mathbb{Q}(\mu_{5^\infty}))$ is finite.

The theorem of Hachimori–Matsuno (Theorem 3.16) tells us that $X(E_{21}/F)$ is Λ_F -torsion and $\mu_{E_{21}/F} = 0$. Moreover, as $m = 2$ is coprime to the conductor $N(E_{21}) = 21$ and $|E_{21}(\mathbb{F}_{16})| = 16$ is coprime to 5, Corollary 3.20 implies that $\lambda_{E_{21}/F} = 0$. Again, we get that the characteristic element $f_{E_{21}/F} = 1$, so that $\text{Sel}_{5^\infty}(E_{21}/\mathbb{Q}(\mu_{5^\infty}, \sqrt[5]{2}))$ is finite.

The reverse implication of Lemma 3.14 now tells us that

$$\mu_{E_{21}/F} = \lambda_{E_{21}/F} = 0 \quad \implies \quad \chi_{cyc}(E_{21}/F) = 1.$$

In particular, by the Euler characteristic formula (3.12),

$$\text{III}(E/F)_{5^\infty} = 1.$$

7.15. Non-abelian Euler characteristics. To test the main conjecture 1.4 for E_{21} , we compute the non-abelian Euler characteristics $\chi_{na}(E_{21}, \sigma)$ and $\chi_{na}(E_{21}, \rho)$. First, the formula of Hachimori–Venjakob (Theorem 4.3) yields

$$\chi_{na}(E_{21}, K) = \chi_{cyc}(E_{21}, K) = 1, \quad \chi_{na}(E_{21}, F) = \chi_{cyc}(E_{21}, F) = 1.$$

This is because $P_1^{(K)}$ and $P_1^{(F)}$ are empty since $m = 2$ is coprime to the conductor $N(E_{21}) = 21$, and $P_2^{(K)}$ and $P_2^{(F)}$ are empty since $|E_{21}(\mathbb{F}_{16})| = 16$ is coprime to 5.

By Artin formalism for non-abelian Euler characteristics (Proposition 4.8),

$$\chi_{na}(E_{21}, \sigma) = \chi_{na}(E_{21}, \rho) = 1.$$

This is consistent with Theorem 5.2, which states that one of the Euler characteristics is trivial if and only if the other one is.

The main conjecture 1.4 predicts that the values of the p -adic L -function $\mathcal{L}_{E_{21}}$, given by (1.3), satisfy

$$\text{ord}_5 \mathcal{L}_{E_{21}}(\sigma) = \text{ord}_5 \chi_{na}(E_{21}, \sigma) = 0$$

and

$$\text{ord}_5 \mathcal{L}_{E_{21}}(\rho) = \text{ord}_5 \chi_{na}(E_{21}, \rho) = 0.$$

Note that Conjecture 5.9, that follows from the main conjecture and Theorem 5.2, asserts that these two 5-adic valuations are either both zero, or neither of them is.

7.16. Verifying the main conjecture. Let us compute $\mathcal{L}_{E_{21}}(\sigma)$ and $\mathcal{L}_{E_{21}}(\rho)$. Recall that

$$\mathcal{L}_{E_{21}}(\rho) = \frac{L_{v, \lambda_0}(E_{21}, \rho, 1)}{\Omega_+(E_{21})^2 \Omega_-(E_{21})^2} \cdot \epsilon_5(\rho) \frac{P_5(\rho, u^{-1})}{P_5(\rho, w^{-1})} \cdot u^{-n(\rho)},$$

and similarly for σ . Here

$$n(\sigma) = \text{ord}_5 N(\sigma) = 3, \quad n(\rho) = \text{ord}_5 N(\rho) = 5.$$

The 5-adic numbers u and w are determined by

$$P_5(E, T) = 1 + 2T + 5T^2 = (1 - uT)(1 - wT) \quad u \in \mathbb{Z}_5^*,$$

so, using Hensel's Lemma,

$$\begin{aligned} u &= 3 + 2 \cdot 5 + 4 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 4 \cdot 5^5 + 2 \cdot 5^7 + 5^8 + 5^9 + O(5^{10}), \\ w &= 2 \cdot 5 + 2 \cdot 5^3 + 3 \cdot 5^4 + 4 \cdot 5^6 + 2 \cdot 5^7 + 3 \cdot 5^8 + 3 \cdot 5^9 + O(5^{10}). \end{aligned}$$

The local ϵ -factors $\epsilon_5(\sigma)$ and $\epsilon_5(\rho)$ can be computed as in §6.10,

$$\epsilon_5(\sigma) = -5^{3/2}, \quad \epsilon_5(\rho) = -5^{5/2}.$$

From (7.10), (7.11) and (7.12) we get numerically the values

$$\begin{aligned} L^*(E_{21}, \sigma) &= \left| \frac{L(E_{21}, \sigma, 1) \sqrt{\Delta_K}}{\Omega_+(E_{21})^2 (2\Omega_-(E_{21}))^2} \right| = 1/8, \\ L^*(E_{21}, \rho) &= \left| \frac{L(E_{21}, \rho, 1) \sqrt{\Delta_L}}{\Omega_+(E_{21})^2 (2\Omega_-(E_{21}))^2} \right| = 2. \end{aligned}$$

We combine all this information to obtain

$$\begin{aligned} \mathcal{L}_{E_{21}}(\rho) &= L^*(E_{21}, \rho) \cdot (-2i)^2 \cdot \prod_{q=2,5} P_q(E, \rho, q^{-1}) \cdot \frac{\epsilon_5(\rho)}{\sqrt{|\Delta_L|}} \frac{P_5(\rho, u^{-1})}{P_5(\rho, w^{-1})} \cdot u^{-n(\rho)} \\ &= 2 \cdot (-4) \cdot (1 \cdot 1) \cdot \frac{-5^{5/2}}{2^2 5^{5/2}} \cdot \frac{1}{1} \cdot u^{-5} \\ &= 4 + 2 \cdot 5 + 2 \cdot 5^2 + 4 \cdot 5^3 + O(5^4). \end{aligned}$$

Similarly, for σ we have

$$\begin{aligned} \mathcal{L}_{E_{21}}(\sigma) &= L^*(E_{21}, \sigma) \cdot (-2i)^2 \cdot \prod_{q=2,5} P_q(E, \sigma, q^{-1}) \cdot \frac{\epsilon_5(\sigma)}{\sqrt{|\Delta_K|}} \frac{P_5(\sigma, u^{-1})}{P_5(\sigma, w^{-1})} \cdot u^{-n(\sigma)} \\ &= \frac{1}{8} \cdot (-4) \cdot (1 - 2^{-4} + 16 \cdot 2^{-8})(1 + 2 \cdot 5^{-1} + 5 \cdot 5^{-2}) \cdot \frac{-5^{3/2}}{5^{3/2}} \cdot \frac{1 - u^{-1}}{1 - w^{-1}} \cdot u^{-3} \\ &= 4 + 3 \cdot 5^2 + 5^3 + O(5^4). \end{aligned}$$

As expected, $\mathcal{L}_{E_{21}}(\sigma)$ and $\mathcal{L}_{E_{21}}(\rho)$ are 5-adic units and are congruent modulo 5, in accordance with (6.1).

7.17. An example with additive reduction. The curve E_{21} from the example in §7 is everywhere semistable. If E/\mathbb{Q} has additive reduction at some prime l of \mathbb{Q} , some care has to be taken when computing the Birch–Swinnerton–Dyer quotient for E/K and $E/\mathbb{Q}(\sqrt[m]{m})$ using a model over \mathbb{Q} . Let us illustrate this with an example.

Consider the elliptic curve $E = 272C1$ in the notation of Cremona [14],

$$E : y^2 = x^3 - x^2 - 4x.$$

The curve has bad reduction at $l = 2$ and $l = 17$ with

$$\begin{aligned} \text{ord}_2 \Delta(E) &= 8, & \text{additive red. at } 2 \text{ of type } I_0^* & & c_2 &= 4, \\ \text{ord}_{17} \Delta(E) &= 1, & \text{split multiplicative red. at } 17 \text{ of type } I_1 & & c_{17} &= 1. \end{aligned}$$

The given Weierstrass model for E/\mathbb{Q} is minimal at all primes and

$$E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}, \quad \Omega^+ \approx 3.47306346 \quad 2\Omega^+\Omega^- \approx 6.75008201i.$$

We can compute the Birch–Swinnerton–Dyer quotient and obtain, numerically,

$$\frac{L(E, 1)}{\Omega^+} \approx \frac{\prod_l c_l}{|E(\mathbb{Q})|^2} \cdot 1.00000000,$$

so the analytic order of $\text{III}(E/\mathbb{Q})$ is 1.

Now let $p = 3$, $m = 2$ and $L = \mathbb{Q}(\sqrt[3]{m}) = \mathbb{Q}(\sqrt[3]{2})$. We use 2-descent to conclude that $E(\mathbb{Q}) = E(L)$, compute the local Tamagawa numbers c_v for E/L and the value $L(E/L, 1)$, leading to

$$(7.18) \quad \frac{L(E/L, 1)}{|\Omega^+\Omega^+2\Omega^-|} \sqrt{|\Delta_L|} \approx \frac{\prod_v c_v}{|E(L)|^2} \cdot 2.00000000.$$

Clearly the real number on the right is not supposed to be the analytic order of III (being not a square of an integer). The reason is that the model that we use for E is no longer minimal and the Néron differential of E/\mathbb{Q} is not the Néron differential for E/L at $l = 2$. Thus, if we use the same differential as over \mathbb{Q} , then the ideal $\mathcal{A}_{E/L}$ is non-trivial. In fact, it is not hard to see that $\mathcal{A}_{E/L} = (\sqrt[3]{2})$ and its norm $N_{L/\mathbb{Q}}(\mathcal{A}_{E/L}) = 2$ contributes to the Birch–Swinnerton–Dyer quotient, leading to $\text{III}_{an}(E/L) = 1$ (in agreement with 2-descent.)

Alternatively, note that over \mathbb{Q} the curve E has another minimal Weierstrass equation,

$$y^2 - 2xy = x^3 - 2x^2 - 4x,$$

which can be written over L as

$$y^2 - \pi^3 xy = x^3 - \pi^3 x^2 - \pi^6 x, \quad \pi = \sqrt[3]{2}.$$

Now let $x = \pi^2 x'$ and $y = \pi^3 y'$, transforming the latter model into

$$E' : (y')^2 - \pi^2 x' y' = (x')^3 - \pi(x')^2 - \pi^2 x'.$$

From Tate’s algorithm, it follows that this is a minimal Weierstrass equation at all primes, so $\mathcal{A}' = 1$ for the differential of this model. On the other hand, from the transformation of coordinates it follows that, with this differential,

$$(\Omega^+)' = \pi \Omega^+, \quad (\Omega^-)' = \pi \Omega^-.$$

With these “corrected” periods the analytic order of III from (7.18) becomes 1, as asserted.

7.19. An example for a GL_2 -extension. One can make L -value computations for an elliptic curve E in p -adic Lie extensions other than F_∞/\mathbb{Q} . For example, one may take an extension of the form $\mathbb{Q}(C[p^\infty])/\mathbb{Q}$ for some elliptic curve C/\mathbb{Q} . For non-CM curves C , the Galois group of this extension is an open subgroup of $GL_2(\mathbb{Z}_p)$ and is the whole of GL_2 for almost all p , see [40]. These extensions (with $E = C$) are the main focus of [9]. One can again define a non-abelian Euler characteristic $\chi_{na}(E, \tau)$ of E twisted by an Artin representation τ that factors through such an extension. There is also a conjectural p -adic L -function \mathcal{L}_E , interpolating the special values of twisted L -functions $L(E, \tau, s)$ ([9], Conjecture 5.7). The “Main Conjecture” ([9], Conjecture 5.8) gives a relation between the values $\mathcal{L}_E(\tau)$ and Euler characteristics $\chi_{na}(E, \tau)$.

Example 7.20. Take $p = 5$ and consider the elliptic curve $E = 11A3 = X_1(11)$ over \mathbb{Q} ,

$$E : y^2 + y = x^3 - x^2 .$$

The field $\mathbb{Q}(E[5^\infty])$ contains the fields k_1, k_2 of 5-torsion points of E and of its isogenous curve 11A2. Both are extensions of degree 5 of $\mathbb{Q}(\mu_5)$ and can be explicitly described as follows. In $\mathbb{Q}(\sqrt{5})$ one can write $-11 = \alpha\beta$ with $\alpha = \frac{9+5\sqrt{5}}{2}$ and $\beta = \frac{9-5\sqrt{5}}{2}$. Then

$$k_1 = \mathbb{Q}\left(\mu_5, \sqrt[5]{\frac{\alpha}{\beta}}\right), \quad k_2 = \mathbb{Q}\left(\mu_5, \sqrt[5]{11}\right) .$$

Their Galois groups are the same as in the first layer of the false Tate curve extension for $p = 5$,

$$\text{Gal}(k_1/\mathbb{Q}) \cong G_{20} \cong \text{Gal}(k_2/\mathbb{Q}) .$$

In particular, we have two 4-dimensional irreducible Artin representations ρ_1, ρ_2 of $\text{Gal}(\mathbb{Q}(E[5^\infty])/\mathbb{Q})$ that factor through k_1 and k_2 respectively.

The Main Conjecture implies that ([9], 5.7-5.10)

(7.21)

$$\text{ord}_5(\chi_{na}(E, \rho_i)) = \text{ord}_5\left(\frac{L_R(E, \rho_i, 1)}{\Omega_+(E)^2 \Omega_-(E)^2} \cdot \frac{P_5(\rho_i, u^{-1})}{P_5(\rho_i, w^{-1})} \epsilon_5(\rho_i) u^{-f_{\rho_i}}\right), \quad i = 1, 2.$$

Here $R = \{5, 11\}$ is the set of primes that contains p and those primes that divide the denominator of the j -invariant $j(E) = -2^{12}/11$. The modified L -function $L_R(E, \rho_i, s)$ is that of E of twisted by ρ_i with the local factors at $l \in R$ removed.

We proceed to verify this conjecture, similarly to the false Tate curve case.

First, the curve E has a 5-torsion point over \mathbb{Q} , and

$$E(\mathbb{Q}(\mu_5))[5^\infty] \cong \mathbb{Z}/5\mathbb{Z}, \quad E(k_1)[5^\infty] \cong (\mathbb{Z}/5\mathbb{Z})^2, \quad E(k_2)[5^\infty] \cong \mathbb{Z}/5\mathbb{Z}.$$

By results of Fisher [20], E has Mordell-Weil rank 0 over $\mathbb{Q}(\mu_5)$, k_1 and k_2 , and

$$\text{III}(E/\mathbb{Q}(\mu_5))[5^\infty] = 0, \quad \text{III}(E/k_1)[5^\infty] \cong (\mathbb{Z}/5\mathbb{Z})^2, \quad \text{III}(E/k_2)[5^\infty] = 0 .$$

Since E has good ordinary reduction at $p = 5$, the module $X(E/\mathbb{Q}(\mu_5)^{cyc})$ is $\Lambda_{\mathbb{Q}(\mu_5)}$ -torsion by Theorem 3.4. Using formula 3.12, it is easy to see that $\chi_{cyc}(E/\mathbb{Q}(\mu_5)) = 1$. It follows from Lemma 3.14 that $\lambda_{E/\mathbb{Q}(\mu_5)} = \mu_{E/\mathbb{Q}(\mu_5)} = 0$. By Theorem 3.16, both $X(E/k_1^{cyc})$ and $X(E/k_2^{cyc})$ are torsion and their μ -invariants are 0.

Next, we compute $\chi_{na}(E, \rho_i)$. By Artin formalism ([9] Theorem 3.10),

$$\chi_{na}(E, \rho_i) = \sqrt[4]{\frac{\chi_{na}(E/k_i)}{\chi_{na}(E/\mathbb{Q}(\mu_5))}} .$$

A formula for χ_{na} of E over number fields in $\mathbb{Q}(E[5^\infty])$ has been worked out in Coates-Howson [10], Theorem 1.1. In our case, for a number field $k \subset \mathbb{Q}(E[5^\infty])$ for which $\text{Sel}_{5^\infty}(E/k)$ is finite, it states

$$\chi_{na}(E/k) = \chi_{cyc}(E/k) \prod_{v|11} |L_v(E, 1)|_5,$$

the product taken over primes of k dividing 11. (That the theorem is applicable here follows from [10], Theorem 6.4.)

Now, the above two formulae together with the values of $|\text{III}(E/k_i)|$ and $|\text{III}(E/\mathbb{Q}(\mu_5))|$ yield

$$(7.22) \quad \chi_{na}(E/k_1) = 5^3, \quad \chi_{na}(E/k_2) = 5^1 .$$

To test the main conjecture numerically, we evaluate the right-hand side of (7.21). We compute the L -values as described in §§6.2-6.7 with mild modifications in the case of E/k_1 . We find

$$L(E, \rho_1, 1) \approx 1.26706100, \quad L(E, \rho_2, 1) \approx 4.05459521.$$

The local ϵ -factors $\epsilon_5(\rho_1)$ and $\epsilon_5(\rho_2)$ can be determined as in §6.10,

$$\epsilon_5(\rho_1) = -5^{3/2}, \quad \epsilon_5(\rho_2) = -5^{5/2}.$$

Next,

$$\begin{aligned} P_5(\rho_1, T) &= 1 - T, & P_5(\rho_2, T) &= 1, \\ P_5(E, \rho_1, T) &= 1 - T + 5T^2, & P_5(E, \rho_2, T) &= 1, \\ P_{11}(E, \rho_1, T) &= 1, & P_{11}(E, \rho_2, T) &= 1. \end{aligned}$$

and

$$\Omega_1 \approx 6.34604652, \quad \Omega_2 \approx 1.45881662i.$$

Finally,

$$u = 1 + 4 \cdot 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + O(5^4), \quad w = 5 + 5^2 + 2 \cdot 5^3 + O(5^7).$$

Combining the above information, we obtain

$$\begin{aligned} \frac{L_R(E, \rho_1, 1)}{\Omega_+(E)^2 \Omega_-(E)^2} \cdot \frac{P_5(\rho_1, u^{-1})}{P_5(\rho_1, w^{-1})} \epsilon_5(\rho_1) u^{-f_{\rho_1}} &= 4 \cdot 5^3 + 2 \cdot 5^4 + O(5^5) \\ \frac{L_R(E, \rho_2, 1)}{\Omega_+(E)^2 \Omega_-(E)^2} \cdot \frac{P_5(\rho_2, u^{-1})}{P_5(\rho_2, w^{-1})} \epsilon_5(\rho_2) u^{-f_{\rho_2}} &= 4 \cdot 5 + 4 \cdot 5^2 + O(5^3). \end{aligned}$$

This agrees with (7.22), as predicted by the main conjecture.

APPENDIX A (BY J. COATES AND R. SUJATHA)

The aim of this appendix is to prove several theoretical results, which are related to, and illustrated by, the numerical calculations carried out in this paper. As earlier, let p be an odd prime number, and put $K = \mathbb{Q}(\mu_p)$, $K^{cyc} = \mathbb{Q}(\mu_{p^\infty})$. Let $m > 1$ be an integer, which is p -power free, and put

$$(A.1) \quad F_\infty = K^{cyc}(\sqrt[n]{m} : n = 1, 2, \dots), \quad G = \text{Gal}(F_\infty/\mathbb{Q}).$$

Let E be an elliptic curve over \mathbb{Q} , which we shall always assume satisfies:

Hypothesis A.2. E has good ordinary reduction at p , and m is not divisible by any prime of additive reduction for E .

The hypothesis on m is made to ensure that the principal results of [25] and [26] remain valid for $p = 3$ in our case.

We write $X(E/F_\infty)$ for the Pontryagin dual of the p^∞ -Selmer group of E over F_∞ . If J is any compact p -adic Lie group, $\Lambda(J)$ will denote the Iwasawa algebra of J . Then $X(E/F_\infty)$ is in fact $\Lambda(G)$ -torsion (see [26]), thanks to Kato's [29] deep theorem that the dual of the Selmer group of E over K^{cyc} is $\Lambda(\Gamma_K)$ -torsion, where $\Gamma_K = \text{Gal}(K^{cyc}/K)$. However, the ideas of [9] suggest that something much stronger should always be true for $X(E/F_\infty)$. Let $H = \text{Gal}(F_\infty/\mathbb{Q}^{cyc})$, where \mathbb{Q}^{cyc} denotes the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} , and let $\mathfrak{M}_H(G)$ denote the category of all finitely generated $\Lambda(G)$ -modules M such that $M/M[p^\infty]$ is finitely generated over $\Lambda(H)$; here $M[p^\infty]$ denotes the p -primary submodule of M .

Conjecture A.3. $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$.

Using similar techniques to those discussed in §5 of [9], it is easy to see that Conjecture A.3 is equivalent to the assertion that

$$(A.4) \quad \mu_{G_K}(X(E/F_\infty)) = \mu_{\Gamma_K}(X(E/K^{cyc})),$$

where G_K is the pro- p group given by $G_K = \text{Gal}(F_\infty/K)$, and $\mu_{G_K}(M)$ denotes the μ -invariant of any finitely generated torsion $\Lambda(G_K)$ -module M as defined in [28, 35], and similarly $\mu_{\Gamma_K}(R)$ denotes the μ -invariant of a finitely generated torsion $\Lambda(\Gamma_K)$ -module R . Moreover, Conjecture A.3 is true if there exists an elliptic curve E' over K , which is isogenous to E over K , such that $X(E'/K^{cyc})$ is a finitely generated \mathbb{Z}_p -module.

As always, we shall find that the deepest arithmetic questions arise from the interplay between the $\Lambda(G)$ -module $X(E/F_\infty)$ and the complex L -functions $L(E, \phi, s)$, which are studied earlier in the paper, where ϕ runs over all Artin representations of G . We recall that the $L(E, \phi, s)$ are known to be entire, and to satisfy the standard functional equation. Moreover, if k denotes any finite extension of \mathbb{Q} contained in F_∞ , we write $L(E/k, s)$ for the complex L -function of E over k , and we define

$$(A.5) \quad g_{E/k} = \text{rk}_{\mathbb{Z}}(E(k)), \quad r_{E/k} = \text{ord}_{s=1}(L(E/k, s)).$$

Of course, the conjecture of Birch and Swinnerton-Dyer predicts that we always have $g_{E/k} = r_{E/k}$.

We first discuss the notion of regularity for E over F_∞ .

Definition A.6. We say E is *regular* over F_∞ if $X(E/F_\infty) = 0$.

As we shall explain below, the calculations in this paper provide many examples when E is regular over F_∞ . But first we note the following theorem.

Theorem A.7. *Assume E is regular over F_∞ . Then $g_{E/k} = r_{E/k}$ for all finite extensions k of \mathbb{Q} contained in F_∞ if and only if $L(E, \phi, 1) \neq 0$ for all Artin representations ϕ of G .*

Proof. As earlier, let $\text{Sel}_{p^\infty}(E/k)$ (resp. $\text{Sel}_{p^\infty}(E/F_\infty)$) denote the p^∞ -Selmer group of E over k (resp. F_∞). Now the kernel of the restriction map

$$\text{Sel}_{p^\infty}(E/k) \rightarrow \text{Sel}_{p^\infty}(E/F_\infty)$$

is always finite (see [26]). Assuming now that $\text{Sel}_{p^\infty}(E/F_\infty) = 0$, it follows that $\text{Sel}_{p^\infty}(E/k)$ must be finite, and so $g_{E/k} = 0$ for all finite extensions k of \mathbb{Q} contained in F_∞ . Thus it remains to show that $r_{E/k} = 0$ for all finite extensions k of \mathbb{Q} contained in F_∞ if and only if $L(E, \phi, 1) \neq 0$ for all irreducible Artin representations ϕ of G . But on the one hand, for any finite extension k of \mathbb{Q} in F_∞ , we have $L(E/k, s) = L(E, \theta, s)$, where θ is the Artin representation of G induced by the trivial representation of $\text{Gal}(F_\infty/k)$, and so one direction is clear. The other direction is plain from the holomorphy of the $L(E, \phi, s)$ at $s = 1$, and the fact that, for any finite Galois extension k of \mathbb{Q} contained in F_∞ with Galois group Δ , we have

$$(A.8) \quad L(E/k, s) = \prod_{\phi \in \tilde{\Delta}} L(E, \phi, s)^{n_\phi},$$

where $\tilde{\Delta}$ denotes the set of all irreducible Artin representations of Δ , and n_ϕ denotes the dimension of ϕ . This completes the proof. \square

The following result leads to a useful numerical criterion for deciding whether or not E is regular over F_∞ . Let M be a compact $\Lambda(G)$ -module, and G' an open subgroup of G . As always, we say M has finite G' -Euler characteristic if the $H_i(G', M)$ ($i = 0, 1, 2$) are finite, and we then define

$$\chi(G', M) = \prod_{i=0}^2 |H_i(G', M)|^{(-1)^i}.$$

Proposition A.9. *Assume that $X(E/F_\infty)$ belongs to the category $\mathfrak{M}_H(G)$. Then E is regular over F_∞ if and only if*

$$(A.10) \quad \chi(G_K, X(E/F_\infty)) = 1, \quad \text{where } G_K = \text{Gal}(F_\infty/K).$$

Proof. Assume (A.10) holds. We show that $X(E/F_\infty) = 0$ by using the general remarks made in [9] in paragraph immediately after the proof of Lemma 3.9. Of course, G_K is pro- p , and entirely analogous arguments to those given in the proof of Lemma 2.5 of [11] show that $H_1(H_K, X(E/F_\infty)) = 0$, where $H_K = \text{Gal}(F_\infty/\mathbb{Q}(\mu_{p^\infty}))$. Finally, it is proven in [26] that $X(E/F_\infty)$ has no non-zero pseudo-null $\Lambda(G)$ -submodule. Hence, as explained in [9], (A.10) implies that $X(E/F_\infty) = 0$, as required. \square

We remark that Proposition A.9 is just a slight generalisation of Theorem 5.7.

The next result enables us to read off many examples when E is regular over F_∞ from the tables in Appendix B. As earlier, let $\sigma = \sigma_1 : G \rightarrow \text{GL}_r(\mathbb{Q}_p)$, where $r = p-1$, be the Artin representation given by the direct sum of the $p-1$ characters of $\text{Gal}(K/\mathbb{Q})$, and let $\mathcal{L}_E(\sigma)$ be defined by formula (1.3) of §4.

Corollary A.11. *Assume that the Mazur–Swinerton–Dyer p -adic L -function of E over K lies in $\Lambda(\Gamma_K)$. If $\mathcal{L}_E(\sigma)$ is a p -adic unit, then E is regular over F_∞ , i.e. $X(E/F_\infty) = 0$.*

Proof. Assume that $\mathcal{L}_E(\sigma)$ is a p -adic unit. Define

$$A_m = \prod_{v|m, v \nmid p} L_v(E/K, 1)^{-1},$$

where the product is taken over all places v of K , which divide m , and which do not divide p , and where $L_v(E/K, s)$ denotes the Euler factor at v of the complex L -function of E over K . Since $\mathcal{L}_E(\sigma)$ is a p -adic unit, it follows easily that A_m is a p -adic unit and the Mazur–Swinerton–Dyer p -adic L -function of E over K is a unit in $\Lambda(\Gamma_K)$. From this latter assertion and the results of Kato [29] and Matsuno [33], it follows that $X(E/K^{cy_c}) = 0$, whence $X(E/F_\infty)$ must belong to $\mathfrak{M}_H(G)$. But it is proven in [26] that, whenever $X(E/K^{cy_c})$ has finite Γ_K -Euler characteristic, we have

$$\chi(G_K, X(E/F_\infty)) = \chi(\Gamma_K, X(E/K^{cy_c})) \cdot |A_m|_p^{-1}.$$

In particular, we conclude that the right hand side of this formula, and so also the left hand side, is equal to 1. Hence the corollary follows from Proposition A.9. \square

Needless to say, it is conjectured that the Mazur–Swinerton–Dyer p -adic L -function of E over K always lies in $\Lambda(\Gamma_K)$. This is known when the Galois module $E[p]$ is either irreducible, or contains either $\mathbb{Z}/p\mathbb{Z}$ or μ_p as a Galois submodule (see [52]).

The next result was motivated by the congruences studied in this paper, together with the “main conjecture” of [9]. If $\phi : G \rightarrow \mathrm{GL}_m(\mathbb{Z}_p)$ is an Artin representation of G , which we assume, for simplicity, can be realised over \mathbb{Q}_p , and M is a compact $\Lambda(G)$ -module, we recall from [9] that

$$(A.12) \quad \mathrm{tw}_\phi(M) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m$$

endowed with the obvious left diagonal action of G . For each integer $n \geq 1$, we let ρ_n be the irreducible Artin representation of G of dimension $p^{n-1}(p-1)$ defined earlier in the paper (see §1). Let σ_n denote the direct sum of the $p^{n-1}(p-1)$ characters of the Galois group of $\mathbb{Q}(\mu_{p^n})$ over \mathbb{Q} . Now, assuming the Mazur–Swinnerton–Dyer p -adic L -function of E over K belongs to $\Lambda(\Gamma_k)$, it follows easily that $\mathcal{L}_E(\sigma)$ is a p -adic unit if and only if $\mathcal{L}_E(\sigma_n)$ is a p -adic unit for any $n \geq 1$. Thus, assuming both the congruence

$$\mathcal{L}_E(\rho_n) \equiv \mathcal{L}_E(\sigma_n) \pmod{p}$$

and Corollary 5.10 of the “main conjecture” of [9], it would follow that E is regular over F_∞ if and only if $\chi(G, \mathrm{tw}_{\rho_n}(X(E/F_\infty))) = 1$ for some $n \geq 1$. We now give an unconditional proof of this last assertion.

Theorem A.13. *Let M be a $\Lambda(G)$ -module such that (i) $M \in \mathfrak{M}_H(G)$, (ii) $H_1(H', M)$ is finite for all open subgroups H' of H_K , and (iii) M has no non-zero pseudo-null $\Lambda(G)$ -submodule. Then $M = 0$ if and only if there exists an integer $n \geq 1$ such that $\chi(G, \mathrm{tw}_{\rho_n}(M)) = 1$.*

Corollary A.14. *Assume that $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. Then $X(E/F_\infty) = 0$ if and only if there exists an integer $n \geq 1$ such that $\chi(G, \mathrm{tw}_{\rho_n}(X(E/F_\infty))) = 1$.*

Indeed, (iii) of Theorem A.13 is proven for $M = X(E/F_\infty)$ in [26], and (ii) is valid by a similar argument to that used to prove Lemma 2.5 of [11].

Prior to proving Theorem A.13, we state a lemma which will be proven at the end of the Appendix, after a general discussion Akashi series. If G' is an open subgroup of G , let $H' = H \cap G'$, $\Gamma' = G'/H'$, and write $Q(\Gamma')$ for the field of quotients of $\Lambda(\Gamma')$. If $M \in \mathfrak{M}_H(G)$, we recall that $\mathrm{Ak}_{H'}(M)$ is defined to be the image in $Q(\Gamma')^\times / \Lambda(\Gamma')^\times$ of f_0/f_1 , where f_0 (resp. f_1) is a characteristic element in $\Lambda(\Gamma')$ for $H_0(H', M)$ (resp. $H_1(H', M)$). For each integer $n \geq 1$, let

$$(A.15) \quad F_n = \mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m}), \quad F'_n = \mathbb{Q}(\mu_{p^n}, \sqrt[p^{n-1}]{m}),$$

and let G_n (resp. G'_n) be the open subgroup of G fixing F_n (resp. F'_n). Put

$$(A.16) \quad H_n = G_n \cap H, \quad H'_n = G'_n \cap H,$$

and note that

$$(A.17) \quad \Gamma_n = G_n/H_n = G'_n/H'_n$$

is, in fact, the unique closed subgroup of $\Gamma = G/H$ of index p^{n-1} .

Lemma A.18. *Assume $M \in \mathfrak{M}_H(G)$. Then, for all $n \geq 1$, we have*

$$(A.19) \quad \mathrm{Ak}_{H_n}(M) = \mathrm{Ak}_{H'_n}(M) \mathrm{Ak}_H(\mathrm{tw}_{\rho_n}(M))^{p-1}.$$

Moreover, if in addition M satisfies condition (ii) of Theorem A.13, then $\mathrm{Ak}_H(\mathrm{tw}_{\rho_n}(M))$ belongs to $\Lambda(\Gamma)$.

Remark A.20. If one accepts that $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$ and Case 2 of Conjecture 4.8 of [9], the above lemma implies that the G -characteristic element of $X(E/F_\infty)$ is in $\Lambda(G)$. In view of the “main conjecture” of [9], the same should hold for the p -adic L -function \mathcal{L}_E .

We now prove Theorem A.13. Let us pick a topological generator of Γ , and so identify $\Lambda(\Gamma)$ with the formal power series ring $\mathbb{Z}_p[[T]]$. We assume that there exists an integer $n \geq 1$ such that

$$(A.21) \quad \chi(G, \text{tw}_{\rho_n}(M)) = 1,$$

and we must deduce that $M = 0$. Put $f_n = \text{Ak}_H(\text{tw}_{\rho_n}(M))$. By Lemma A.18, f_n belongs to $\mathbb{Z}_p[[T]]$. Moreover, by the connexion between Euler characteristics and Akashi series ([9], Theorem 3.6), we have $f_n(0) \in \mathbb{Z}_p^\times$ by virtue of (A.21). Hence f_n belongs to $\Lambda(\Gamma)^\times$. Let g_n (resp. g'_n) denote a characteristic power series for the $\Lambda(\Gamma_n)$ -module $(M)_{H_n}$ (resp. $(M)_{H'_n}$). Since f_n is a unit in $\Lambda(\Gamma)$, and M satisfies condition (ii) of Theorem A.13, it follows from Lemma A.18 that we can assume that

$$(A.22) \quad g_n = g'_n.$$

We show that (A.22) forces M to be zero.

Let $M[p^\infty]$ denote the p -primary submodule of M . We first show that necessarily $M[p^\infty] = 0$. If G' is a pro- p open subgroup of G , we recall that $\mu_{G'}(M)$ denotes the μ -invariant of M viewed as a $\Lambda(G')$ -module. Since $G_n \subset G'_n \subset G_1$ are all pro- p open subgroups of G , and $[G'_n : G_n] = p$, we have

$$(A.23) \quad \mu_{G_n}(M) = p \cdot \mu_{G'_n}(M).$$

On the other hand, we claim that

$$(A.24) \quad \mu_{G_n}(M) = \mu_{\Gamma_n}((M)_{H_n}), \quad \mu_{G'_n}(M) = \mu_{\Gamma_n}((M)_{H'_n}).$$

We give the proof of the first equation in (A.24), and the proof of the second is entirely similar. Put $Y(M) = M/M[p^\infty]$. Then we have the exact sequence

$$(A.25) \quad H_1(H_n, Y(M)) \rightarrow (M[p^\infty])_{H_n} \rightarrow (M)_{H_n} \rightarrow (Y(M))_{H_n} \rightarrow 0.$$

Since M belongs to $\mathfrak{M}_H(G)$ and H_n is open in H , the homology groups $H_i(H_n, Y(M))$ ($i = 0, 1$) are finitely generated \mathbb{Z}_p -modules, whence (A.25) implies that

$$(A.26) \quad \mu_{\Gamma_n}((M)_{H_n}) = \mu_{\Gamma_n}((M[p^\infty])_{H_n}).$$

But, since G_n is pro- p , a standard argument with the Hochschild-Serre spectral sequence (see [11]) shows that

$$(A.27) \quad \mu_{G_n}(M) = \mu_{\Gamma_n}((M[p^\infty])_{H_n}) - \mu_{\Gamma_n}(H_1(H_n, M[p^\infty])).$$

But $H_1(H_n, M[p^\infty])$ injects into $H_1(H_n, M)$, and thus $H_1(H_n, M[p^\infty])$ is finite because of hypothesis (ii) on M . Thus the term on the extreme right of (A.27) is 0, and the first equation in (A.24) follows on combining (A.26) and (A.27). If we now combine (A.22) and (A.24), we conclude that $\mu_{G_n}(M) = \mu_{G'_n}(M)$. Comparing this last equation with (A.23), it follows that $\mu_{G_n}(M) = 0$. Thus $M[p^\infty]$ must be pseudo-null as a $\Lambda(G)$ -module, and so $M[p^\infty] = 0$ because of hypothesis (iii) on M .

Since $M[p^\infty] = 0$, M is finitely generated over $\Lambda(H)$, and so also over $\Lambda(H_n)$ and $\Lambda(H'_n)$. If A is a ring with no zero divisors, and W is a finitely generated A -module,

we write $r_A(W)$ for the rank of W over A . Since $H_n \subset H'_n \subset H_1$ are all pro- p open subgroups of H , and $[H'_n : H_n] = p$, we have

$$(A.28) \quad r_{\Lambda(H_n)}(M) = p \cdot r_{\Lambda(H'_n)}(M).$$

On the other hand, we have the well known formula

$$(A.29) \quad r_{\Lambda(H_n)}(M) = \sum_{i=0}^1 (-1)^i r_{\mathbb{Z}_p}(H_i(H_n, M)),$$

and similarly for $r_{\Lambda(H'_n)}(M)$. But then, as M satisfies hypothesis (ii), (A.22) and (A.29) together imply that $r_{\Lambda(H_n)}(M) = r_{\Lambda(H'_n)}(M)$. Hence, in view of (A.28), we have $r_{\Lambda(H_n)}(M) = 0$. By a result of Venjakob [53], we then conclude that M is pseudo-null, whence $M = 0$ by hypothesis (iii). This completes the proof of Theorem A.13.

We next discuss some remarkable arithmetic phenomena which arise from the interplay between root numbers and Iwasawa theory for our false Tate curve extension F_∞ . For each integer $n \geq 1$, we write $w(E, \rho_n) = \pm 1$ for the sign in the functional equation of $L(E, \rho_n, s)$. We also write $w(E/K)$ for the sign in the functional equation for $L(E/K, s)$. It is an important fact, first proven in [19], that the value $w(E, \rho_n)$ is independent of n (it is assumed in [19] that the conductor N_E of E is cube free, but the assertion remains valid for all E provided m is not divisible by a prime of additive reduction for E , see (A.33) below.) Suppose that we have

$$(A.30) \quad w(E, \rho_n) = -1 \quad \forall n \geq 1.$$

Since all $L(E, \rho_n, s)$ are holomorphic at $s = 1$ and ρ_n has dimension $p^{n-1}(p-1)$, it follows immediately on applying (A.8) to F_n/\mathbb{Q} that, if (A.30) is valid, then

$$(A.31) \quad r_{E/F_n} \geq p^n - 1 + r_{E/K_n} \quad \forall n \geq 1,$$

where F_n is given by (A.15) and $K_n = \mathbb{Q}(\mu_{p^n})$ (note that r_{E/K_n} is bounded as $n \rightarrow \infty$ by an important theorem of Rohrlich). We identify $\Lambda(\Gamma_K)$ with $\mathbb{Z}_p[[T]]$ by mapping a fixed topological generator γ_0 of Γ_K to $1 + T$. By Kato's theorem, $X(E/K^{cyc})$ is a torsion $\Lambda(\Gamma_K)$ -module, and we write $t_{E/K}$ for the multiplicity of the zero at $T = 0$ of a characteristic power series of $X(E/K^{cyc})$. We have $t_{E/K} \geq g_{E/K}$, and conjecturally there is always equality.

Theorem A.32. *Assume that $X(E/K^{cyc})$ is a finitely generated \mathbb{Z}_p -module, and that $t_{E/K} \equiv r_{E/K} \pmod{2}$. Then (A.30) is valid if and only if $X(E/F_\infty)$ has odd $\Lambda(H_K)$ -rank. In particular, (A.30) implies that E is not regular over F_∞ .*

Proof. Let S be the set of primes of multiplicative reduction for E which divide m . Then, as was remarked to us by V. Dokchitser, similar arguments to those used to prove Propositions 9 and 11 of [19] show that

$$(A.33) \quad w(E, \rho_n) = w(E/K) \prod_{q \in S} \left(\frac{q}{p} \right), \quad n = 1, 2, \dots$$

On the other hand, let $h(E/F_\infty)$ denote the $\Lambda(H_K)$ -rank of $X(E/F_\infty)$. We can determine $h(E/F_\infty)$ modulo 2 as follows. Let λ denote the \mathbb{Z}_p -corank of $X(E/K^{cyc})$. For each prime $q \neq p$, let s_q denote the number of primes of K above q . Since K/\mathbb{Q} is a cyclic extension of conductor p , one sees easily that s_q is odd if and only if $\left(\frac{q}{p} \right) = -1$. We define S_1 to be the subset of S consisting of all primes q dividing

m such that E has split multiplicative reduction at all primes of K above q . Let U denote the set of all primes $q \neq p$ satisfying (i) q divides m , (ii) E has good reduction at q , and (iii) p divides the order of $\tilde{E}_q(k_v)$ for each prime v of K above q , where \tilde{E}_q denotes the reduction of E modulo q , and k_v denotes the residue field at v . Then a standard argument applying the formula (A.29) for H_K and the fact that $H_1(H_K, X(E/F_\infty)) = 0$ shows that

$$(A.34) \quad h(E/F_\infty) = \lambda + \sum_{q \in S_1} s_q + 2 \sum_{q \in U} s_q$$

(see [26], Theorem 3.1 for an alternative proof when $p \geq 5$). Define S_2 to be the set of all primes q of multiplicative reduction such that $\left(\frac{q}{p}\right) = -1$. Clearly $S_2 \subset S_1$ since a q in S_2 is inert in the quadratic subfield of K , and so has split multiplicative reduction at all primes of K above q . Hence we have

$$(A.35) \quad \sum_{q \in S_1} s_q \equiv \sum_{q \in S_2} 1 \pmod{2}.$$

To determine the parity of λ , we use Greenberg's theorem [23] which asserts that the characteristic ideal of $X(E/K^{cy_c})$ is invariant under the involution of $\Lambda(\Gamma_K)$ which sends each γ in Γ_K to γ^{-1} . Hence, if $f(T)$ denotes the monic distinguished polynomial which generates the characteristic ideal of $X(E/K^{cy_c})$, and if α is any non-zero root of $f(T)$ lying in the algebraic closure of \mathbb{Q}_p , then $\frac{1}{1+\alpha} - 1$ must also be a root distinct from α . Hence the degree λ of $f(T)$ must satisfy

$$(A.36) \quad \lambda \equiv t_{E/K} \pmod{2}.$$

Combining (A.33), (A.34), (A.35) and (A.36), and recalling the hypothesis that $t_{E/K} \equiv r_{E/K} \pmod{2}$, we conclude that (A.30) holds if and only if $h(E/F_\infty)$ is odd. This completes the proof of Theorem A.32. \square

Proposition A.37. *Assume that $X(E/K^{cy_c})$ is a finitely generated \mathbb{Z}_p -module, and take $m = q$, where q is a prime of multiplicative reduction for E . Then $X(E/F_\infty)$ has $\Lambda(H_K)$ -rank 1 if and only if either (i) q is inert in K , and $X(E/K^{cy_c}) = 0$, or (ii) E has non-split multiplicative reduction at the primes of K above q , and $X(E/K^{cy_c}) = \mathbb{Z}_p$, with trivial action of Γ_K .*

Proof. The sufficiency follows from (A.34) above. Conversely, assume that $X(E/F_\infty)$ has $\Lambda(H_K)$ -rank 1. Since $X(E/K^{cy_c})$ is assumed to be a finitely generated \mathbb{Z}_p -module, Matsuno's theorem [33] shows that $X(E/K^{cy_c})$ has no non-zero finite Γ_K -submodule. Hence (A.34) implies that either (i) there is a single prime of K above q and $X(E/K^{cy_c}) = 0$, or (ii) E has non-split multiplicative reduction at the primes of K above q , and $X(E/K^{cy_c})$ is a free \mathbb{Z}_p -module of rank 1. In the second case, the same argument with Greenberg's theorem as in the proof of Theorem A.32 shows that Γ_K must act trivially on $X(E/K^{cy_c})$. \square

We now deal with the two different cases occurring in Proposition A.37 separately. Recall that $F_n = \mathbb{Q}(\mu_{p^n}, \sqrt[n]{q})$.

Theorem A.38. *Assume that $X(E/K^{cy_c}) = 0$ and $r_{E/K} = 0$. Take $m = q$, where q is a prime of multiplicative reduction for E , which is inert in K^{cy_c} . Then we have*

$$(A.39) \quad g_{E/F_n} \leq p^n - 1 \leq r_{E/F_n}, \quad \forall n \geq 1.$$

Moreover, if we assume that $g_{E/F_n} = r_{E/F_n}$ for a given integer $n \geq 1$, then

- (i) $g_{E/F_n} = g_{E/F_n^{cy_c}} = p^n - 1$,
- (ii) $\text{III}(E/F_n)[p^\infty]$ is finite,
- (iii) $\text{III}(E/F_n^{cy_c})[p^\infty] = 0$,
- (iv) $L(E, \rho_k, s)$ has a simple zero at $s = 1$ for $1 \leq k \leq n$, and
- (v) $r_{E/\mathbb{Q}(\sqrt[n]{q})} = n$.

In particular, under the hypotheses of Theorem A.38, we see that if we assume the Birch–Swinnerton–Dyer conjecture $g_{E/F_n} = r_{E/F_n}$ for all $n \geq 1$, then $\text{III}(E/F_\infty)[p^\infty] = 0$, and $X(E/F_\infty)$ is dual to $E(F_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Remarkably, if one assumes the hypothesis of Theorem A.38 and, in addition, that E has prime conductor, H. Darmon and Y. Tian have informed us that they can prove that $g_{E/F_n} = r_{E/F_n}$ for all $n \geq 1$. We also mention that the calculations carried out in this paper give numerical examples of Theorem A.38, for example

$$\begin{aligned} E &= 11A3, p = 3, q = 11, \\ E &= 38B1, p = 3, q = 2, \\ E &= 21A4, p = 5, q = 3 \text{ or } q = 7, \\ E &= 24A4, p = 5, q = 3, \\ E &= 26A1, p = 5, q = 7, \\ E &= 84B1, p = 5, q = 3 \text{ or } q = 7, \\ E &= 17A1, p = 7, q = 17. \end{aligned}$$

The hypothesis that $X(E/K^{cy_c}) = 0$ in each of these cases can be verified by noting from the tables that $\mathcal{L}_E(\sigma)$ is a p -adic unit for some choice of m (cf. the proof of Corollary A.11).

We now prove Theorem A.38. By Theorem A.32 and Proposition A.37, we have $w(E, \rho_n) = -1$ for all $n \geq 1$. Hence, by (A.31) and the fact that $r_{E/K} = 0$, we have $r_{E/F_n} \geq p^n - 1$ for all $n \geq 1$; and if there is equality for a given n , $L(E, \rho_k, s)$ must have a simple zero at $s = 1$ for $1 \leq k \leq n$. Also we have

$$(A.40) \quad L(E/\mathbb{Q}(\sqrt[n]{q}), s) = L(E/\mathbb{Q}, s) \prod_{k=1}^n L(E, \rho_k, s),$$

whence assertion (v) is then clear.

To establish the upper bound for g_{E/F_n} in (A.39), we apply the theorem of Hachimori–Matsuno [25] to the Galois extension $F_n^{cy_c}/K^{cy_c}$ of degree p^n . In this extension, E has split multiplicative reduction at the unique prime of K^{cy_c} above q , which is totally ramified, and no other prime of K^{cy_c} not dividing p ramifies. Hence $X(E/F_n^{cy_c})$ is a finitely generated \mathbb{Z}_p -module of rank $p^n - 1$, and, by Matsuno’s theorem [33], it is a free \mathbb{Z}_p -module. All the remaining assertions of Theorem A.38 are now clear if we note that the restriction map from the p^∞ -Selmer group of E over F_n to the p^∞ -Selmer group of E over $F_n^{cy_c}$ has finite kernel. This completes the proof.

Theorem A.41. *Assume that $r_{E/K} = 1$, and that $X(E/K^{cy_c}) = \mathbb{Z}_p$, with trivial action of Γ_K . Take $m = q$, where q is a prime number such that E has non-split multiplicative reduction at the primes of K above q . Then we have*

$$(A.42) \quad g_{E/F_n} \leq p^n \leq r_{E/F_n} \quad \forall n \geq 1.$$

Moreover, if we assume that $g_{E/F_n} = r_{E/F_n}$ for a given integer $n \geq 1$, then

- (i) $g_{E/F_n} = g_{E/F_n^{cyc}} = p^n$,
- (ii) $\text{III}(E/F_n)[p^\infty]$ is finite,
- (iii) $\text{III}(E/F_n^{cyc})[p^\infty] = 0$,
- (iv) $L(E, \rho_k, s)$ has a simple zero at $s = 1$ for $1 \leq k \leq n$, and
- (v) $r_{E/\mathbb{Q}(\sqrt[n]{q})} = n + r_{E/\mathbb{Q}}$.

Again, under the hypotheses of Theorem A.41, we see that if we assume that $g_{E/F_n} = r_{E/F_n}$ for all $n \geq 1$, then $\text{III}(E/F_\infty)[p^\infty] = 0$, and $X(E/F_\infty)$ is dual to $E(F_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. We believe that there must be many numerical examples of curves E satisfying the hypotheses of Theorem A.41. However, the condition that $X(E/K^{cyc})$ is a free \mathbb{Z}_p -module of rank 1 is more delicate to verify. We are very grateful to C. Wuthrich for providing us with the following example. Let E be the elliptic curve 79A1 of Cremona's tables given by

$$(A.43) \quad y^2 + xy + y = x^3 + x^2 - 2x,$$

which has non-split multiplicative reduction at $q = 79$. Moreover, E has good ordinary reduction at $p = 3$, $r_{E/K} = 1$, and Wuthrich has shown that $E(K^{cyc})$ is a free \mathbb{Z}_p -module generated by $P = (0, 0)$, and that $X(E/K^{cyc})$ is a free \mathbb{Z}_3 -module of rank 1 (we omit his proof). Finally, 79 splits in K , so that E has non-split multiplicative reduction at both primes of K above 79. Thus all the hypotheses of Theorem A.38 are valid in this case with $p = 3$ and $q = 79$.

The proof of Theorem A.41 is entirely parallel to that of Theorem A.38. As $r_{E/K} = 1$, (A.31) shows that $r_{E/F_n} \geq p^n$, and that if there is equality the $L(E, \rho_k, s)$ must have a simple zero at $s = 1$ for $1 \leq k \leq n$. On the other hand, as $X(E/K^{cyc})$ is a free \mathbb{Z}_p -module of rank 1 and E has non-split multiplicative reduction at the primes of K above q , the theorems of Hachimori-Matsuno and Matsuno applied to the extension F_n^{cyc}/K_n^{cyc} of degree p^n show that $X(E/F_n^{cyc})$ is a free \mathbb{Z}_p -module of rank p^n . The assertions of Theorem A.41 now follow as in the proof of Theorem A.38.

We remark that variants of Theorems A.38 and A.41 hold for certain other choices of the integer m defining the false Tate curve extension. For example, with the same hypotheses and the same choice of a prime q of multiplicative reduction of E as before, the conclusions of Theorems A.38 and A.41 remain valid if we take $m = m'q$, where m' is a p -power free integer all of whose prime factors are either p or primes l such that E has good ordinary reduction modulo l , and the order of $\tilde{E}_l(k_v)$ is prime to p ; where \tilde{E}_l denotes the reduction of E modulo l , v denotes any prime of K above l , and k_v is the residue field of v . Similarly, if we assume that $r_{E/K} = 1$ and that $X(E/K^{cyc}) = \mathbb{Z}_p$ with trivial action of Γ_K , and now take $m = m'$ as just defined, the conclusions of Theorem A.41 again hold. The proofs are entirely analogous to those given before.

We end this appendix by establishing an analogue of the Artin formalism for Akashi series for an arbitrary compact p -adic Lie group G with a closed normal subgroup H such that $\Gamma = G/H$ is isomorphic to \mathbb{Z}_p . Let G' be an open normal subgroup of G , and put

$$H' = H \cap G', \quad \Gamma' = G'/H',$$

so that there is a natural inclusion of Γ' as an open subgroup in Γ . Let $\Delta = G/G'$, and write L for some fixed finite extension of \mathbb{Q}_p such that all absolutely

irreducible representations of Δ can be realized over L . Let \mathcal{O} denote the ring of integers of L , and write $\Lambda_{\mathcal{O}}(\Gamma)$ for the Iwasawa algebra of Γ with coefficients in \mathcal{O} . If M is a compact $\Lambda(G)$ -module, we define $M_{\mathcal{O}} = M \otimes_{\mathbb{Z}_p} \mathcal{O}$. Assume now that G has no element of order p . For $M \in \mathfrak{M}_H(G)$, we recall (see §3 of [9]) that $\text{Ak}_H(M_{\mathcal{O}})$ (resp. $\text{Ak}_{H'}(M_{\mathcal{O}})$) is defined to be the image in $Q_{\mathcal{O}}(\Gamma)^{\times}/\Lambda_{\mathcal{O}}(\Gamma)^{\times}$ (resp. $Q_{\mathcal{O}}(\Gamma')^{\times}/\Lambda_{\mathcal{O}}(\Gamma')^{\times}$) of the alternating product of the characteristic elements in $\Lambda_{\mathcal{O}}(\Gamma)$ (resp. in $\Lambda_{\mathcal{O}}(\Gamma')$) of the $H_i(H, M_{\mathcal{O}})$ (resp. $H_i(H', M_{\mathcal{O}})$) for all $i \geq 0$. Again, $\hat{\Delta}$ will denote the set of all irreducible representations of Δ , and we write n_{ρ} for the dimension of ρ in $\hat{\Delta}$.

Theorem A.44. *For each M in $\mathfrak{M}_H(G)$, we have*

$$(A.45) \quad \text{Ak}_{H'}(M_{\mathcal{O}})^{[\Gamma:\Gamma']} = \prod_{\rho \in \hat{\Delta}} \text{Ak}_H(\text{tw}_{\rho}(M_{\mathcal{O}}))^{n_{\rho}}.$$

Before proving Theorem A.44, we note that Lemma A.18 follows on applying it to the two Galois extensions F_n/\mathbb{Q} and F'_n/\mathbb{Q} . Since the set of all irreducible representations of $\text{Gal}(F_n/\mathbb{Q})$ consists of ρ_n together with all irreducible representations of $\text{Gal}(F'_n/\mathbb{Q})$, we conclude that, for a suitable choice of L , we have

$$\text{Ak}_{H_n}(M_{\mathcal{O}})^{p^{n-1}} = \text{Ak}_{H'_n}(M_{\mathcal{O}})^{p^{n-1}} \text{Ak}_H(\text{tw}_{\rho_n}(M_{\mathcal{O}}))^{p^{n-1}(p-1)}.$$

We then obtain Lemma A.18 on extracting p^{n-1} -th roots of both sides, taking the norm map from $\mathbb{Q}_{\mathcal{O}}(\Gamma)^{\times}$ to $\mathbb{Q}(\Gamma)^{\times}$, and recalling that $\Lambda(\Gamma)$ is a unique factorization domain.

We end this Appendix by proving Theorem A.44. Let $K_0(\mathfrak{M}_H(G))$ be the Grothendieck group of the category $\mathfrak{M}_H(G)$, and write $[M]$ for the class of a module in this group. As usual, we define

$$\text{Ind}_{G'}^G(M_{\mathcal{O}}) = \Lambda(G) \otimes_{\Lambda(G')} M_{\mathcal{O}} = R \otimes_{\mathcal{O}} M_{\mathcal{O}},$$

where $R = \mathcal{O}[\Delta]$ is the \mathcal{O} -group ring of Δ . Let \mathcal{C}_R be the category of all finitely generated R -modules, and let $K_0(\mathcal{C}_R)$ be the Grothendieck group in this category. For each irreducible representation ρ of Δ , let L_{ρ} be a free \mathcal{O} -module of rank n_{ρ} realizing ρ , and put

$$W = \bigoplus_{\rho \in \hat{\Delta}} L_{\rho}^{n_{\rho}}.$$

Since $W \otimes_{\mathcal{O}} L$ is isomorphic to $R \otimes_{\mathcal{O}} L$ as $L[\Delta]$ -modules, a theorem of Swan ([46], Theorem 3) implies that

$$(A.46) \quad [W] = [R] \quad \text{in} \quad K_0(\mathcal{C}_R).$$

But Swan ([46], Theorem 1.2) has shown that the natural inclusion of the category \mathcal{D}_R of all finitely generated R -modules which are \mathcal{O} -free in \mathcal{C}_R induces an isomorphism from $K_0(\mathcal{D}_R)$ to $K_0(\mathcal{C}_R)$. As W and R are \mathcal{O} -free, it follows from (A.46) that there exist a finite number of exact sequences

$$(A.47) \quad 0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0 \quad (i = 1, \dots, n)$$

in \mathcal{D}_R such that

$$R - W = \sum_{i=1}^n (B_i - A_i - C_i)$$

in the free abelian group on the isomorphism classes of \mathcal{D}_R . Since the modules in the exact sequence (A.47) are \mathcal{O} -free, it remains exact when we tensor it over \mathcal{O} with any \mathcal{O} -module. It follows that, for each M in $\mathfrak{M}_H(G)$, we must have

$$(A.48) \quad [R \otimes_{\mathcal{O}} M_{\mathcal{O}}] = [W \otimes_{\mathcal{O}} M_{\mathcal{O}}]$$

in $K_0(\mathfrak{M}_H(G))$. Now $\text{Ak}_H(\cdot)$ is well defined on $K_0(\mathfrak{M}_H(G))$. By Shapiro's lemma

$$H_i(H, R \otimes M_{\mathcal{O}}) = \text{Tor}^{\Lambda(G)}(\Lambda(G/H), \text{Ind}_{G'}^G(M_{\mathcal{O}})) = H_i(H', M_{\mathcal{O}})^{[\Gamma, \Gamma']},$$

and so $\text{Ak}_H(R \otimes_{\mathcal{O}} M_{\mathcal{O}})$ is the left hand side of (A.45). But $\text{Ak}_H(W \otimes_{\mathcal{O}} M_{\mathcal{O}})$ is, by definition, the right hand side of (A.48). Hence Theorem A.44 follows from (A.48).

Finally, we remark without proof (see Theorem 6.8 of [1] for the case of p -primary modules in $\mathfrak{M}_H(G)$) that (A.48) can also be used to establish the Artin formalism for the characteristic elements of any module in $\mathfrak{M}_H(G)$.

APPENDIX B. TABLES

Take an odd prime p and an elliptic curve E/\mathbb{Q} with good ordinary reduction at p . Define $K = \mathbb{Q}(\mu_p)$ and let σ be the regular representation of $\text{Gal}(K/\mathbb{Q})$. Assume that E/K has Mordell-Weil rank 0.

In each table we list the prime p , the name (as in Cremona's tables [14]) and the equation of E , the local polynomial $P_p(E/\mathbb{Q}, T)$ of E at p (see 2.7), the conductors $N(E)$ and $N(E, \sigma)$ (see 2.4) and the orders of the torsion groups $E(\mathbb{Q})$ and $E(K)$. Next, we compute numerically the Birch-Swinnerton-Dyer quotients

$$L^*(E/\mathbb{Q}) = \frac{L(E, 1)}{\Omega_+(E)}, \quad L^*(E/K) = \left| \frac{L(E, \sigma, 1) \sqrt{\Delta_K}}{(2\Omega_+(E)\Omega_-(E))^{(p-1)/2}} \right|$$

and the analytic orders of the Tate-Shafarevich groups $\text{III}(E/\mathbb{Q})$ and $\text{III}(E/K)$. We also list the reduction types of E/\mathbb{Q}_l and E/K_v at bad primes $l \in \mathbb{Q}$ and primes $v|l$ of K .

Finally, for varying m that defines the extension $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})/\mathbb{Q}$, we tabulate the following data:

Table columns:

m	p -th power free integer, $m > 1$. It specifies the false Tate curve extension $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})/\mathbb{Q}$. (When m and m' define the same field, e.g. 20 and 50 for $p = 3$, we take the smaller one.)
$N(\rho)$	conductor of ρ (the representation of $\text{Gal}(\mathbb{Q}(\mu_p, \sqrt[p]{m})/\mathbb{Q})$ in §1).
$N(E, \rho)$	conductor of the twist of E by ρ .
L^*	$\frac{L^*(E/\mathbb{Q}(\sqrt[p]{m}))}{L^*(E/\mathbb{Q})} = \left \frac{L(E, \rho, 1) \sqrt{\Delta_{\mathbb{Q}(\sqrt[p]{m})}}}{(2\Omega_+(E)\Omega_-(E))^{(p-1)/2}} \right .$
III	Analytic order of $\text{III}(E/\mathbb{Q}(\sqrt[p]{m}))$. We write “-” if the analytic rank of $E/\mathbb{Q}(\sqrt[p]{m})$ is non-zero.
$\mathcal{L}_E(\sigma)$	The quantity in (1.3). Conjecturally, this is the value of the non-abelian p -adic L -function for F_∞/\mathbb{Q} at σ .
$\mathcal{L}_E(\rho)$	The quantity in (1.3). Conjecturally, this is the value of the non-abelian p -adic L -function for F_∞/\mathbb{Q} at ρ .

We do not include m for which $L(E, \rho, s)$ has sign -1 in the functional equation.

In the table below we give a list of our tables specifying whether the curve has one of the contributions in (1), (2) and (3) of §6.13, and whether the curve is semistable. We also list what we can say concerning the μ -invariant $\mu_{E/K}$. We write “0” if we can prove that $\text{Sel}_{p^\infty}(E/K)$ is finite and $\mu_{E/K} = 0$. We write “0 $^\Gamma$ ” if we can deduce this from the Birch–Swinnerton-Dyer conjecture, and write “ Γ ” otherwise.

	Table	$E(K)[p]$	$E(\mathbb{F}_p)[p]$	$p \mid \prod_v c_v$	semistable	$\mu_{E/K}$	page
p=3	3-11A3	.	.	.	×	0	47
	3-20A3	×	×	×	.	Γ	48
	3-26A1	×	×	×	×	Γ	49
	3-35A1	×	×	×	×	Γ	50
	3-38B1	.	.	.	×	0	50
	3-50B1	0 $^\Gamma$	49
	3-56B1	0	51
	3-77C1	.	.	.	×	0	52
	3-80B2	0 $^\Gamma$	52
	3-92A1	×	×	×	.	0 $^\Gamma$	54
	3-116C2	.	.	×	.	0 $^\Gamma$	54
	3-128B2	.	×	.	.	Γ	51
	3-152B1	.	×	.	.	Γ	55
	3-176B1	.	×	.	.	Γ	55
	3-224B1	0	48
	3-260A1	.	.	×	.	Γ	53
	3-272C1	0	54
3-275B1	.	×	.	.	Γ	53	
3-395B1	.	.	×	×	Γ	55	
3-800E1	.	×	×	.	Γ	56	
p=5	5-11A3	×	×	.	×	0	56
	5-19A3	.	.	.	×	0 $^\Gamma$	56
	5-21A4	.	.	.	×	0 $^\Gamma$	57
	5-24A4	0 $^\Gamma$	57
	5-44A1	0 $^\Gamma$	58
	5-56A1	0 $^\Gamma$	57
	5-56B1	.	×	.	.	Γ	57
	5-84B1	0 $^\Gamma$	58
p=7	7-17A1	.	.	.	×	0 $^\Gamma$	58
	7-19A3	.	.	.	×	0 $^\Gamma$	58

$E = 11A3$ $y^2 + y = x^3 - x^2$ **Table 3-11A3**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 + T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 11$ $L^*(E/\mathbb{Q}) = 5^{-2}$ $|E(\mathbb{Q})| = 5$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^2 \cdot 11^2$ $L^*(E/K) = 5^{-2}$ $|E(K)| = 5$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_{11}$ I_1 split, $c_{11} = 1$ $/K_v$ ($v|11$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^4 3^6 11^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
3	3^5	$3^{10} 11^2$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^6)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
5	$3^3 5^2$	$3^6 5^4 11^2$	2^2	2^2	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
6	$2^2 3^5$	$2^4 3^{10} 11^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
7	$3^3 7^2$	$3^6 7^4 11^2$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
10	$2^2 3 \cdot 5^2$	$2^4 3^2 5^4 11^2$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
12	$2^2 3^5$	$2^4 3^{10} 11^2$	2^2	2^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + O(3^3)$
13	$3^3 13^2$	$3^6 11^2 13^4$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^4 3^6 7^4 11^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
15	$3^5 5^2$	$3^{10} 5^4 11^2$	1	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
17	$3 \cdot 17^2$	$3^2 11^2 17^4$	1	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
19	$3 \cdot 19^2$	$3^2 11^2 19^4$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
20	$2^2 3^3 5^2$	$2^4 3^6 5^4 11^2$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + O(3^3)$
21	$3^5 7^2$	$3^{10} 7^4 11^2$	2^2	2^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
23	$3^3 23^2$	$3^6 11^2 23^4$	2^4	4^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + O(3^3)$
26	$2^2 3 \cdot 13^2$	$2^4 3^2 11^2 13^4$	2^2	2^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$
28	$2^2 3 \cdot 7^2$	$2^4 3^2 7^4 11^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
29	$3^3 29^2$	$3^6 11^2 29^4$	3^2	3^2	$2 \cdot 3^2 + O(3^5)$	$1 \cdot 3^2 + 1 \cdot 3^4 + O(3^5)$
30	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
31	$3^3 31^2$	$3^6 11^2 31^4$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$
34	$2^2 3^3 17^2$	$2^4 3^6 11^2 17^4$	5^2	5^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + O(3^6)$
35	$3 \cdot 5^2 7^2$	$3^2 5^4 7^4 11^2$	1	1^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$
37	$3 \cdot 37^2$	$3^2 11^2 37^4$	2^2	2^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^5)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
45	$3^5 5^2$	$3^{10} 5^4 11^2$	1	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
46	$2^2 3 \cdot 23^2$	$2^4 3^2 11^2 23^4$	2^2	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
52	$2^2 3^3 13^2$	$2^4 3^6 11^2 13^4$	2^4	4^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
53	$3 \cdot 53^2$	$3^2 11^2 53^4$	3^2	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^9)$	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
60	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	11^2	11^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + O(3^3)$
62	$2^2 3 \cdot 31^2$	$2^4 3^2 11^2 31^4$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
63	$3^5 7^2$	$3^{10} 7^4 11^2$	2^4	4^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$
68	$2^2 3^3 17^2$	$2^4 3^6 11^2 17^4$	2^4	4^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^5)$
71	$3 \cdot 71^2$	$3^2 11^2 71^4$	0	—	$1 \cdot 3^2 + O(3^5)$	0
73	$3 \cdot 73^2$	$3^2 11^2 73^4$	2^2	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
82	$2^2 3 \cdot 41^2$	$2^4 3^2 11^2 41^4$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + O(3^5)$
89	$3 \cdot 89^2$	$3^2 11^2 89^4$	3^2	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$
90	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	2^2	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^9)$
91	$3 \cdot 7^2 13^2$	$3^2 7^4 11^2 13^4$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
116	$2^2 3 \cdot 29^2$	$2^4 3^2 11^2 29^4$	3^2	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^8)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
150	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
172	$2^2 3 \cdot 43^2$	$2^4 3^2 11^2 43^4$	1	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
188	$2^2 3 \cdot 47^2$	$2^4 3^2 11^2 47^4$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
325	$3 \cdot 5^2 13^2$	$3^2 5^4 11^2 13^4$	5^2	5^2	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^5)$
350	$2^2 3 \cdot 5^2 7^2$	$2^4 3^2 5^4 7^4 11^2$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$

$$E = 20A3 \quad y^2 = x^3 + x^2 - 36x - 140$$

Table 3-20A3

$$p = 3 \quad P_3(E/\mathbb{Q}, T) = 1 + 2T + 3T^2$$

/ \mathbb{Q}	$N(E) = 2^2 \cdot 5$	$L^*(E/\mathbb{Q}) = 2^{-1}$	$ E(\mathbb{Q}) = 2$	$ \text{III}(E/\mathbb{Q}) = 1^2$
/ K	$N(E, \sigma) = 2^4 \cdot 3^2 \cdot 5^2$	$L^*(E/K) = 2^{-1}$	$ E(K) = 6$	$ \text{III}(E/K) = 1^2$
/ \mathbb{Q}_2	$\text{IV}^*, c_2 = 1$	$/K_v (v 2)$	$\text{IV}^*, c_v = 3$	
/ \mathbb{Q}_5	I_6 non-split, $c_5 = 2$	$/K_v (v 5)$	I_6 split, $c_v = 6$	

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^4 3^{10} 5^2$	$2 \cdot 3^2$	1^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + O(3^7)$
7	$3^3 7^2$	$2^4 3^6 5^2 7^4$	0	—	$2 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	0
10	$2^2 3 \cdot 5^2$	$2^2 3^2 5^4$	2^2	1^2	$1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
11	$3^3 11^2$	$2^4 3^6 5^2 11^4$	0	—	$2 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	0
13	$3^3 13^2$	$2^4 3^6 5^2 13^4$	$2 \cdot 3^4$	3^2	$2 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$
17	$3 \cdot 17^2$	$2^4 3^2 5^2 17^4$	$2 \cdot 3^2$	1^2	$1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$
19	$3 \cdot 19^2$	$2^4 3^2 5^2 19^4$	$2 \cdot 3^2$	1^2	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$	$2 \cdot 3^4 + 1 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$
21	$3^5 7^2$	$2^4 3^{10} 5^2 7^4$	$2 \cdot 3^4$	3^2	$2 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	$2 \cdot 3^4 + 2 \cdot 3^5 + O(3^7)$
23	$3^3 23^2$	$2^4 3^6 5^2 23^4$	0	—	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	0
30	$2^2 3^5 5^2$	$2^2 3^{10} 5^4$	$2^2 3^2$	3^2	$1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$
37	$3 \cdot 37^2$	$2^4 3^2 5^2 37^4$	0	—	$2 \cdot 3^6 + O(3^{10})$	0
53	$3 \cdot 53^2$	$2^4 3^2 5^2 53^4$	$2 \cdot 3^2$	1^2	$1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	$1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$
63	$3^5 7^2$	$2^4 3^{10} 5^2 7^4$	0	—	$2 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	0
70	$2^2 3^5 5^2 7^2$	$2^2 3^6 5^4 7^4$	0	—	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^9)$	0
71	$3 \cdot 71^2$	$2^4 3^2 5^2 71^4$	$2^3 3^2$	2^2	$1 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	$1 \cdot 3^4 + 2 \cdot 3^5 + O(3^7)$
73	$3 \cdot 73^2$	$2^4 3^2 5^2 73^4$	$2 \cdot 3^4$	3^2	$2 \cdot 3^6 + 2 \cdot 3^7 + 2 \cdot 3^8 + O(3^9)$	$2 \cdot 3^6 + 1 \cdot 3^7 + 1 \cdot 3^8 + O(3^9)$
90	$2^2 3^5 5^2$	$2^2 3^{10} 5^4$	$2^4 3^2$	6^2	$1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 2 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$
110	$2^2 3^5 5^2 11^2$	$2^2 3^6 5^4 11^4$	$2^2 3^4$	9^2	$1 \cdot 3^5 + 2 \cdot 3^6 + O(3^9)$	$1 \cdot 3^5 + 2 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$
130	$2^2 3^5 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2^2 3^4$	9^2	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$	$2 \cdot 3^5 + 2 \cdot 3^6 + 2 \cdot 3^7 + O(3^8)$
150	$2^2 3^5 5^2$	$2^2 3^{10} 5^4$	$2^2 3^2$	3^2	$1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$
170	$2^2 3 \cdot 5^2 17^2$	$2^2 3^2 5^4 17^4$	$2^2 3^2$	3^2	$2 \cdot 3^5 + 1 \cdot 3^7 + O(3^8)$	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$
190	$2^2 3 \cdot 5^2 19^2$	$2^2 3^2 5^4 19^4$	$2^2 3^2$	3^2	$1 \cdot 3^5 + 1 \cdot 3^6 + O(3^9)$	$2 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^{10})$
350	$2^2 3 \cdot 5^2 7^2$	$2^2 3^2 5^4 7^4$	$2^2 3^2$	3^2	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^9)$	$2 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$
370	$2^2 3 \cdot 5^2 37^2$	$2^2 3^2 5^4 37^4$	$2^2 3^4$	9^2	$1 \cdot 3^7 + 2 \cdot 3^9 + O(3^{10})$	$2 \cdot 3^7 + 1 \cdot 3^8 + 1 \cdot 3^9 + O(3^{11})$
490	$2^2 3^5 5^2 7^2$	$2^2 3^6 5^4 7^4$	$2^2 3^4$	9^2	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^9)$	$2 \cdot 3^5 + O(3^{10})$
539	$3 \cdot 7^2 11^2$	$2^4 3^2 5^2 7^4 11^4$	$2 \cdot 3^4$	3^2	$2 \cdot 3^6 + 1 \cdot 3^7 + 2 \cdot 3^8 + O(3^9)$	$1 \cdot 3^6 + O(3^{10})$
550	$2^2 3 \cdot 5^2 11^2$	$2^2 3^2 5^4 11^4$	$2^2 3^2$	3^2	$1 \cdot 3^5 + 2 \cdot 3^6 + O(3^9)$	$1 \cdot 3^5 + 2 \cdot 3^6 + 2 \cdot 3^7 + O(3^9)$
650	$2^2 3^5 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2^4 3^4$	18^2	$1 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$	$2 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^8)$
1450	$2^2 3 \cdot 5^2 29^2$	$2^2 3^2 5^4 29^4$	$2^4 3^2$	6^2	$2 \cdot 3^6 + 1 \cdot 3^7 + O(3^{10})$	$1 \cdot 3^5 + 1 \cdot 3^7 + O(3^{10})$
2150	$2^2 3 \cdot 5^2 43^2$	$2^2 3^2 5^4 43^4$	$2^2 3^2 5^2$	15^2	$1 \cdot 3^9 + 1 \cdot 3^{10} + 2 \cdot 3^{11} + O(3^{12})$	$2 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^8)$
2350	$2^2 3 \cdot 5^2 47^2$	$2^2 3^2 5^4 47^4$	$2^2 3^4$	9^2	$1 \cdot 3^7 + 1 \cdot 3^9 + O(3^{10})$	$1 \cdot 3^7 + 2 \cdot 3^8 + O(3^{10})$

$$E = 224B1 \quad y^2 = x^3 - x^2 + 2x$$

Table 3-224B1

$$p = 3 \quad P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$$

/ \mathbb{Q}	$N(E) = 2^5 \cdot 7$	$L^*(E/\mathbb{Q}) = 2^{-1}$	$ E(\mathbb{Q}) = 2$	$ \text{III}(E/\mathbb{Q}) = 1^2$
/ K	$N(E, \sigma) = 2^{10} \cdot 3^2 \cdot 7^2$	$L^*(E/K) = 2^{-1}$	$ E(K) = 2$	$ \text{III}(E/K) = 1^2$
/ \mathbb{Q}_2	$\text{III}, c_2 = 2$	$/K_v (v 2)$	$\text{III}, c_v = 2$	
/ \mathbb{Q}_7	I_1 split, $c_7 = 1$	$/K_v (v 7)$	I_1 split, $c_v = 1$	

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^{10} 3^6 7^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$
6	$2^2 3^5$	$2^{10} 3^{10} 7^2$	2^2	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
10	$2^2 3 \cdot 5^2$	$2^{10} 3^2 5^4 7^2$	3^2	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
12	$2^2 3^5$	$2^{10} 3^{10} 7^2$	1	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^{10} 3^6 7^4$	$2^2 3$	2^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
20	$2^2 3^3 5^2$	$2^{10} 3^6 5^4 7^2$	$2^2 3^2$	6^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
26	$2^2 3 \cdot 13^2$	$2^{10} 3^2 7^2 13^4$	3^2	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + 1 \cdot 3^6 + O(3^8)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^7)$
28	$2^2 3 \cdot 7^2$	$2^{10} 3^2 7^4$	3	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 1 \cdot 3^3 + O(3^4)$
44	$2^2 3 \cdot 11^2$	$2^{10} 3^2 7^2 11^4$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^5)$
46	$2^2 3 \cdot 23^2$	$2^{10} 3^2 7^2 23^4$	2^6	8^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^5)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^9)$
350	$2^2 3 \cdot 5^2 7^2$	$2^{10} 3^2 5^4 7^4$	3^3	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + O(3^8)$	$2 \cdot 3^3 + O(3^6)$

$E = 26A1$ $y^2 + xy + y = x^3 - 5x - 8$ **Table 3-26A1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2 \cdot 13$ $L^*(E/\mathbb{Q}) = 3^{-1}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^2 \cdot 3^2 \cdot 13^2$ $L^*(E/K) = 3^{-1}$ $|E(K)| = 9$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ I_3 non-split, $c_2 = 1$ $/K_v$ ($v|2$) I_3 split, $c_v = 3$
 $/\mathbb{Q}_{13}$ I_3 split, $c_{13} = 3$ $/K_v$ ($v|13$) I_3 split, $c_v = 3$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^2 3^{10} 13^2$	$2^2 3$	2^2	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	$1 \cdot 3^1 + 2 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$
5	$3^3 5^2$	$2^2 3^6 5^4 13^2$	3^3	1^2	$1 \cdot 3^4 + 2 \cdot 3^5 + O(3^8)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$
7	$3^3 7^2$	$2^2 3^6 7^4 13^2$	3^3	3^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$1 \cdot 3^3 + O(3^8)$
11	$3^3 11^2$	$2^2 3^6 11^4 13^2$	3^3	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^7)$
13	$3^3 13^2$	$2^2 3^6 13^4$	3^2	1^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
15	$3^5 5^2$	$2^2 3^{10} 5^4 13^2$	0	—	$1 \cdot 3^4 + 2 \cdot 3^5 + O(3^8)$	0
17	$3 \cdot 17^2$	$2^2 3^2 13^2 17^4$	$2^2 3$	2^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$
19	$3 \cdot 19^2$	$2^2 3^2 13^2 19^4$	3^3	3^2	$1 \cdot 3^5 + 2 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$	$1 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^9)$
21	$3^5 7^2$	$2^2 3^{10} 7^4 13^2$	0	—	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	0
25	$3 \cdot 5^2 7^2$	$2^2 3^2 5^4 7^4 13^2$	3^3	3^2	$1 \cdot 3^8 + 1 \cdot 3^9 + O(3^{11})$	$2 \cdot 3^5 + 1 \cdot 3^6 + O(3^9)$
37	$3 \cdot 37^2$	$2^2 3^2 13^2 37^4$	3^3	3^2	$1 \cdot 3^5 + 1 \cdot 3^7 + O(3^9)$	$1 \cdot 3^5 + 1 \cdot 3^6 + O(3^8)$
39	$3^5 13^2$	$2^2 3^{10} 13^4$	3^4	3^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^4 + O(3^8)$
45	$3^5 5^2$	$2^2 3^{10} 5^4 13^2$	3^3	3^2	$1 \cdot 3^4 + 2 \cdot 3^5 + O(3^8)$	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$
53	$3 \cdot 53^2$	$2^2 3^2 13^2 53^4$	$2^2 3^3$	2^2	$1 \cdot 3^7 + 2 \cdot 3^8 + 2 \cdot 3^9 + O(3^{10})$	$2 \cdot 3^5 + O(3^8)$
55	$3 \cdot 5^2 11^2$	$2^2 3^2 5^4 11^4 13^2$	0	—	$1 \cdot 3^7 + 2 \cdot 3^8 + 2 \cdot 3^9 + O(3^{11})$	0
63	$3^5 7^2$	$2^2 3^{10} 7^4 13^2$	3^3	3^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$1 \cdot 3^3 + 2 \cdot 3^4 + 2 \cdot 3^5 + O(3^7)$
65	$3^3 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2^2 3^4$	6^2	$1 \cdot 3^6 + 2 \cdot 3^7 + 1 \cdot 3^8 + O(3^{11})$	$2 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$
91	$3 \cdot 7^2 13^2$	$2^2 3^2 7^4 13^4$	3^4	3^2	$1 \cdot 3^7 + O(3^{10})$	$1 \cdot 3^6 + 1 \cdot 3^7 + O(3^9)$
117	$3^5 13^2$	$2^2 3^{10} 13^4$	3^2	1^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + O(3^6)$
143	$3 \cdot 11^2 13^2$	$2^2 3^2 11^4 13^4$	$2^2 3^4$	6^2	$1 \cdot 3^6 + 1 \cdot 3^7 + 1 \cdot 3^8 + O(3^9)$	$2 \cdot 3^6 + 2 \cdot 3^8 + O(3^9)$
325	$3 \cdot 5^2 13^2$	$2^2 3^2 5^4 13^4$	0	—	$1 \cdot 3^6 + 2 \cdot 3^7 + 1 \cdot 3^8 + O(3^{11})$	0

$E = 50B1$ $y^2 + xy + y = x^3 + x^2 - 3x + 1$ **Table 3-50B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 + T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2 \cdot 5^2$ $L^*(E/\mathbb{Q}) = 5^{-1}$ $|E(\mathbb{Q})| = 5$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^2 \cdot 3^2 \cdot 5^4$ $L^*(E/K) = 5^{-1}$ $|E(K)| = 5$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ I_5 split, $c_2 = 5$ $/K_v$ ($v|2$) I_5 split, $c_v = 5$
 $/\mathbb{Q}_5$ II , $c_5 = 1$ $/K_v$ ($v|5$) II , $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^2 3^{10} 5^4$	5	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$
5	$3^3 5^2$	$2^2 3^6 5^4$	2.5	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
7	$3^3 7^2$	$2^2 3^6 5^4 7^4$	5	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
11	$3^3 11^2$	$2^2 3^6 5^4 11^4$	$3^2 5$	3^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
13	$3^3 13^2$	$2^2 3^6 5^4 13^4$	5	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$
15	$3^5 5^2$	$2^2 3^{10} 5^4$	2.5	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + O(3^3)$
17	$3 \cdot 17^2$	$2^2 3^2 5^4 17^4$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	0
19	$3 \cdot 19^2$	$2^2 3^2 5^4 19^4$	0	—	$1 \cdot 3^2 + O(3^5)$	0
35	$3 \cdot 5^2 7^2$	$2^2 3^2 5^4 7^4$	2.5	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
37	$3 \cdot 37^2$	$2^2 3^2 5^4 37^4$	5	1^2	$1 \cdot 3^0 + O(3^4)$	$1 \cdot 3^0 + O(3^3)$
45	$3^5 5^2$	$2^2 3^{10} 5^4$	$2^3 5$	2^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
55	$3 \cdot 5^2 11^2$	$2^2 3^2 5^4 11^4$	0	—	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	0
65	$3^3 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2^3 5$	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
145	$3 \cdot 5^2 29^2$	$2^2 3^2 5^4 29^4$	$2 \cdot 3^2 5$	3^2	$1 \cdot 3^2 + 1 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + O(3^5)$
175	$3^3 5^2 7^2$	$2^2 3^6 5^4 7^4$	2.5	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
215	$3 \cdot 5^2 43^2$	$2^2 3^2 5^4 43^4$	2.5	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
235	$3 \cdot 5^2 47^2$	$2^2 3^2 5^4 47^4$	$2^3 3^2 5$	6^2	$2 \cdot 3^3 + 2 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
275	$3^3 5^2 11^2$	$2^2 3^6 5^4 11^4$	$2 \cdot 3^2 5$	3^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$2 \cdot 3^2 + 1 \cdot 3^4 + O(3^5)$
325	$3 \cdot 5^2 13^2$	$2^2 3^2 5^4 13^4$	2.5	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$
575	$3 \cdot 5^2 23^2$	$2^2 3^2 5^4 23^4$	0	—	$2 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$	0
775	$3 \cdot 5^2 31^2$	$2^2 3^2 5^4 31^4$	$2 \cdot 3^2 5$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + O(3^5)$
1025	$3 \cdot 5^2 41^2$	$2^2 3^2 5^4 41^4$	$2 \cdot 3^2 5$	3^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^7)$	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^7)$

$E = 35A1$ $y^2 + y = x^3 + x^2 + 9x + 1$ **Table 3-35A1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 5 \cdot 7$ $L^*(E/\mathbb{Q}) = 3^{-1}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^2 \cdot 5^2 \cdot 7^2$ $L^*(E/K) = 3^{-1}$ $|E(K)| = 9$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_5$ I_3 non-split, $c_5 = 1$ $/K_v$ ($v|5$) I_3 split, $c_v = 3$
 $/\mathbb{Q}_7$ I_3 split, $c_7 = 3$ $/K_v$ ($v|7$) I_3 split, $c_v = 3$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^4 3^6 5^2 7^2$	0	—	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	0
3	3^5	$3^{10} 5^2 7^2$	3	1^2	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
6	$2^2 3^5$	$2^4 3^{10} 5^2 7^2$	3^3	1^2	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 2 \cdot 3^5 + O(3^6)$
7	$3^3 7^2$	$3^6 5^2 7^4$	3^2	1^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + O(3^7)$
11	$3^3 11^2$	$3^6 5^2 7^2 11^4$	0	—	$2 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	0
12	$2^2 3^5$	$2^4 3^{10} 5^2 7^2$	3^3	3^2	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 2 \cdot 3^5 + O(3^6)$
13	$3^3 13^2$	$3^6 5^2 7^2 13^4$	0	—	$1 \cdot 3^5 + 2 \cdot 3^6 + 2 \cdot 3^7 + O(3^{10})$	0
14	$2^2 3^3 7^2$	$2^4 3^6 5^2 7^4$	3^4	3^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$2 \cdot 3^4 + 1 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$
17	$3 \cdot 17^2$	$3^2 5^2 7^2 17^4$	3	1^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$
19	$3 \cdot 19^2$	$3^2 5^2 7^2 19^4$	0	—	$1 \cdot 3^5 + 2 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$	0
21	$3^5 7^2$	$3^{10} 5^2 7^4$	$2 \cdot 3^2$	2^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^4 + O(3^7)$
26	$2^2 3 \cdot 13^2$	$2^4 3^2 5^2 7^2 13^4$	3^3	3^2	$1 \cdot 3^7 + 1 \cdot 3^8 + 1 \cdot 3^9 + O(3^{10})$	$2 \cdot 3^5 + 1 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$
28	$2^2 3 \cdot 7^2$	$2^4 3^2 5^2 7^4$	3^2	1^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$
37	$3 \cdot 37^2$	$3^2 5^2 7^2 37^4$	3^3	3^2	$1 \cdot 3^5 + O(3^8)$	$1 \cdot 3^5 + 1 \cdot 3^6 + O(3^8)$
42	$2^2 3^5 7^2$	$2^4 3^{10} 5^2 7^4$	0	—	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	0
44	$2^2 3 \cdot 11^2$	$2^4 3^2 5^2 7^2 11^4$	3^3	3^2	$2 \cdot 3^6 + 2 \cdot 3^7 + O(3^{12})$	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^9)$
46	$2^2 3 \cdot 23^2$	$2^4 3^2 5^2 7^2 23^4$	3^3	3^2	$2 \cdot 3^6 + 2 \cdot 3^7 + 2 \cdot 3^8 + O(3^{10})$	$1 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^8)$
53	$3 \cdot 53^2$	$3^2 5^2 7^2 53^4$	$2 \cdot 3$	2^2	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^7)$	$2 \cdot 3^3 + O(3^6)$
63	$3^5 7^2$	$3^{10} 5^2 7^4$	0	—	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	0
84	$2^2 3^5 7^2$	$2^4 3^{10} 5^2 7^4$	3^4	3^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$
91	$3 \cdot 7^2 13^2$	$3^2 5^2 7^4 13^4$	0	—	$1 \cdot 3^7 + 2 \cdot 3^8 + O(3^{12})$	0
116	$2^2 3 \cdot 29^2$	$2^4 3^2 5^2 7^2 29^4$	3^3	3^2	$2 \cdot 3^7 + 1 \cdot 3^8 + 1 \cdot 3^9 + O(3^{10})$	$1 \cdot 3^5 + 1 \cdot 3^7 + O(3^{10})$
126	$2^2 3^5 7^2$	$2^4 3^{10} 5^2 7^4$	0	—	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	0
154	$2^2 3 \cdot 7^2 11^2$	$2^4 3^2 5^2 7^4 11^4$	3^4	3^2	$2 \cdot 3^8 + 2 \cdot 3^9 + 2 \cdot 3^{10} + O(3^{11})$	$1 \cdot 3^6 + 1 \cdot 3^7 + 2 \cdot 3^8 + O(3^9)$
252	$2^2 3^5 7^2$	$2^4 3^{10} 5^2 7^4$	3^4	3^2	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$
539	$3 \cdot 7^2 11^2$	$3^2 5^2 7^4 11^4$	0	—	$2 \cdot 3^6 + 1 \cdot 3^7 + 2 \cdot 3^8 + O(3^9)$	0

$E = 38B1$ $y^2 + xy + y = x^3 + x^2 + 1$ **Table 3-38B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 + T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2 \cdot 19$ $L^*(E/\mathbb{Q}) = 5^{-1}$ $|E(\mathbb{Q})| = 5$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^2 \cdot 3^2 \cdot 19^2$ $L^*(E/K) = 5^{-1}$ $|E(K)| = 5$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ I_5 split, $c_2 = 5$ $/K_v$ ($v|2$) I_5 split, $c_v = 5$
 $/\mathbb{Q}_{19}$ I_1 non-split, $c_{19} = 1$ $/K_v$ ($v|19$) I_1 non-split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^2 3^{10} 19^2$	5	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$
5	$3^3 5^2$	$2^2 3^6 5^4 19^2$	5	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
7	$3^3 7^2$	$2^2 3^6 7^4 19^2$	$2^2 5$	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
11	$3^3 11^2$	$2^2 3^6 11^4 19^2$	5	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
13	$3^3 13^2$	$2^2 3^6 13^4 19^2$	$3^2 5$	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^7)$
15	$3^5 5^2$	$2^2 3^{10} 5^4 19^2$	$2^2 5$	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + O(3^3)$
17	$3 \cdot 17^2$	$2^2 3^2 17^4 19^2$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	0
19	$3 \cdot 19^2$	$2^2 3^2 19^4$	5	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
35	$3 \cdot 5^2 7^2$	$2^2 3^2 5^4 7^4 19^2$	5	1^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
37	$3 \cdot 37^2$	$2^2 3^2 19^2 37^4$	5	1^2	$1 \cdot 3^0 + O(3^4)$	$1 \cdot 3^0 + O(3^3)$
45	$3^5 5^2$	$2^2 3^{10} 5^4 19^2$	5	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
53	$3 \cdot 53^2$	$2^2 3^2 19^2 53^4$	$2^2 5$	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
55	$3 \cdot 5^2 11^2$	$2^2 3^2 5^4 11^4 19^2$	5	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
57	$3^5 19^2$	$2^2 3^{10} 19^4$	$2^2 5$	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
171	$3^5 19^2$	$2^2 3^{10} 19^4$	$2^2 5$	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$

$E = 56B1$ $y^2 = x^3 - x^2 - 4$ **Table 3-56B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^3 \cdot 7$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^6 \cdot 3^2 \cdot 7^2$ $L^*(E/K) = 2^{-1}$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ $\text{III}^*, c_2 = 2$ $/K_v (v|2)$ $\text{III}^*, c_v = 2$
 $/\mathbb{Q}_7$ I_1 split, $c_7 = 1$ $/K_v (v|7)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^6 3^6 7^2$	2^2	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
6	$2^2 3^5$	$2^6 3^{10} 7^2$	2^4	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
10	$2^2 3 \cdot 5^2$	$2^6 3^2 5^4 7^2$	2^2	1^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + O(3^4)$
12	$2^2 3^5$	$2^6 3^{10} 7^2$	2^2	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^6 3^6 7^4$	0	—	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	0
20	$2^2 3^5 5^2$	$2^6 3^5 5^4 7^2$	2^4	2^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
22	$2^2 3^3 11^2$	$2^6 3^6 7^2 11^4$	0	—	$1 \cdot 3^2 + O(3^6)$	0
26	$2^2 3 \cdot 13^2$	$2^6 3^2 7^2 13^4$	2^2	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$
28	$2^2 3 \cdot 7^2$	$2^6 3^2 7^4$	$2^2 3$	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
30	$2^2 3^5 5^2$	$2^6 3^{10} 5^4 7^2$	2^2	1^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
42	$2^2 3^5 7^2$	$2^6 3^{10} 7^4$	$2^4 3$	2^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$
44	$2^2 3 \cdot 11^2$	$2^6 3^2 7^2 11^4$	$2^2 3^2$	3^2	$1 \cdot 3^2 + O(3^6)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$
46	$2^2 3 \cdot 23^2$	$2^6 3^2 7^2 23^4$	2^4	2^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
52	$2^2 3^3 13^2$	$2^6 3^6 7^2 13^4$	2^4	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
60	$2^2 3^5 5^2$	$2^6 3^{10} 5^4 7^2$	2^4	2^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
62	$2^2 3 \cdot 31^2$	$2^6 3^2 7^2 31^4$	2^6	4^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$
70	$2^2 3^3 5^2 7^2$	$2^6 3^6 5^4 7^4$	$2^4 3$	2^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^1 + 1 \cdot 3^3 + O(3^6)$
82	$2^2 3 \cdot 41^2$	$2^6 3^2 7^2 41^4$	$2^2 5^2$	5^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$
84	$2^2 3^5 7^2$	$2^6 3^{10} 7^4$	$2^2 3^3$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^3 + 2 \cdot 3^5 + O(3^6)$
90	$2^2 3^5 5^2$	$2^6 3^{10} 5^4 7^2$	2^6	4^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
116	$2^2 3 \cdot 29^2$	$2^6 3^2 7^2 29^4$	2^2	1^2	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
126	$2^2 3^5 7^2$	$2^6 3^{10} 7^4$	$2^2 3$	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 2 \cdot 3^3 + O(3^4)$
140	$2^2 3^3 5^2 7^2$	$2^6 3^6 5^4 7^4$	$2^4 3$	2^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^1 + 1 \cdot 3^3 + O(3^6)$
150	$2^2 3^5 5^2$	$2^6 3^{10} 5^4 7^2$	$2^2 5^2$	5^2	$2 \cdot 3^0 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
154	$2^2 3 \cdot 7^2 11^2$	$2^6 3^2 7^4 11^4$	$2^2 3^3$	3^2	$1 \cdot 3^4 + 1 \cdot 3^6 + O(3^8)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$
172	$2^2 3 \cdot 43^2$	$2^6 3^2 7^2 43^4$	$2^2 3^2$	3^2	$1 \cdot 3^4 + O(3^7)$	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^7)$
252	$2^2 3^5 7^2$	$2^6 3^{10} 7^4$	$2^6 3$	4^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^1 + 2 \cdot 3^2 + O(3^5)$
350	$2^2 3 \cdot 5^2 7^2$	$2^6 3^2 5^4 7^4$	$2^2 3$	1^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
476	$2^2 3 \cdot 7^2 17^2$	$2^6 3^2 7^4 17^4$	$2^2 3^3$	3^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$
490	$2^2 3^3 5^2 7^2$	$2^6 3^6 5^4 7^4$	$2^2 3^3$	3^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^7)$
1666	$2^2 3 \cdot 7^2 17^2$	$2^6 3^2 7^4 17^4$	$2^2 3$	1^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^1 + 1 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$
2366	$2^2 3 \cdot 7^2 13^2$	$2^6 3^2 7^4 13^4$	$2^2 3^3$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^3 + O(3^7)$

$E = 128B2$ $y^2 = x^3 + x^2 - 2x - 2$ **Table 3-128B2**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 + 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^7$ $L^*(E/\mathbb{Q}) = 2^{-2}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^{14} \cdot 3^2$ $L^*(E/K) = 2^{-2}$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ $\text{II}, c_2 = 1$ $/K_v (v|2)$ $\text{II}, c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^{14} 3^6$	0	—	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	0
6	$2^2 3^5$	$2^{14} 3^{10}$	$2 \cdot 3^2$	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + O(3^7)$
10	$2^2 3 \cdot 5^2$	$2^{14} 3^2 5^4$	2	1^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
12	$2^2 3^5$	$2^{14} 3^{10}$	0	—	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	0
14	$2^2 3^3 7^2$	$2^{14} 3^6 7^4$	$2 \cdot 3^2$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$1 \cdot 3^2 + 1 \cdot 3^4 + O(3^5)$
20	$2^2 3^3 5^2$	$2^{14} 3^6 5^4$	0	—	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	0
26	$2^2 3 \cdot 13^2$	$2^{14} 3^2 13^4$	0	—	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^8)$	0
28	$2^2 3 \cdot 7^2$	$2^{14} 3^2 7^4$	2	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$1 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
44	$2^2 3 \cdot 11^2$	$2^{14} 3^2 11^4$	2	1^2	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
46	$2^2 3 \cdot 23^2$	$2^{14} 3^2 23^4$	2	1^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
116	$2^2 3 \cdot 29^2$	$2^{14} 3^2 29^4$	0	—	$2 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + O(3^9)$	0

$E = 77C1$ $y^2 + xy = x^3 + x^2 + 4x + 11$ **Table 3-77C1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 7 \cdot 11$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^2 \cdot 7^2 \cdot 11^2$ $L^*(E/K) = 2^{-1}$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_7$ I_3 non-split, $c_7 = 1$ $/K_v$ ($v|7$) I_3 non-split, $c_v = 1$
 $/\mathbb{Q}_{11}$ I_2 split, $c_{11} = 2$ $/K_v$ ($v|11$) I_2 split, $c_v = 2$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^4 3^6 7^2 11^2$	2^3	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
3	3^5	$3^{10} 7^2 11^2$	2	1^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
5	$3^3 5^2$	$3^6 5^4 7^2 11^2$	2^3	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
6	$2^2 3^5$	$2^4 3^{10} 7^2 11^2$	2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
7	$3^3 7^2$	$3^6 7^4 11^2$	2	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$
10	$2^2 3 \cdot 5^2$	$2^4 3^2 5^4 7^2 11^2$	2	1^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + O(3^4)$
12	$2^2 3^5$	$2^4 3^{10} 7^2 11^2$	2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
13	$3^3 13^2$	$3^6 7^2 11^2 13^4$	2	1^2	$1 \cdot 3^0 + O(3^5)$	$1 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^4 3^6 7^4 11^2$	$2 \cdot 5^2$	5^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
17	$3 \cdot 17^2$	$3^2 7^2 11^2 17^4$	2^3	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
19	$3 \cdot 19^2$	$3^2 7^2 11^2 19^4$	2	1^2	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
20	$2^2 3^3 5^2$	$2^4 3^6 5^4 7^2 11^2$	2	1^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
21	$3^5 7^2$	$3^{10} 7^4 11^2$	$2 \cdot 5^2$	5^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
26	$2^2 3 \cdot 13^2$	$2^4 3^2 7^2 11^2 13^4$	2	1^2	$2 \cdot 3^0 + O(3^5)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
28	$2^2 3 \cdot 7^2$	$2^4 3^2 7^4 11^2$	2	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
35	$3 \cdot 5^2 7^2$	$3^2 5^4 7^4 11^2$	$2 \cdot 5^2$	5^2	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
37	$3 \cdot 37^2$	$3^2 7^2 11^2 37^4$	2^3	2^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$
63	$3^5 7^2$	$3^{10} 7^4 11^2$	2	1^2	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
91	$3 \cdot 7^2 13^2$	$3^2 7^4 11^2 13^4$	$2 \cdot 7^2$	7^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
350	$2^2 3 \cdot 5^2 7^2$	$2^4 3^2 5^4 7^4 11^2$	$2 \cdot 7^2$	7^2	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$

$E = 80B2$ $y^2 = x^3 - x^2 - x$ **Table 3-80B2**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^4 \cdot 5$ $L^*(E/\mathbb{Q}) = 2^{-2}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^8 \cdot 3^2 \cdot 5^2$ $L^*(E/K) = 2^{-2}$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ II , $c_2 = 1$ $/K_v$ ($v|2$) II , $c_v = 1$
 $/\mathbb{Q}_5$ I_1 non-split, $c_5 = 1$ $/K_v$ ($v|5$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^8 3^6 5^2$	2	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
3	3^5	$2^8 3^{10} 5^2$	1	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
6	$2^2 3^5$	$2^8 3^{10} 5^2$	2	1^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
7	$3^3 7^2$	$2^8 3^6 5^2 7^4$	2^2	2^2	$2 \cdot 3^0 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
11	$3^3 11^2$	$2^8 3^6 5^2 11^4$	$2^2 3^2$	6^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
12	$2^2 3^5$	$2^8 3^{10} 5^2$	2^3	2^2	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^8 3^6 5^2 7^4$	2	1^2	$2 \cdot 3^0 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$
17	$3 \cdot 17^2$	$2^8 3^2 5^2 17^4$	3^2	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
19	$3 \cdot 19^2$	$2^8 3^2 5^2 19^4$	1	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^3)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
22	$2^2 3^3 11^2$	$2^8 3^6 5^2 11^4$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	0
26	$2^2 3 \cdot 13^2$	$2^8 3^2 5^2 13^4$	$2 \cdot 3^2$	3^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
28	$2^2 3 \cdot 7^2$	$2^8 3^2 5^2 7^4$	2	1^2	$2 \cdot 3^0 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
37	$3 \cdot 37^2$	$2^8 3^2 5^2 37^4$	$2^2 3^2$	6^2	$2 \cdot 3^4 + 1 \cdot 3^6 + O(3^7)$	$2 \cdot 3^2 + O(3^5)$
44	$2^2 3 \cdot 11^2$	$2^8 3^2 5^2 11^4$	$2 \cdot 3^2$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
46	$2^2 3 \cdot 23^2$	$2^8 3^2 5^2 23^4$	$2 \cdot 3^2$	3^2	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
62	$2^2 3 \cdot 31^2$	$2^8 3^2 5^2 31^4$	2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + O(3^4)$
116	$2^2 3 \cdot 29^2$	$2^8 3^2 5^2 29^4$	$2 \cdot 3^2$	3^2	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$

$E = 260A1$ $y^2 = x^3 - x^2 - 281x + 1910$ **Table 3-260A1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^2 \cdot 5 \cdot 13$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^4 \cdot 3^2 \cdot 5^2 \cdot 13^2$ $L^*(E/K) = 3$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ $\text{IV}, c_2 = 1$ $/K_v (v|2)$ $\text{IV}, c_v = 3$
 $/\mathbb{Q}_5$ I_1 non-split, $c_5 = 1$ $/K_v (v|5)$ I_1 split, $c_v = 1$
 $/\mathbb{Q}_{13}$ I_2 non-split, $c_{13} = 2$ $/K_v (v|13)$ I_2 non-split, $c_v = 2$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^4 3^{10} 5^2 13^2$	3	1^2	$2 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$	$2 \cdot 3^1 + 2 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$
7	$3^3 7^2$	$2^4 3^6 5^2 7^4 13^2$	0	—	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	0
10	$2^2 3 \cdot 5^2$	$2^2 3^2 5^4 13^2$	$2 \cdot 3^2$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$
13	$3^3 13^2$	$2^4 3^6 5^2 13^4$	$3 \cdot 7^2$	7^2	$2 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$	$2 \cdot 3^1 + O(3^4)$
17	$3 \cdot 17^2$	$2^4 3^2 5^2 13^2 17^4$	3	1^2	$1 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$
19	$3 \cdot 19^2$	$2^4 3^2 5^2 13^2 19^4$	$3 \cdot 5^2$	5^2	$2 \cdot 3^1 + 1 \cdot 3^3 + O(3^4)$	$2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$
20	$2^2 3^3 5^2$	$2^2 3^6 5^4 13^2$	$2 \cdot 3^2$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^3 + 2 \cdot 3^5 + O(3^{11})$
30	$2^2 3^5 5^2$	$2^2 3^{10} 5^4 13^2$	$2 \cdot 3^4$	9^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^5 + 2 \cdot 3^6 + O(3^9)$
60	$2^2 3^5 5^2$	$2^2 3^{10} 5^4 13^2$	0	—	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	0
90	$2^2 3^5 5^2$	$2^2 3^{10} 5^4 13^2$	$2^5 3^2$	6^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^3 + 2 \cdot 3^5 + O(3^6)$
130	$2^2 3^3 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2 \cdot 3^2 7^2$	21^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$
150	$2^2 3^5 5^2$	$2^2 3^{10} 5^4 13^2$	$2 \cdot 3^2$	3^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^7)$
350	$2^2 3 \cdot 5^2 7^2$	$2^2 3^2 5^4 7^4 13^2$	$2^3 3^2$	3^2	$1 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + O(3^8)$	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$
550	$2^2 3 \cdot 5^2 11^2$	$2^2 3^2 5^4 11^4 13^2$	$2 \cdot 3^4$	9^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^5 + 2 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$
650	$2^2 3^3 5^2 13^2$	$2^2 3^6 5^4 13^4$	$2^3 3^4$	18^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	$2 \cdot 3^5 + O(3^8)$
1300	$2^2 3^3 5^2 13^2$	$2^2 3^6 5^4 13^4$	0	—	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$	0

$E = 275B1$ $y^2 + y = x^3 + x^2 - 8x + 19$ **Table 3-275B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 5^2 \cdot 11$ $L^*(E/\mathbb{Q}) = 1$ $|E(\mathbb{Q})| = 1$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^2 \cdot 5^4 \cdot 11^2$ $L^*(E/K) = 1$ $|E(K)| = 1$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_5$ I_0^* , $c_5 = 1$ $/K_v (v|5)$ I_0^* , $c_v = 1$
 $/\mathbb{Q}_{11}$ I_1 split, $c_{11} = 1$ $/K_v (v|11)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^4 3^6 5^4 11^2$	3^2	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
3	3^5	$3^{10} 5^4 11^2$	3^2	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
5	$3^3 5^2$	$3^6 5^4 11^2$	0	—	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
6	$2^2 3^5$	$2^4 3^{10} 5^4 11^2$	3^2	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
10	$2^2 3 \cdot 5^2$	$2^4 3^2 5^4 11^2$	5	1^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
12	$2^2 3^5$	$2^4 3^{10} 5^4 11^2$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	0
15	$3^5 5^2$	$3^{10} 5^4 11^2$	$3^2 5$	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
17	$3 \cdot 17^2$	$3^2 5^4 11^2 17^4$	1	1^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^8)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
19	$3 \cdot 19^2$	$3^2 5^4 11^2 19^4$	1	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$
20	$2^2 3^3 5^2$	$2^4 3^6 5^4 11^2$	$3^2 5$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
28	$2^2 3 \cdot 7^2$	$2^4 3^2 5^4 7^4 11^2$	3^2	3^2	$2 \cdot 3^4 + 1 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$
30	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	$3^2 5$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
35	$3 \cdot 5^2 7^2$	$3^2 5^4 7^4 11^2$	$3^2 5$	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$	$1 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^8)$
45	$3^5 5^2$	$3^{10} 5^4 11^2$	$3^2 5$	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
60	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	$3^2 5$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
90	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	0
150	$2^2 3^5 5^2$	$2^4 3^{10} 5^4 11^2$	$3^2 5$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$
325	$3 \cdot 5^2 13^2$	$3^2 5^4 11^2 13^4$	$3^2 5$	3^2	$1 \cdot 3^6 + 2 \cdot 3^8 + O(3^{10})$	$1 \cdot 3^4 + O(3^8)$
350	$2^2 3 \cdot 5^2 7^2$	$2^4 3^2 5^4 7^4 11^2$	$3^2 5$	3^2	$2 \cdot 3^4 + 1 \cdot 3^5 + 1 \cdot 3^6 + O(3^7)$	$2 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$

$E = 272C1$ $y^2 = x^3 - x^2 - 4x$ **Table 3-272C1**

$p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$

$/\mathbb{Q}$ $N(E) = 2^4 \cdot 17$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$

$/K$ $N(E, \sigma) = 2^8 \cdot 3^2 \cdot 17^2$ $L^*(E/K) = 1$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$

$/\mathbb{Q}_2$ I_0^* , $c_2 = 2$ $/K_v$ $(v|2)$ I_0^* , $c_v = 4$

$/\mathbb{Q}_{17}$ I_1 split, $c_{17} = 1$ $/K_v$ $(v|17)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^8 3^6 17^2$	2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$
3	3^5	$2^8 3^{10} 17^2$	2^2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
5	$3^3 5^2$	$2^8 3^6 5^4 17^2$	2^2	1^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
6	$2^2 3^5$	$2^8 3^{10} 17^2$	2	1^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
7	$3^3 7^2$	$2^8 3^6 7^4 17^2$	0	—	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	0
10	$2^2 3 \cdot 5^2$	$2^8 3^2 5^4 17^2$	2	1^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + O(3^4)$
12	$2^2 3^5$	$2^8 3^{10} 17^2$	2^3	2^2	$2 \cdot 3^0 + 1 \cdot 3^1 + O(3^4)$	$2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + O(3^3)$
14	$2^2 3^3 7^2$	$2^8 3^6 7^4 17^2$	$2 \cdot 3^2$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
19	$3 \cdot 19^2$	$2^8 3^2 17^2 19^4$	2^4	2^2	$2 \cdot 3^0 + 1 \cdot 3^2 + O(3^3)$	$2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$
20	$2^2 3^3 5^2$	$2^8 3^6 5^4 17^2$	2^5	4^2	$1 \cdot 3^0 + O(3^3)$	$1 \cdot 3^0 + 2 \cdot 3^2 + O(3^3)$
26	$2^2 3 \cdot 13^2$	$2^8 3^2 13^4 17^2$	$2 \cdot 3^2$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
28	$2^2 3 \cdot 7^2$	$2^8 3^2 7^4 17^2$	$2 \cdot 3^2$	3^2	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
44	$2^2 3 \cdot 11^2$	$2^8 3^2 11^4 17^2$	0	—	$2 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$	0

$E = 92A1$ $y^2 = x^3 + x^2 + 2x + 1$ **Table 3-92A1**

$p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$

$/\mathbb{Q}$ $N(E) = 2^2 \cdot 23$ $L^*(E/\mathbb{Q}) = 3^{-1}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$

$/K$ $N(E, \sigma) = 2^4 \cdot 3^2 \cdot 23^2$ $L^*(E/K) = 3^{-1}$ $|E(K)| = 3$ $|\text{III}(E/K)| = 1^2$

$/\mathbb{Q}_2$ IV , $c_2 = 3$ $/K_v$ $(v|2)$ IV , $c_v = 3$

$/\mathbb{Q}_{23}$ I_1 non-split, $c_{23} = 1$ $/K_v$ $(v|23)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^4 3^{10} 23^2$	3	1^2	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
5	$3^3 5^2$	$2^4 3^6 5^4 23^2$	0	—	$1 \cdot 3^3 + O(3^6)$	0
7	$3^3 7^2$	$2^4 3^6 7^4 23^2$	0	—	$1 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	0
11	$3^3 11^2$	$2^4 3^6 11^4 23^2$	3^3	3^2	$1 \cdot 3^3 + 2 \cdot 3^4 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 2 \cdot 3^4 + O(3^7)$
17	$3 \cdot 17^2$	$2^4 3^2 17^4 23^2$	3	1^2	$2 \cdot 3^3 + O(3^9)$	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$
19	$3 \cdot 19^2$	$2^4 3^2 19^4 23^2$	0	—	$1 \cdot 3^5 + 2 \cdot 3^6 + 1 \cdot 3^7 + O(3^8)$	0
35	$3 \cdot 5^2 7^2$	$2^4 3^2 5^4 7^4 23^2$	3^3	3^2	$1 \cdot 3^5 + 1 \cdot 3^7 + O(3^8)$	$2 \cdot 3^5 + 1 \cdot 3^6 + O(3^9)$
46	$2^2 3 \cdot 23^2$	$2^2 3^2 23^4$	$2 \cdot 3^{-1}$	1^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
92	$2^2 3^3 23^2$	$2^2 3^6 23^4$	$2 \cdot 3$	3^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^7)$
138	$2^2 3^5 23^2$	$2^2 3^{10} 23^4$	$2 \cdot 3$	3^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
276	$2^2 3^5 23^2$	$2^2 3^{10} 23^4$	$2 \cdot 3$	3^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
414	$2^2 3^5 23^2$	$2^2 3^{10} 23^4$	$2^5 3$	12^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + O(3^7)$
828	$2^2 3^5 23^2$	$2^2 3^{10} 23^4$	$2^3 3$	6^2	$2 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^5)$

$E = 116C2$ $y^2 = x^3 - x^2 - 9x + 14$ **Table 3-116C2**

$p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$

$/\mathbb{Q}$ $N(E) = 2^2 \cdot 29$ $L^*(E/\mathbb{Q}) = 2^{-2}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$

$/K$ $N(E, \sigma) = 2^4 \cdot 3^2 \cdot 29^2$ $L^*(E/K) = 2^{-2} \cdot 3$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$

$/\mathbb{Q}_2$ IV , $c_2 = 1$ $/K_v$ $(v|2)$ IV , $c_v = 3$

$/\mathbb{Q}_{29}$ I_1 non-split, $c_{29} = 1$ $/K_v$ $(v|29)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
3	3^5	$2^4 3^{10} 29^2$	3	1^2	$2 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$2 \cdot 3^1 + 2 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$
5	$3^3 5^2$	$2^4 3^6 5^4 29^2$	3	1^2	$1 \cdot 3^1 + 2 \cdot 3^2 + O(3^4)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$
7	$3^3 7^2$	$2^4 3^6 7^4 29^2$	3	1^2	$2 \cdot 3^1 + 1 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$	$2 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
17	$3 \cdot 17^2$	$2^4 3^2 17^4 29^2$	3	1^2	$1 \cdot 3^1 + 1 \cdot 3^2 + O(3^4)$	$1 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$
19	$3 \cdot 19^2$	$2^4 3^2 19^4 29^2$	$2 \cdot 23$	2^2	$2 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$	$2 \cdot 3^1 + 1 \cdot 3^3 + O(3^4)$
58	$2^2 3^3 29^2$	$2^2 3^6 29^4$	0	—	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$	0
406	$2^2 3 \cdot 7^2 29^2$	$2^2 3^2 7^4 29^4$	$2 \cdot 3^4$	9^2	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$2 \cdot 3^5 + 1 \cdot 3^6 + O(3^{10})$
1450	$2^2 3 \cdot 5^2 29^2$	$2^2 3^2 5^4 29^4$	$2^3 3^2$	6^2	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^6)$	$1 \cdot 3^3 + O(3^6)$

$E = 152B1$ $y^2 = x^3 + x^2 - 8x - 16$ **Table 3-152B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^3 \cdot 19$ $L^*(E/\mathbb{Q}) = 1$ $|E(\mathbb{Q})| = 1$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^6 \cdot 3^2 \cdot 19^2$ $L^*(E/K) = 1$ $|E(K)| = 1$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ II^* , $c_2 = 1$ $/K_v$ $(v|2)$ II^* , $c_v = 1$
 $/\mathbb{Q}_{19}$ I_1 split, $c_{19} = 1$ $/K_v$ $(v|19)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^6 3^6 19^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
6	$2^2 3^5$	$2^6 3^{10} 19^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
10	$2^2 3 \cdot 5^2$	$2^6 3^2 5^4 19^2$	0	-	$1 \cdot 3^4 + O(3^7)$	0
12	$2^2 3^5$	$2^6 3^{10} 19^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
14	$2^2 3^3 7^2$	$2^6 3^6 7^4 19^2$	0	-	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	0
20	$2^2 3^3 5^2$	$2^6 3^6 5^4 19^2$	$2^3 3^2$	3^2	$1 \cdot 3^4 + O(3^7)$	$2 \cdot 3^2 + 1 \cdot 3^4 + O(3^5)$
26	$2^2 3 \cdot 13^2$	$2^6 3^2 13^4 19^2$	2^3	1^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^6)$
28	$2^2 3 \cdot 7^2$	$2^6 3^2 7^4 19^2$	2^3	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + 2 \cdot 3^4 + O(3^7)$
38	$2^2 3^3 19^2$	$2^6 3^6 19^4$	0	-	$1 \cdot 3^6 + 2 \cdot 3^7 + 1 \cdot 3^8 + O(3^9)$	0
44	$2^2 3 \cdot 11^2$	$2^6 3^2 11^4 19^2$	2^3	1^2	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^6)$	$2 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
46	$2^2 3 \cdot 23^2$	$2^6 3^2 23^4 19^2$	2^3	1^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$
76	$2^2 3^3 19^2$	$2^6 3^6 19^4$	$2^3 3^3$	3^2	$1 \cdot 3^6 + 2 \cdot 3^7 + 1 \cdot 3^8 + O(3^9)$	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^7)$
190	$2^2 3 \cdot 5^2 19^2$	$2^6 3^2 5^4 19^4$	0	-	$1 \cdot 3^8 + 1 \cdot 3^9 + 2 \cdot 3^{10} + O(3^{11})$	0
1900	$2^2 3 \cdot 5^2 19^2$	$2^6 3^2 5^4 19^4$	0	-	$1 \cdot 3^8 + 1 \cdot 3^9 + 2 \cdot 3^{10} + O(3^{11})$	0

$E = 176B1$ $y^2 = x^3 + x^2 - 5x - 13$ **Table 3-176B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^4 \cdot 11$ $L^*(E/\mathbb{Q}) = 1$ $|E(\mathbb{Q})| = 1$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^8 \cdot 3^2 \cdot 11^2$ $L^*(E/K) = 1$ $|E(K)| = 1$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ II^* , $c_2 = 1$ $/K_v$ $(v|2)$ II^* , $c_v = 1$
 $/\mathbb{Q}_{11}$ I_1 non-split, $c_{11} = 1$ $/K_v$ $(v|11)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^8 3^6 11^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
3	3^5	$2^8 3^{10} 11^2$	3^2	3^2	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$
5	$3^3 5^2$	$2^8 3^6 5^4 11^2$	0	-	$2 \cdot 3^2 + O(3^6)$	0
6	$2^2 3^5$	$2^8 3^{10} 11^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
7	$3^3 7^2$	$2^8 3^6 7^4 11^2$	3^2	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$	$1 \cdot 3^2 + O(3^7)$
10	$2^2 3 \cdot 5^2$	$2^8 3^2 5^4 11^2$	2^3	1^2	$2 \cdot 3^2 + O(3^6)$	$2 \cdot 3^2 + 1 \cdot 3^3 + O(3^5)$
12	$2^2 3^5$	$2^8 3^{10} 11^2$	0	-	$1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^6)$	0
14	$2^2 3^3 7^2$	$2^8 3^6 7^4 11^2$	$2^3 3^2$	3^2	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^7)$
17	$3 \cdot 17^2$	$2^8 3^2 11^2 17^4$	1	1^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^6)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^5)$
19	$3 \cdot 19^2$	$2^8 3^2 11^2 19^4$	1	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + O(3^6)$	$1 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + O(3^6)$
20	$2^2 3^3 5^2$	$2^8 3^6 5^4 11^2$	0	-	$2 \cdot 3^2 + O(3^6)$	0
26	$2^2 3 \cdot 13^2$	$2^8 3^2 11^2 13^4$	2^3	1^2	$1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$	$1 \cdot 3^2 + 1 \cdot 3^3 + O(3^6)$
28	$2^2 3 \cdot 7^2$	$2^8 3^2 7^4 11^2$	0	-	$1 \cdot 3^4 + 1 \cdot 3^5 + 2 \cdot 3^6 + O(3^7)$	0
46	$2^2 3 \cdot 23^2$	$2^8 3^2 11^2 23^4$	2^3	1^2	$2 \cdot 3^2 + 2 \cdot 3^4 + O(3^5)$	$2 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + O(3^5)$

$E = 395B1$ $y^2 + xy + y = x^3 + x^2 - 40x - 128$ **Table 3-395B1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - 2T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 5 \cdot 79$ $L^*(E/\mathbb{Q}) = 2^{-1} \cdot 3$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^2 \cdot 5^2 \cdot 79^2$ $L^*(E/K) = 2^{-1} \cdot 3$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_5$ I_6 split, $c_5 = 6$ $/K_v$ $(v|5)$ I_6 split, $c_v = 6$
 $/\mathbb{Q}_{79}$ I_1 non-split, $c_{79} = 1$ $/K_v$ $(v|79)$ I_1 non-split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^4 3^6 5^2 79^2$	$2^3 3$	2^2	$2 \cdot 3^1 + 1 \cdot 3^2 + O(3^5)$	$2 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$
3	3^5	$3^{10} 5^2 79^2$	$2 \cdot 3$	1^2	$1 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + O(3^4)$	$1 \cdot 3^1 + 2 \cdot 3^2 + 2 \cdot 3^3 + O(3^4)$
17	$3 \cdot 17^2$	$3^2 5^2 17^4 79^2$	0	-	$1 \cdot 3^5 + 2 \cdot 3^7 + O(3^8)$	0
28	$2^2 3 \cdot 7^2$	$2^4 3^2 5^2 7^4 79^2$	$2 \cdot 3^3$	3^2	$2 \cdot 3^3 + 1 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 2 \cdot 3^5 + O(3^6)$

$E = 800E1$ $y^2 = x^3 + x^2 - 208x - 1412$ **Table 3-800E1**
 $p = 3$ $P_3(E/\mathbb{Q}, T) = 1 - T + 3T^2$
 $/\mathbb{Q}$ $N(E) = 2^5 \cdot 5^2$ $L^*(E/\mathbb{Q}) = 3$ $|E(\mathbb{Q})| = 1$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^{10} \cdot 3^2 \cdot 5^4$ $L^*(E/K) = 3$ $|E(K)| = 1$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ $I_0^*, c_2 = 1$ $/K_v (v|2)$ $I_0^*, c_v = 1$
 $/\mathbb{Q}_5$ $\text{IV}^*, c_5 = 3$ $/K_v (v|5)$ $\text{IV}^*, c_v = 3$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^2 3^3$	$2^{10} 3^6 5^4$	$2 \cdot 3^3$	3^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	$1 \cdot 3^3 + 2 \cdot 3^4 + O(3^8)$
6	$2^2 3^5$	$2^{10} 3^{10} 5^4$	0	—	$1 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	0
12	$2^2 3^5$	$2^{10} 3^{10} 5^4$	$2^3 3^3$	6^2	$1 \cdot 3^3 + 1 \cdot 3^4 + 1 \cdot 3^5 + O(3^7)$	$1 \cdot 3^3 + 2 \cdot 3^4 + 2 \cdot 3^5 + O(3^6)$
28	$2^2 3 \cdot 7^2$	$2^{10} 3^2 5^4 7^4$	$2^3 3$	2^2	$1 \cdot 3^3 + 1 \cdot 3^5 + O(3^7)$	$1 \cdot 3^3 + 2 \cdot 3^5 + O(3^8)$
44	$2^2 3 \cdot 11^2$	$2^{10} 3^2 5^4 11^4$	$2 \cdot 3$	1^2	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$	$2 \cdot 3^3 + 1 \cdot 3^5 + O(3^6)$

$E = 11A3$ $y^2 + y = x^3 - x^2$ **Table 5-11A3**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 - T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 11$ $L^*(E/\mathbb{Q}) = 5^{-2}$ $|E(\mathbb{Q})| = 5$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 5^6 \cdot 11^4$ $L^*(E/K) = 5^{-2}$ $|E(K)| = 5$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_{11}$ I_1 split, $c_{11} = 1$ $/K_v (v|11)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^8 5^{10} 11^4$	0	—	$4 \cdot 5^2 + 1 \cdot 5^3 + O(5^5)$	0
3	$3^4 5^5$	$3^8 5^{10} 11^4$	0	—	$2 \cdot 5^2 + 4 \cdot 5^3 + 1 \cdot 5^4 + O(5^7)$	0
5	5^9	$5^{18} 11^4$	1	1^2	$4 \cdot 5^0 + 3 \cdot 5^1 + O(5^4)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$
6	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	5^2	5^2	$2 \cdot 5^4 + 3 \cdot 5^5 + 1 \cdot 5^6 + O(5^8)$	$4 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$
7	$5^3 7^4$	$5^6 7^8 11^4$	2^2	2^2	$4 \cdot 5^2 + 4 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	$4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$
11	$5^5 11^4$	$5^{10} 11^8$	$2^4 5$	4^2	$4 \cdot 5^4 + 3 \cdot 5^5 + O(5^7)$	$4 \cdot 5^1 + 1 \cdot 5^2 + 3 \cdot 5^3 + O(5^4)$
12	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	0	—	$2 \cdot 5^4 + 3 \cdot 5^5 + 1 \cdot 5^6 + O(5^8)$	0
18	$2^4 3^4 5^3$	$2^8 3^8 5^6 11^4$	0	—	$2 \cdot 5^4 + 3 \cdot 5^5 + 1 \cdot 5^6 + O(5^8)$	0
22	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	5^3	5^2	$4 \cdot 5^6 + 1 \cdot 5^7 + O(5^9)$	$1 \cdot 5^3 + 1 \cdot 5^4 + 2 \cdot 5^5 + O(5^8)$
44	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	0	—	$4 \cdot 5^6 + 1 \cdot 5^7 + O(5^9)$	0
48	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	5^2	5^2	$2 \cdot 5^4 + 3 \cdot 5^5 + 1 \cdot 5^6 + O(5^8)$	$4 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$
88	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	5^3	5^2	$4 \cdot 5^6 + 1 \cdot 5^7 + O(5^9)$	$1 \cdot 5^3 + 1 \cdot 5^4 + 2 \cdot 5^5 + O(5^8)$
99	$3^4 5^3 11^4$	$3^8 5^6 11^8$	$3^2 5$	3^2	$2 \cdot 5^6 + 4 \cdot 5^7 + 1 \cdot 5^8 + O(5^9)$	$4 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + O(5^6)$
176	$2^4 5^3 11^4$	$2^8 5^6 11^8$	5	1^2	$4 \cdot 5^6 + 1 \cdot 5^7 + O(5^9)$	$1 \cdot 5^3 + 3 \cdot 5^4 + 4 \cdot 5^5 + O(5^6)$

$E = 19A3$ $y^2 + y = x^3 + x^2 + x$ **Table 5-19A3**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 - 3T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 19$ $L^*(E/\mathbb{Q}) = 3^{-2}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 5^6 \cdot 19^4$ $L^*(E/K) = 3^{-2}$ $|E(K)| = 3$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_{19}$ I_1 split, $c_{19} = 1$ $/K_v (v|19)$ I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^8 5^{10} 19^4$	1	1^2	$2 \cdot 5^0 + 2 \cdot 5^1 + 4 \cdot 5^2 + O(5^3)$	$2 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$
3	$3^4 5^5$	$3^8 5^{10} 19^4$	2^2	2^2	$3 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$	$3 \cdot 5^0 + 3 \cdot 5^1 + O(5^3)$
5	5^9	$5^{18} 19^4$	2^4	4^2	$3 \cdot 5^0 + 2 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$3 \cdot 5^0 + 4 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
6	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 19^4$	2^2	2^2	$2 \cdot 5^0 + 3 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 3 \cdot 5^1 + 2 \cdot 5^2 + O(5^5)$
7	$5^3 7^4$	$5^6 7^8 19^4$	2^4	4^2	$3 \cdot 5^0 + 2 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$	$3 \cdot 5^0 + 4 \cdot 5^1 + O(5^3)$
12	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 19^4$	7^2	7^2	$2 \cdot 5^0 + 3 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 4 \cdot 5^1 + O(5^3)$
18	$2^4 3^4 5^3$	$2^8 3^8 5^6 19^4$	1	1^2	$2 \cdot 5^0 + 3 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 4 \cdot 5^1 + O(5^3)$
19	$5^5 19^4$	$5^{10} 19^8$	$2^4 5$	4^2	$2 \cdot 5^2 + 2 \cdot 5^4 + O(5^5)$	$3 \cdot 5^1 + 3 \cdot 5^2 + 1 \cdot 5^3 + O(5^4)$
48	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 19^4$	2^2	2^2	$2 \cdot 5^0 + 3 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 3 \cdot 5^1 + 2 \cdot 5^2 + O(5^5)$
57	$3^4 5^3 19^4$	$3^8 5^6 19^8$	$3^4 5$	9^2	$2 \cdot 5^2 + 1 \cdot 5^3 + 2 \cdot 5^4 + O(5^6)$	$3 \cdot 5^1 + 4 \cdot 5^2 + 3 \cdot 5^3 + O(5^4)$
76	$2^4 5^3 19^4$	$2^8 5^6 19^8$	$2^6 5$	8^2	$3 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$	$2 \cdot 5^1 + 3 \cdot 5^3 + O(5^5)$

$E = 21A4$ $y^2 + xy = x^3 + x$ **Table 5-21A4**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 + 2T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 3 \cdot 7$ $L^*(E/\mathbb{Q}) = 2^{-3}$ $|E(\mathbb{Q})| = 4$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 3^4 \cdot 5^6 \cdot 7^4$ $L^*(E/K) = 2^{-3}$ $|E(K)| = 4$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_3$ I_2 split, $c_3 = 2$ $/K_v$ ($v|3$) I_2 split, $c_v = 2$
 $/\mathbb{Q}_7$ I_1 non-split, $c_7 = 1$ $/K_v$ ($v|7$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^8 3^4 5^{10} 7^4$	2	1^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 2 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$
5	5^9	$3^4 5^{18} 7^4$	2^3	2^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
21	$3^4 5^5 7^4$	$3^8 5^{10} 7^8$	$5 \cdot 17^2$	17^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$2 \cdot 5^1 + 2 \cdot 5^2 + 3 \cdot 5^3 + O(5^4)$
63	$3^4 5^5 7^4$	$3^8 5^{10} 7^8$	$2^4 5$	4^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$3 \cdot 5^1 + 4 \cdot 5^3 + O(5^4)$
126	$2^4 3^4 5^3 7^4$	$2^8 3^8 5^6 7^8$	$2^4 5$	4^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$3 \cdot 5^1 + 1 \cdot 5^2 + O(5^4)$
147	$3^4 5^5 7^4$	$3^8 5^{10} 7^8$	5	1^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$3 \cdot 5^1 + 1 \cdot 5^2 + 4 \cdot 5^3 + O(5^4)$
168	$2^4 3^4 5^3 7^4$	$2^8 3^8 5^6 7^8$	$2^2 5$	2^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$2 \cdot 5^1 + O(5^4)$
567	$3^4 5^5 7^4$	$3^8 5^{10} 7^8$	$2^6 5$	8^2	$4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7)$	$2 \cdot 5^1 + 2 \cdot 5^2 + 1 \cdot 5^3 + O(5^4)$

$E = 24A4$ $y^2 = x^3 - x^2 + x$ **Table 5-24A4**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 + 2T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 2^3 \cdot 3$ $L^*(E/\mathbb{Q}) = 2^{-3}$ $|E(\mathbb{Q})| = 4$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^{12} \cdot 3^4 \cdot 5^6$ $L^*(E/K) = 2^{-3}$ $|E(K)| = 4$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ III, $c_2 = 2$ $/K_v$ ($v|2$) III, $c_v = 2$
 $/\mathbb{Q}_3$ I_1 non-split, $c_3 = 1$ $/K_v$ ($v|3$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^{12} 3^4 5^{10}$	2	1^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 2 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$
10	$2^4 5^9$	$2^{12} 3^4 5^{18}$	2^5	4^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
20	$2^4 5^9$	$2^{12} 3^4 5^{18}$	2^5	4^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
40	$2^4 5^9$	$2^{12} 3^4 5^{18}$	2^5	4^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
80	$2^4 5^9$	$2^{12} 3^4 5^{18}$	2	1^2	$4 \cdot 5^0 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 4 \cdot 5^1 + 1 \cdot 5^2 + O(5^4)$
176	$2^4 5^3 11^4$	$2^{12} 3^4 5^6 11^8$	2^3	2^2	$4 \cdot 5^0 + 4 \cdot 5^1 + 3 \cdot 5^2 + O(5^4)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2 + O(5^4)$

$E = 56A1$ $y^2 = x^3 + x + 2$ **Table 5-56A1**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 - 2T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 2^3 \cdot 7$ $L^*(E/\mathbb{Q}) = 2^{-2}$ $|E(\mathbb{Q})| = 4$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^{12} \cdot 5^6 \cdot 7^4$ $L^*(E/K) = 2^{-2}$ $|E(K)| = 4$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ I_1^* , $c_2 = 4$ $/K_v$ ($v|2$) I_1^* , $c_v = 4$
 $/\mathbb{Q}_7$ I_1 non-split, $c_7 = 1$ $/K_v$ ($v|7$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^{12} 5^{10} 7^4$	2^4	2^2	$3 \cdot 5^0 + O(5^3)$	$3 \cdot 5^0 + 2 \cdot 5^1 + 4 \cdot 5^2 + O(5^3)$
6	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	$2^2 3^2$	3^2	$2 \cdot 5^0 + 1 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 4 \cdot 5^1 + 4 \cdot 5^2 + O(5^4)$
12	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	2^4	2^2	$2 \cdot 5^0 + 1 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 1 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$
18	$2^4 3^4 5^3$	$2^{12} 3^8 5^6 7^4$	$2^2 3^2$	3^2	$2 \cdot 5^0 + 1 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 2 \cdot 5^1 + O(5^3)$
48	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	$2^2 3^2$	3^2	$2 \cdot 5^0 + 1 \cdot 5^1 + O(5^3)$	$2 \cdot 5^0 + 4 \cdot 5^1 + 4 \cdot 5^2 + O(5^4)$

$E = 56B1$ $y^2 = x^3 - x^2 - 4$ **Table 5-56B1**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 + 4T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 2^3 \cdot 7$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^{12} \cdot 5^6 \cdot 7^4$ $L^*(E/K) = 2^{-1}$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ III^* , $c_2 = 2$ $/K_v$ ($v|2$) III^* , $c_v = 2$
 $/\mathbb{Q}_7$ I_1 split, $c_7 = 1$ $/K_v$ ($v|7$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^{12} 5^{10} 7^4$	0	—	$3 \cdot 5^2 + 2 \cdot 5^3 + O(5^5)$	0
6	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	$2^3 5^2$	5^2	$3 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$	$2 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$
12	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	0	—	$3 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$	0
18	$2^4 3^4 5^3$	$2^{12} 3^8 5^6 7^4$	2^3	1^2	$3 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$	$3 \cdot 5^2 + 4 \cdot 5^4 + O(5^6)$
48	$2^4 3^4 5^5$	$2^{12} 3^8 5^{10} 7^4$	$2^3 5^2$	5^2	$3 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$	$2 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$

$E = 44A1$ $y^2 = x^3 + x^2 + 3x - 1$ **Table 5-44A1**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 + 3T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 2^2 \cdot 11$ $L^*(E/\mathbb{Q}) = 3^{-1}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^8 \cdot 5^6 \cdot 11^4$ $L^*(E/K) = 3^{-1}$ $|E(K)| = 3$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ IV^* , $c_2 = 3$ $/K_v$ ($v|2$) IV^* , $c_v = 3$
 $/\mathbb{Q}_{11}$ I_1 non-split, $c_{11} = 1$ $/K_v$ ($v|11$) I_1 non-split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^8 5^{10} 11^4$	2^3	1^2	$4 \cdot 5^0 + 2 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 3 \cdot 5^1 + 1 \cdot 5^2 + O(5^4)$
3	$3^4 5^5$	$2^8 3^8 5^{10} 11^4$	0	—	$2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	0
6	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	$2^3 5^2$	5^2	$2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	$1 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$
7	$5^3 7^4$	$2^8 5^6 7^8 11^4$	0	—	$4 \cdot 5^2 + 3 \cdot 5^3 + 2 \cdot 5^4 + O(5^5)$	0
11	$5^5 11^4$	$2^8 5^{10} 11^8$	$2^6 3$	8^2	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 3 \cdot 5^1 + 2 \cdot 5^2 + O(5^3)$
12	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	0	—	$2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	0
18	$2^4 3^4 5^3$	$2^8 3^8 5^6 11^4$	0	—	$2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	0
22	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	$2^3 3^4$	9^2	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 4 \cdot 5^2 + O(5^3)$
44	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	$2^3 3^4$	9^2	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 1 \cdot 5^1 + 4 \cdot 5^2 + O(5^3)$
48	$2^4 3^4 5^5$	$2^8 3^8 5^{10} 11^4$	$2^3 5^2$	5^2	$2 \cdot 5^2 + 1 \cdot 5^3 + 3 \cdot 5^4 + O(5^5)$	$1 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + O(5^5)$
88	$2^4 5^5 11^4$	$2^8 5^{10} 11^8$	$2^5 7^2$	14^2	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$
176	$2^4 5^3 11^4$	$2^8 5^6 11^8$	$2^5 3^2$	6^2	$4 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + O(5^3)$	$4 \cdot 5^0 + 3 \cdot 5^1 + 2 \cdot 5^2 + O(5^4)$

$E = 84B1$ $y^2 = x^3 - x^2 - x - 2$ **Table 5-84B1**
 $p = 5$ $P_5(E/\mathbb{Q}, T) = 1 - 4T + 5T^2$
 $/\mathbb{Q}$ $N(E) = 2^2 \cdot 3 \cdot 7$ $L^*(E/\mathbb{Q}) = 2^{-1}$ $|E(\mathbb{Q})| = 2$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 2^8 \cdot 3^4 \cdot 5^6 \cdot 7^4$ $L^*(E/K) = 2^{-1} \cdot 3$ $|E(K)| = 2$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_2$ IV , $c_2 = 1$ $/K_v$ ($v|2$) IV , $c_v = 3$
 $/\mathbb{Q}_3$ I_1 non-split, $c_3 = 1$ $/K_v$ ($v|3$) I_1 split, $c_v = 1$
 $/\mathbb{Q}_7$ I_2 non-split, $c_7 = 2$ $/K_v$ ($v|7$) I_2 split, $c_v = 2$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^4 5^5$	$2^8 3^4 5^{10} 7^4$	2^2	1^2	$1 \cdot 5^0 + 3 \cdot 5^1 + 4 \cdot 5^2 + O(5^3)$	$1 \cdot 5^0 + 4 \cdot 5^1 + 1 \cdot 5^2 + O(5^3)$
126	$2^4 3^4 5^3 7^4$	$2^8 3^8 5^6 7^8$	$2 \cdot 3^2 5^2$	15^2	$1 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^5 + O(5^6)$	$3 \cdot 5^2 + 1 \cdot 5^4 + O(5^8)$
168	$2^4 3^4 5^3 7^4$	$2^8 3^8 5^6 7^8$	$2 \cdot 3^2 5^2$	15^2	$1 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^5 + O(5^6)$	$3 \cdot 5^2 + 1 \cdot 5^4 + O(5^8)$

$E = 17A1$ $y^2 + xy + y = x^3 - x^2 - x - 14$ **Table 7-17A1**
 $p = 7$ $P_7(E/\mathbb{Q}, T) = 1 - 4T + 7T^2$
 $/\mathbb{Q}$ $N(E) = 17$ $L^*(E/\mathbb{Q}) = 2^{-2}$ $|E(\mathbb{Q})| = 4$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 7^{10} \cdot 17^6$ $L^*(E/K) = 2^4$ $|E(K)| = 4$ $|\text{III}(E/K)| = 8^2$
 $/\mathbb{Q}_{17}$ I_4 split, $c_{17} = 4$ $/K_v$ ($v|17$) I_4 split, $c_v = 4$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^6 7^7$	$2^{12} 7^{14} 17^6$	2^6	4^2	$5 \cdot 7^0 + 2 \cdot 7^1 + 2 \cdot 7^2 + O(7^3)$	$5 \cdot 7^0 + 5 \cdot 7^1 + 1 \cdot 7^2 + O(7^3)$
3	$3^6 7^7$	$3^{12} 7^{14} 17^6$	$2^6 7^2$	28^2	$5 \cdot 7^2 + 3 \cdot 7^3 + 4 \cdot 7^4 + O(7^5)$	$2 \cdot 7^2 + 4 \cdot 7^3 + 2 \cdot 7^4 + O(7^5)$

$E = 19A3$ $y^2 + y = x^3 + x^2 + x$ **Table 7-19A3**
 $p = 7$ $P_7(E/\mathbb{Q}, T) = 1 + T + 7T^2$
 $/\mathbb{Q}$ $N(E) = 19$ $L^*(E/\mathbb{Q}) = 3^{-2}$ $|E(\mathbb{Q})| = 3$ $|\text{III}(E/\mathbb{Q})| = 1^2$
 $/K$ $N(E, \sigma) = 7^{10} \cdot 19^6$ $L^*(E/K) = 3^{-2}$ $|E(K)| = 3$ $|\text{III}(E/K)| = 1^2$
 $/\mathbb{Q}_{19}$ I_1 split, $c_{19} = 1$ $/K_v$ ($v|19$) I_1 split, $c_v = 1$

m	$N(\rho)$	$N(E, \rho)$	L^*	III	$\mathcal{L}_E(\sigma)$	$\mathcal{L}_E(\rho)$
2	$2^6 7^7$	$2^{12} 7^{14} 19^6$	1	1^2	$1 \cdot 7^0 + 3 \cdot 7^1 + 1 \cdot 7^2 + O(7^3)$	$1 \cdot 7^0 + 1 \cdot 7^2 + O(7^3)$
3	$3^6 7^7$	$3^{12} 7^{14} 19^6$	2^2	2^2	$3 \cdot 7^0 + 4 \cdot 7^1 + 3 \cdot 7^2 + O(7^3)$	$3 \cdot 7^0 + 4 \cdot 7^2 + O(7^3)$

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