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Fermion Masses and Unification
(Lecture 1)

S. King
University of Southampton, UK

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Fermion Masses and Unification

Steve King
University of Southampton

Lecture I

Fermion Masses and Mixings

1. The Flavour Problem and See-Saw
2. From low energy data to high energy data
3. Textures in a basis

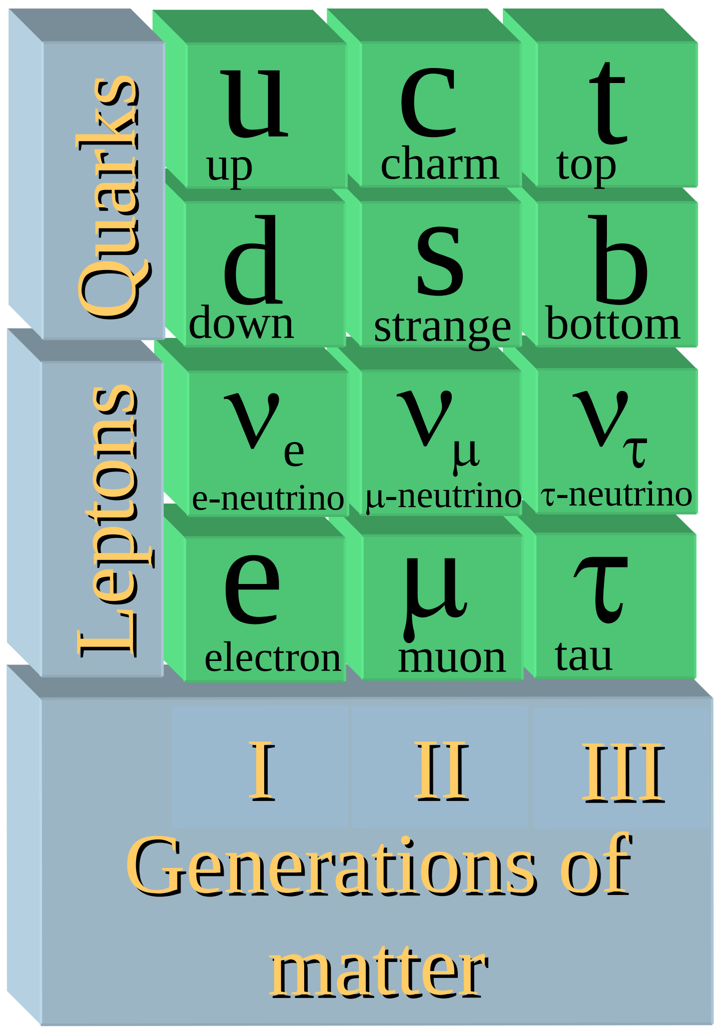
Appendix 1 References

Appendix 2 Basis Changing

1. The Flavour Problem and See-Saw

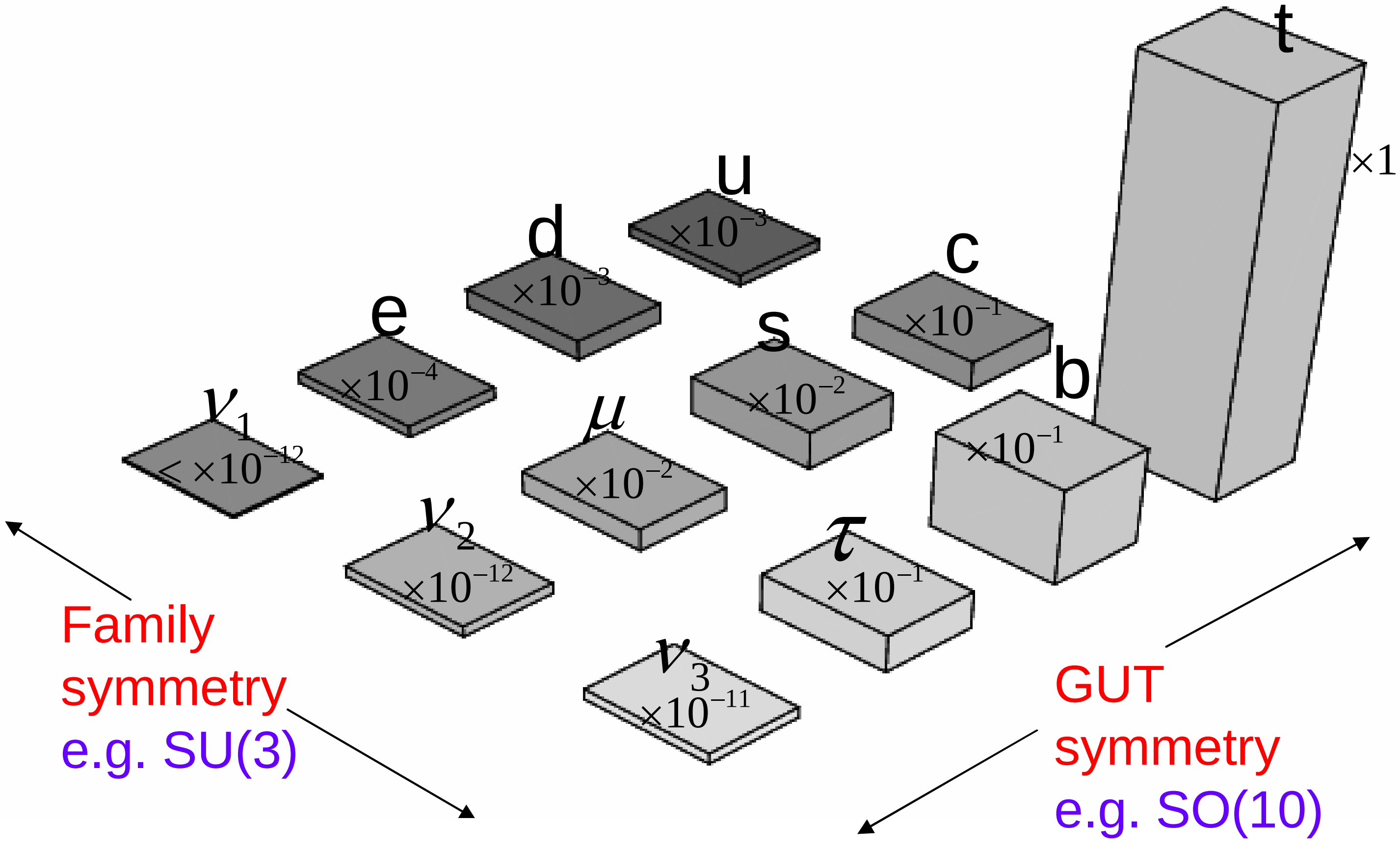
The Flavour Problem

1. Why are there three families of quarks and leptons?



The Flavour Problem

2. Why are quark and charged lepton masses so peculiar?



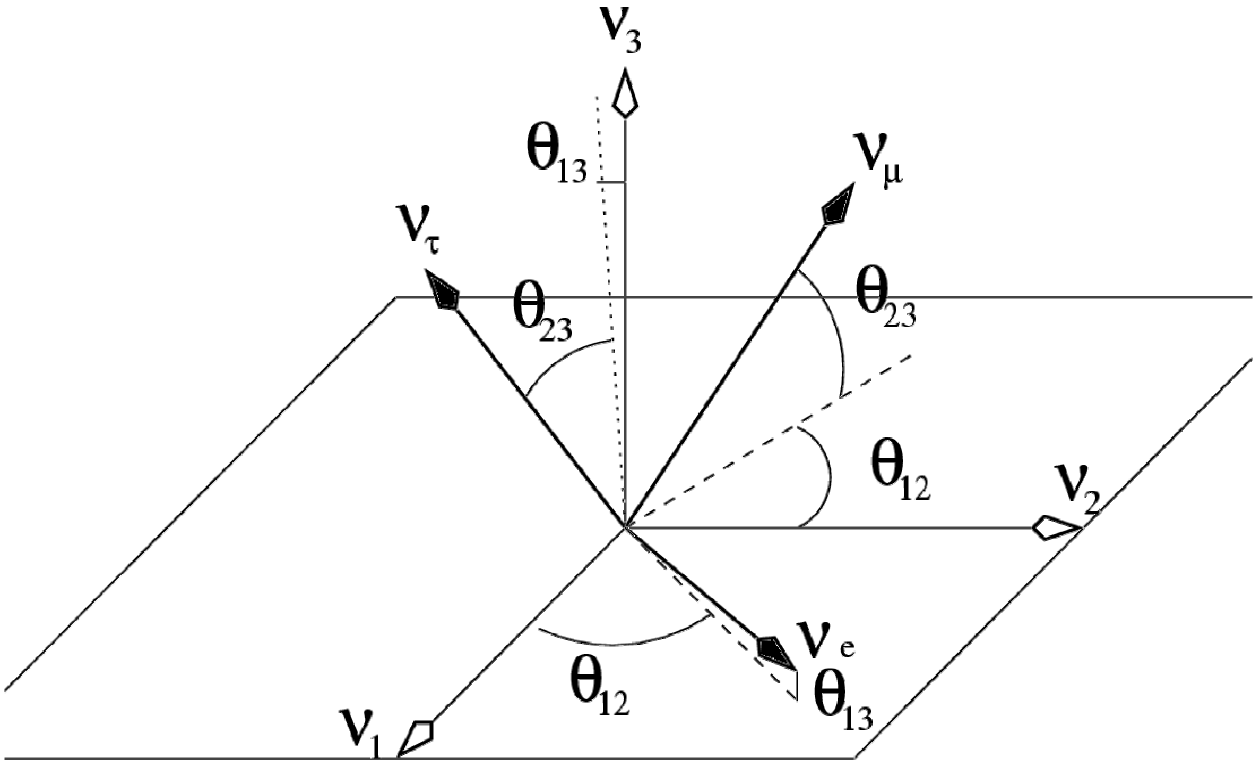
The Flavour Problem

3. Why is lepton mixing so large?

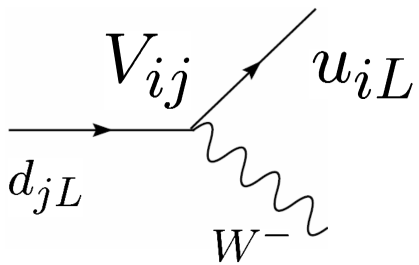
$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

e.g.Tri-bimaximal

Harrison, Perkins, Scott



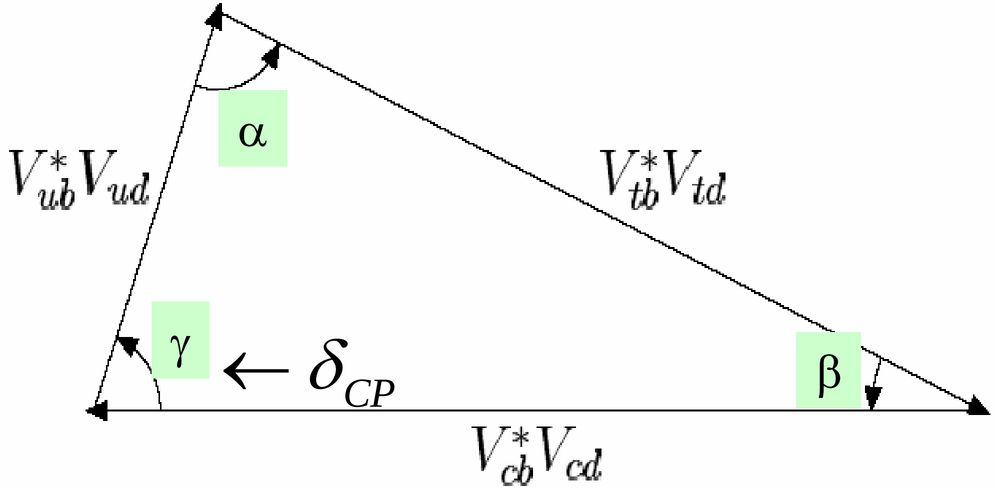
c.f. small quark mixing

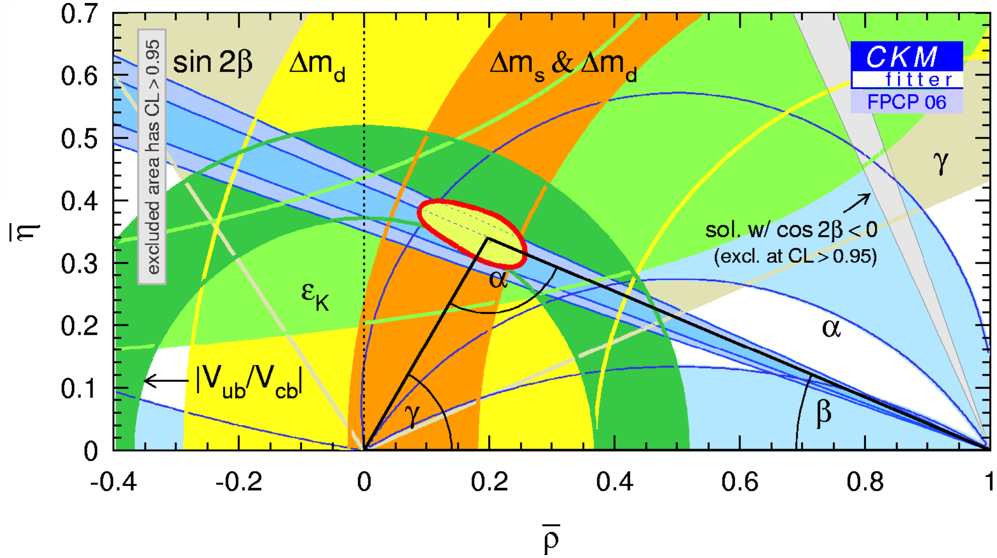


$$V_{ij} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \lambda^3 & -\lambda^2 & 1 \end{pmatrix}$$

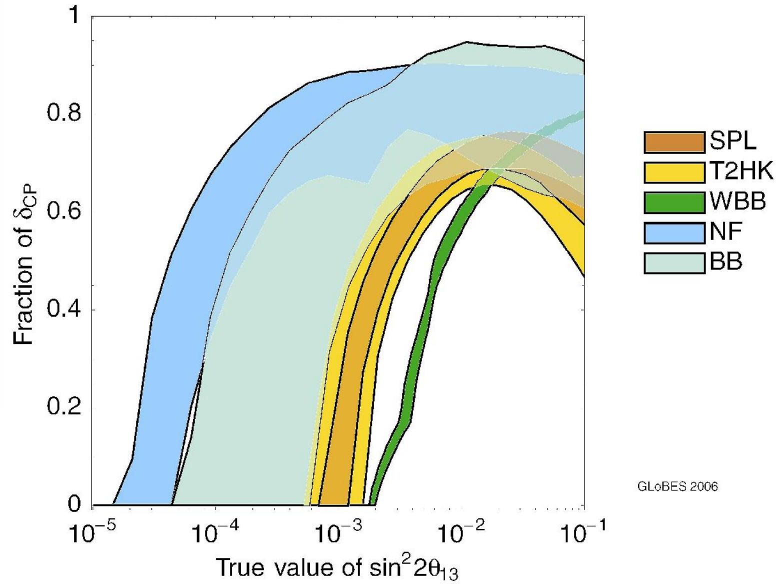
The Flavour Problem

4. What is the origin of CP violation?



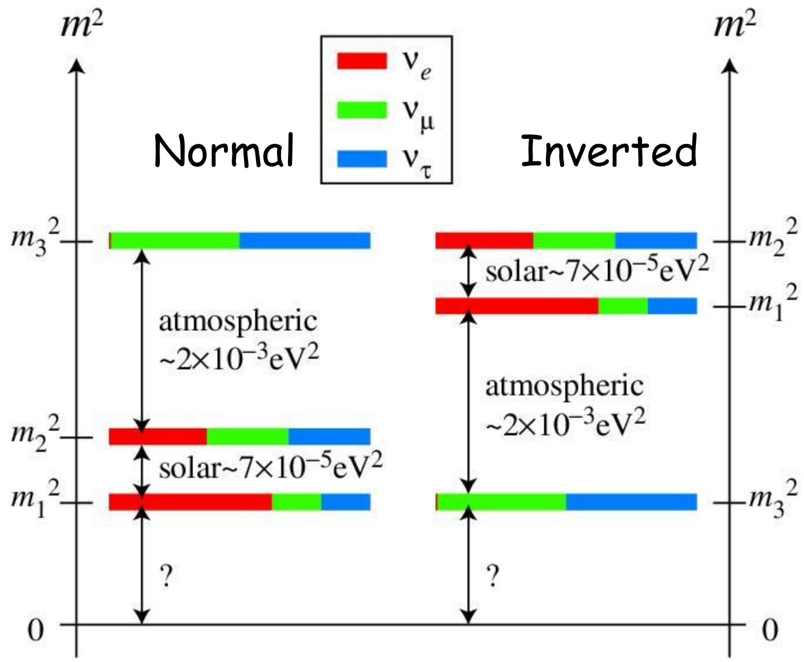


Lepton CP Violation?



The Flavour Problem

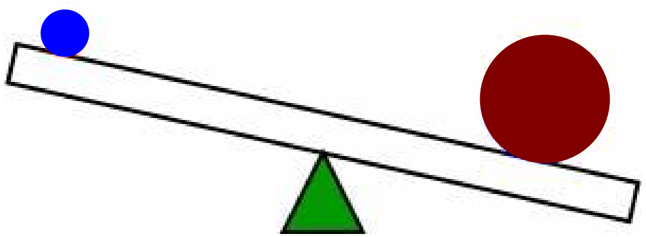
5. Why are neutrino masses so small?



→ See-saw mechanism is most elegant solution

The See-Saw Mechanism

Light neutrinos

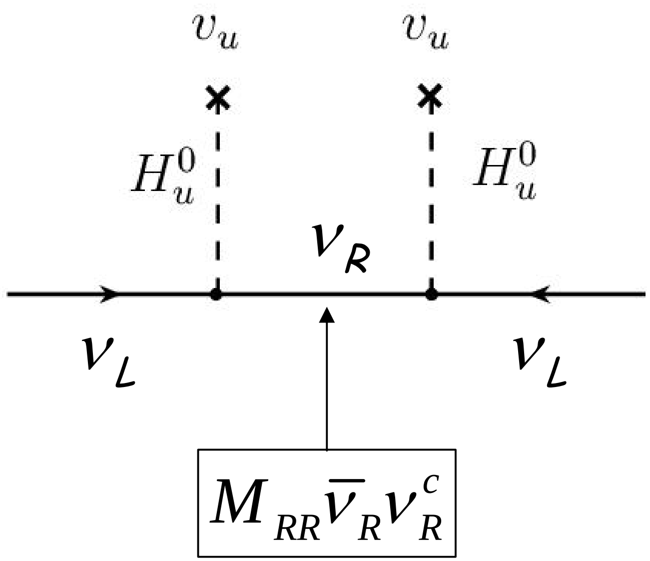


Heavy particles

The see-saw mechanism

Type I see-saw mechanism

P. Minkowski (1977), Gell-Mann, Glashow, Mohapatra, Ramond, Senjanovic, Slanski, Yanagida (1979/1980)

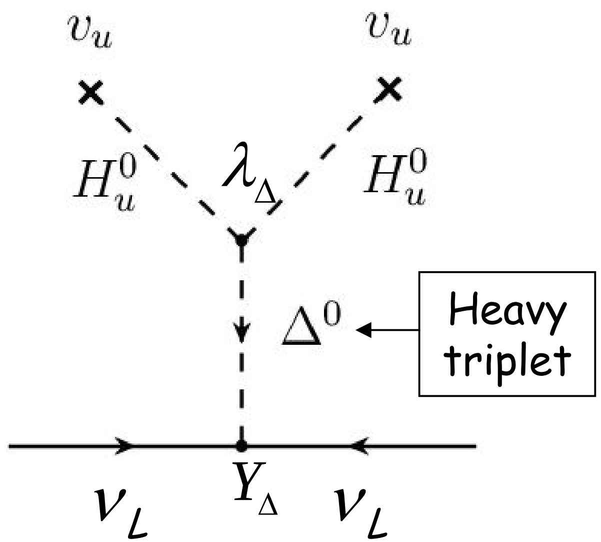


$$m_{LL}^{I} \approx -m_{LR} M_{RR}^{-1} m_{LR}^{T}$$

Type I

Type II see-saw mechanism (SUSY)

Lazarides, Magg, Mohapatra, Senjanovic, Shafi, Wetterich (1981)



$$m_{LL}^{II} \approx \lambda_\Delta Y_\Delta \frac{v_u^2}{M_\Delta}$$

Type II

See-Saw Standard Model (type I)

Yukawa couplings to 2 Higgs doublets (or one with $H_d \equiv H_u^c$)

$$\mathcal{L}_{mass} = -\epsilon_{ab}\left[\tilde{Y}^e_{ij} H^a_d L^b_i e^c_j - \tilde{Y}^\nu_{ij} H^a_u L^b_i \nu^c_j + \frac{1}{2}\nu^c_i \tilde{M}^{ij}_{RR}\nu^c_j\right] + H.c.$$

$$< H^2_u >= v_2, < H^1_d >= v_1 \quad \tan\beta \equiv v_2/v_1 \quad \epsilon_{ab} = -\epsilon_{ba},\ \epsilon_{12} = 1$$

Insert the vevs

$$\mathcal{L}_{mass} = -v_1 \tilde{Y}^e_{ij} e_i e^c_j - v_2 \tilde{Y}^\nu_{ij} \nu_i \nu^c_j - \frac{1}{2}\tilde{M}^{ij}_{RR}\nu^c_i \nu^c_j + H.c.$$

Rewrite in terms of L and R chiral fields, in matrix notation

$$\mathcal{L}_{mass} = -\bar{e}_L v_1 \tilde{Y^e}^* e_R - \bar{\nu}_L v_2 \tilde{Y^\nu}^* \nu_R - \frac{1}{2}\nu^T_R \tilde{M}^*_{RR}\nu_R + H.c.$$

$$= -\overline{e}_L m^E_{LR} e_R - \overline{\nu}_L m^\nu_{LR} \nu_R - \tfrac{1}{2}\overline{\nu}^c_R M_{RR} \nu_R + H.c.$$

The See-Saw Matrix

Type II contribution (ignored here)

Dirac matrix

$$\begin{pmatrix} \overline{\nu_L} & \overline{\nu_R^c} \end{pmatrix} \begin{pmatrix} m_{LL}^{II} & m_{LR} \\ m_{LR}^T & M_{RR} \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}$$

Heavy Majorana matrix

Diagonalise to give effective mass $\rightarrow m_{LL} \overline{\nu}_L \nu_L^c$

Light Majorana matrix $\longrightarrow$ $m_{LL}^{\nu} = m_{LL}^{II} - m_{LR} M_{RR}^{-1} m_{LR}^T$

Lepton mixing matrix V_{MNS}

Neutrino mass matrix (Majorana)

$$V^{E_L} m^E_{LR} V^{E_R \dagger} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad V^{\nu_L} m^\nu_{LL} V^{\nu_L T} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

Defined as $V_{MNS} = V^{E_L} V^{\nu_L \dagger}$

Can be parametrised as $V_{MNS} = V_{23} V_{13} V_{12}$

$$V_{MNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{-i\delta_{23}} \\ 0 & -s_{23} e^{i\delta_{23}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} e^{-i\delta_{12}} & 0 \\ -s_{12} e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Atmospheric — Reactor — Solar

Oscillation phase $\delta = \delta_{13} - \delta_{23} - \delta_{12}$

Quark mixing matrix V_{CKM}

$$V^{U_L} m^U_{LR} V^{U_R \dagger} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \qquad V^{D_L} m^D_{LR} V^{D_R \dagger} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$$

Defined as $V_{CKM} = V^{U_L} V^{D_L \dagger}$

Can be parametrised as $V_{CKM} = R_{23} V_{13} R_{12}$

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Im}\left[V_{ij} V_{kl} V^*_{il} V^*_{kj}\right] = J \sum_{m,n} \varepsilon_{ikm} \varepsilon_{jln}$$

$$J_{CP} = c^2_{13} c_{12} c_{23} \sin\delta\, s_{12} s_{23} s_{13}$$ Phase convention independent

2. From low energy data to high energy data

This Cabibbo-Kobayashi-Maskawa (CKM) matrix [1,2] is a 3×3 unitary matrix. It can be parameterized by three mixing angles and a CP-violating phase. Of the many possible parameterizations, a standard choice is [3]

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \qquad (11.3)$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, and δ is the KM phase [2] responsible for all CP-violating phenomena in flavor changing processes in the SM. The angles θ_{ij} can be chosen to lie in the first quadrant, so $s_{ij}, c_{ij} \geq 0$.

It is known experimentally that $s_{13} \ll s_{23} \ll s_{12} \ll 1$, and it is convenient to exhibit this hierarchy using the Wolfenstein parameterization. We define [4–6]

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \qquad s_{23} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right|,$$

$$s_{13}e^{i\delta} = V_{ub}^* = A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar\rho + i\bar\eta)\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}[1 - A^2\lambda^4(\bar\rho + i\bar\eta)]}. \qquad (11.4)$$

These ensure that $\bar\rho + i\bar\eta = -(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)$ is phase-convention independent and the CKM matrix written in terms of λ, A, $\bar\rho$ and $\bar\eta$ is unitary to all orders in λ. The definitions of $\bar\rho, \bar\eta$ reproduce all approximate results in the literature. *E.g.*, $\bar\rho = \rho(1 - \lambda^2/2 + \ldots)$ and we can write V_{CKM} to $\mathcal{O}(\lambda^4)$ either in terms of $\bar\rho, \bar\eta$ or, traditionally,

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \qquad (11.5)$$

(From Particle Data Book)

Quark data (low energy)

Ross and Serna

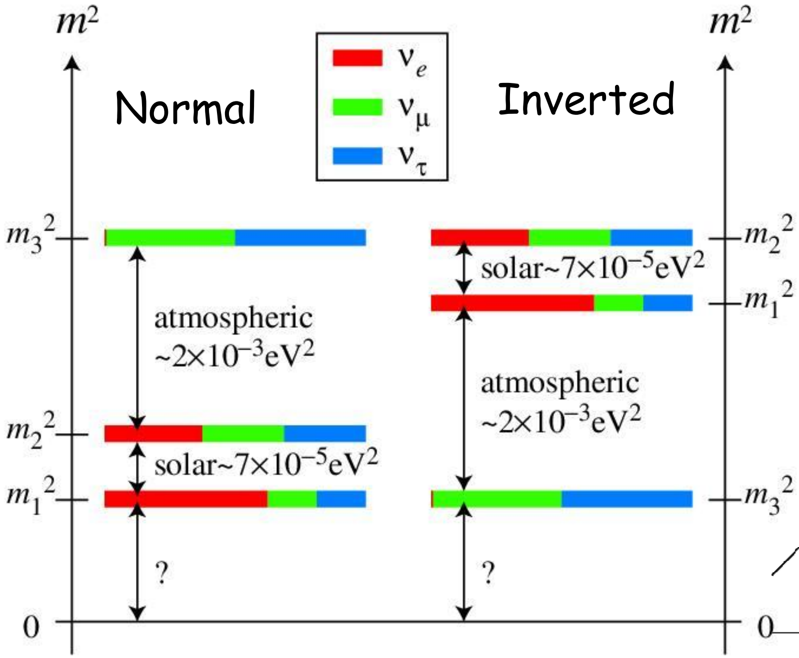
Low-Energy Parameter	Value(Uncertainty in last digit(s))	Notes and Reference
$m_u(\mu_L)/m_d(\mu_L)$	0.45(15)	PDB Estimation [1]
$m_s(\mu_L)/m_d(\mu_L)$	19.5(1.5)	PDB Estimation [1]
$m_u(\mu_L)+m_d(\mu_L)$	[8.8(3.0), 7.6(1.6)] MeV	PDB, Quark Masses, pg 15 [1]. (Non-lattice, Lattice)
$Q=\sqrt{\frac{m_s^2-(m_d+m_u)^2/4}{m_d^2-m_u^2}}$	22.8(4)	Martemyanov and Sopov [2]
$m_s(\mu_L)$	[103(20) , 95(20)] MeV	PDB, Quark Masses, pg 15 [1]. [Non-lattice, lattice]
$m_u(\mu_L)$	3(1) MeV	PDB, Quark Masses, pg 15 [1]. Non-lattice.
$m_d(\mu_L)$	6.0(1.5) MeV	PDB, Quark Masses, pg 15 [1]. Non-lattice.
$m_c(m_c)$	1.24(09) GeV	PDB, Quark Masses, pg 16 [1]. Non-lattice.
$m_b(m_b)$	4.20(07) GeV	PDB, Quark Masses, pg 16,19 [1]. Non-lattice.
M_t	170.9 (1.9) GeV	CDF & D0 [3] Pole Mass
(M_e, M_μ, M_τ)	(0.511(15), 105.6(3.1), 1777(53)) MeV	3% uncertainty from neglecting Y^e thresholds.
A Wolfenstein parameter	0.818(17)	PDB Ch 11 Eq. 11.25 [1]
$\overline{\rho}$ Wolfenstein parameter	0.221(64)	PDB Ch 11 Eq. 11.25 [1]
λ Wolfenstein parameter	0.2272(10)	PDB Ch 11 Eq. 11.25 [1]
$\overline{\eta}$ Wolfenstein parameter	0.340(45)	PDB Ch 11 Eq. 11.25 [1]
$\lvert V_{CKM}\rvert$	$\begin{pmatrix} 0.97383(24) & 0.2272(10) & 0.00396(09) \\ 0.2271(10) & 0.97296(24) & 0.04221(80) \\ 0.00814(64) & 0.04161(78) & 0.999100(34) \end{pmatrix}$	PDB Ch 11 Eq. 11.26 [1]
$\sin 2\beta$ from CKM	0.687(32)	PDB Ch 11 Eq. 11.19 [1]
Jarlskog Invariant	$3.08(18)\times 10^{-5}$	PDB Ch 11 Eq. 11.26 [1]
$v_{Higgs}(M_Z)$	246.221(20) GeV	Uncertainty expanded. [1]
$(\ \alpha_{EM}^{-1}(M_Z),\ \alpha_s(M_Z),\ \sin^2\theta_W(M_Z)\ )$	(127.904(19), 0.1216(17), 0.23122(15))	PDB Sec 10.6 [1]

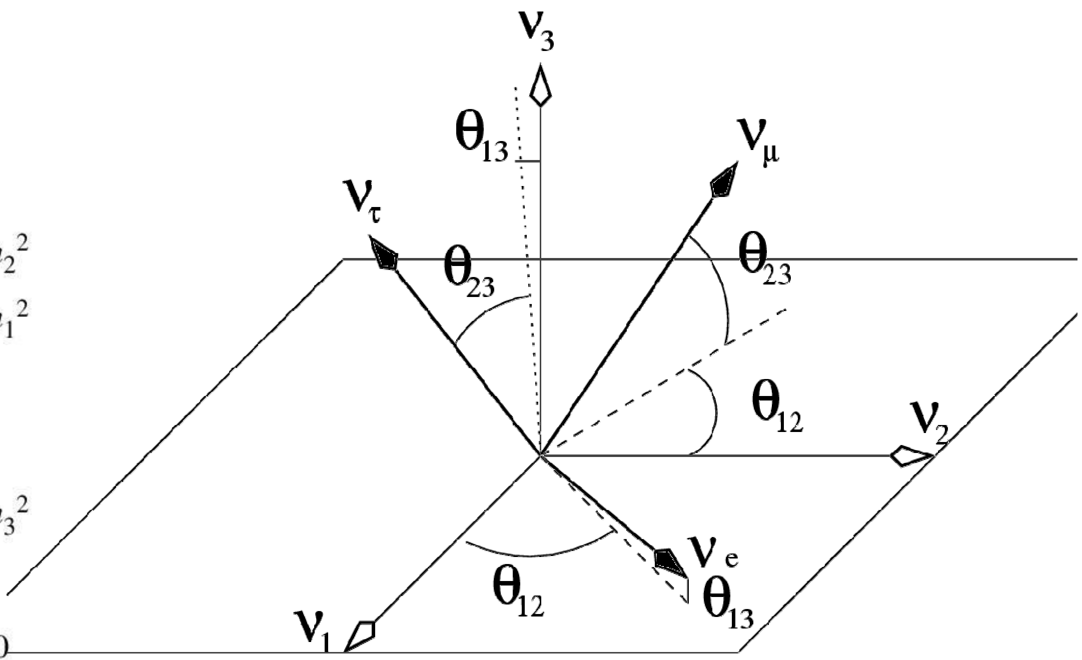
Neutrino Masses and Mixings

Andre de Gouvea

parameter	best fit	2σ
Δm^2_{21} $[10^{-5}\text{eV}^2]$	7.9	7.3–8.5
Δm^2_{31} $[10^{-3}\text{eV}^2]$	2.2	1.7–2.9

Parameter	Best-fit value	3σ range
θ_{12}	33.2°	28.7° .. 38.1°
θ_{23}	45.0°	35.7° .. 55.6°
θ_{13}	2.6°	0° .. 12.5°

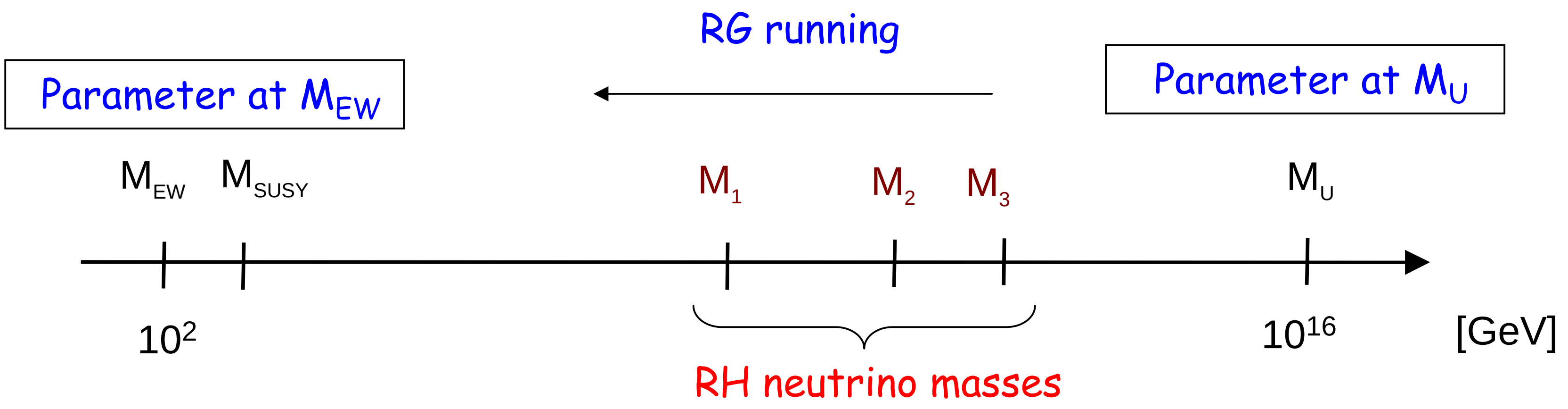




c.f. quark mixing angles

$$\theta_{12} = 13^\circ \pm 0.1^\circ$$
$$\theta_{23} = 2.4^\circ \pm 0.1^\circ$$
$$\theta_{13} = 0.20^\circ \pm 0.05^\circ$$

Renormalisation Group running



RGEs for gauge couplings
(to one loop accuracy)

$$\frac{dg_a}{dt} = \frac{g_a^3}{16\pi^2} b_a$$

where $t = \ln(\mu/M_X)$ (μ is the $\overline{MS}$ scale and M_X is the high energy scale)

SM beta functions

$$b_i = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix} + N_{Fam} \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + N_{Higgs} \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix}$$

$$b_a = (\tfrac{41}{10}, -\tfrac{19}{6}, -7)$$

MSSM beta functions

$$b_i = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + N_{Fam} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + N_{Higgs} \begin{pmatrix} 3/10 \\ 1/2 \\ 0 \end{pmatrix}$$

$$b_a = (\tfrac{33}{5}, 1, -3)$$

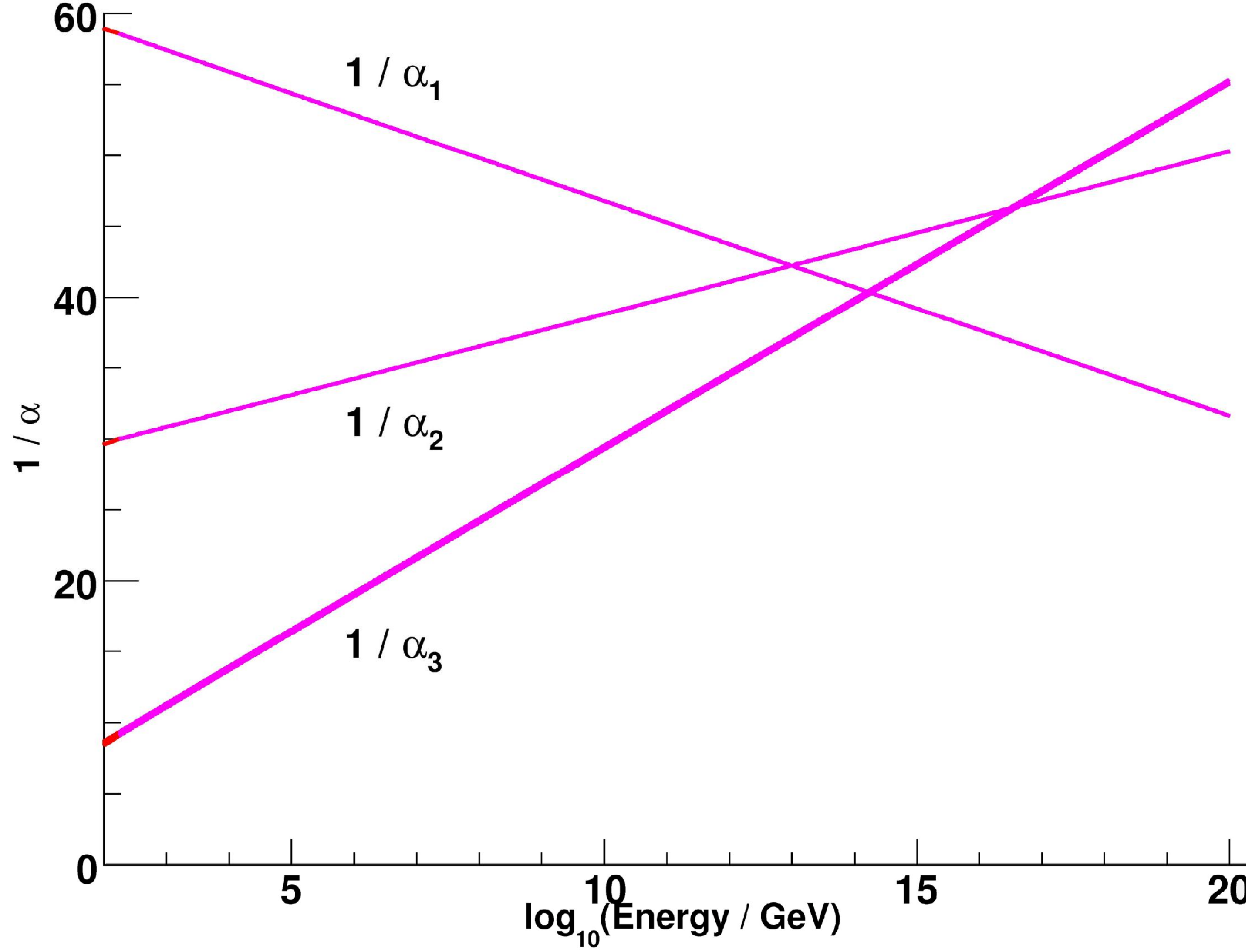
SM couplings at low energy

Latest coupling constant measurements at M_Z energy scale:

- $\alpha_1\ (M_Z)\ (\overline{MS}) = 0.016947(6)\ (RPP\ 2006)$
- $\alpha_2\ (M_Z)\ (\overline{MS}) = 0.033813(27)\ (RPP\ 2006)$
- $\alpha_3\ (M_Z)\ (\overline{MS}) = 0.1187(20)\ (RPP\ 2006)$

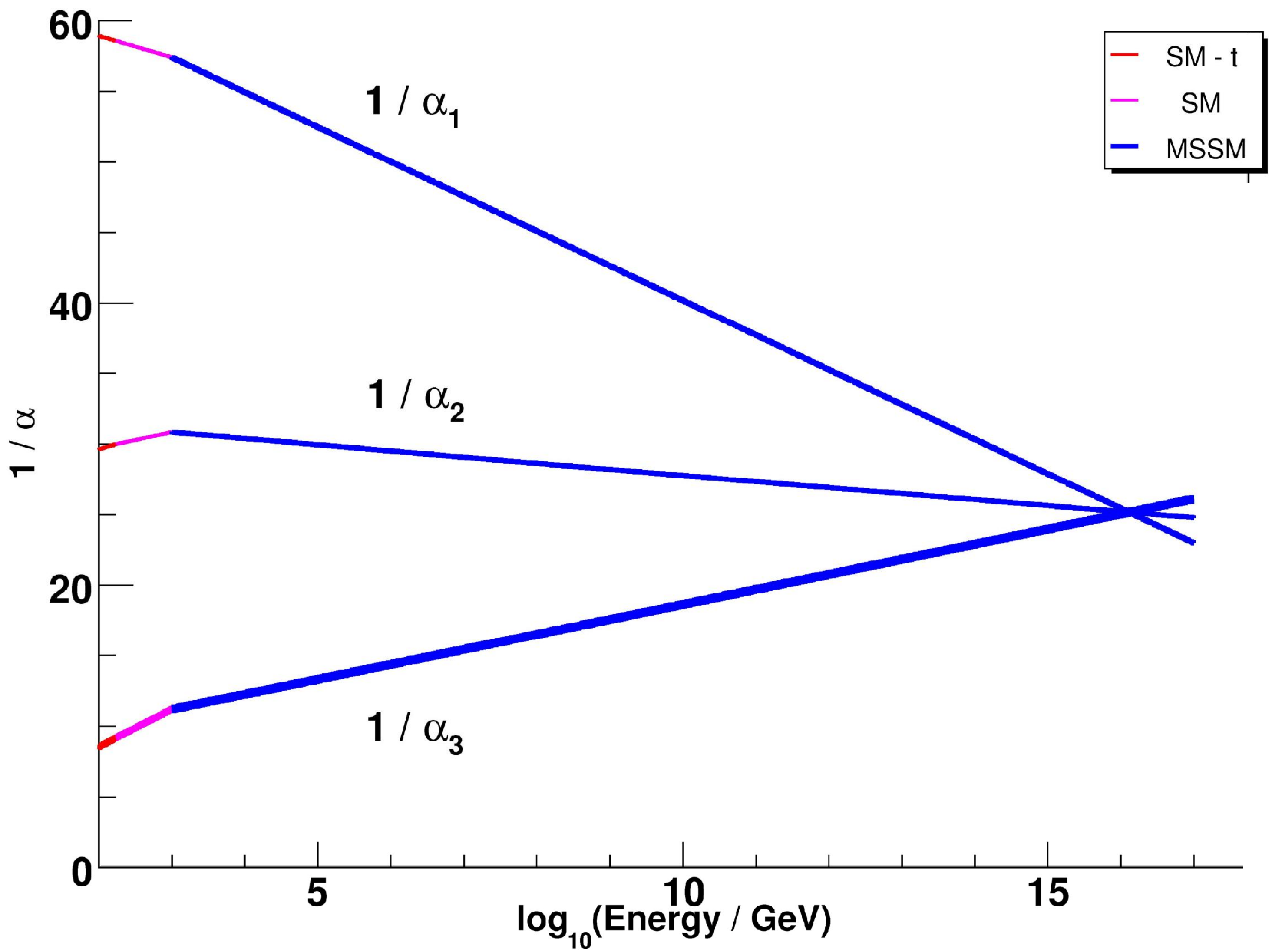
Evolution of SM couplings

Two-loop RGEs for the SM:



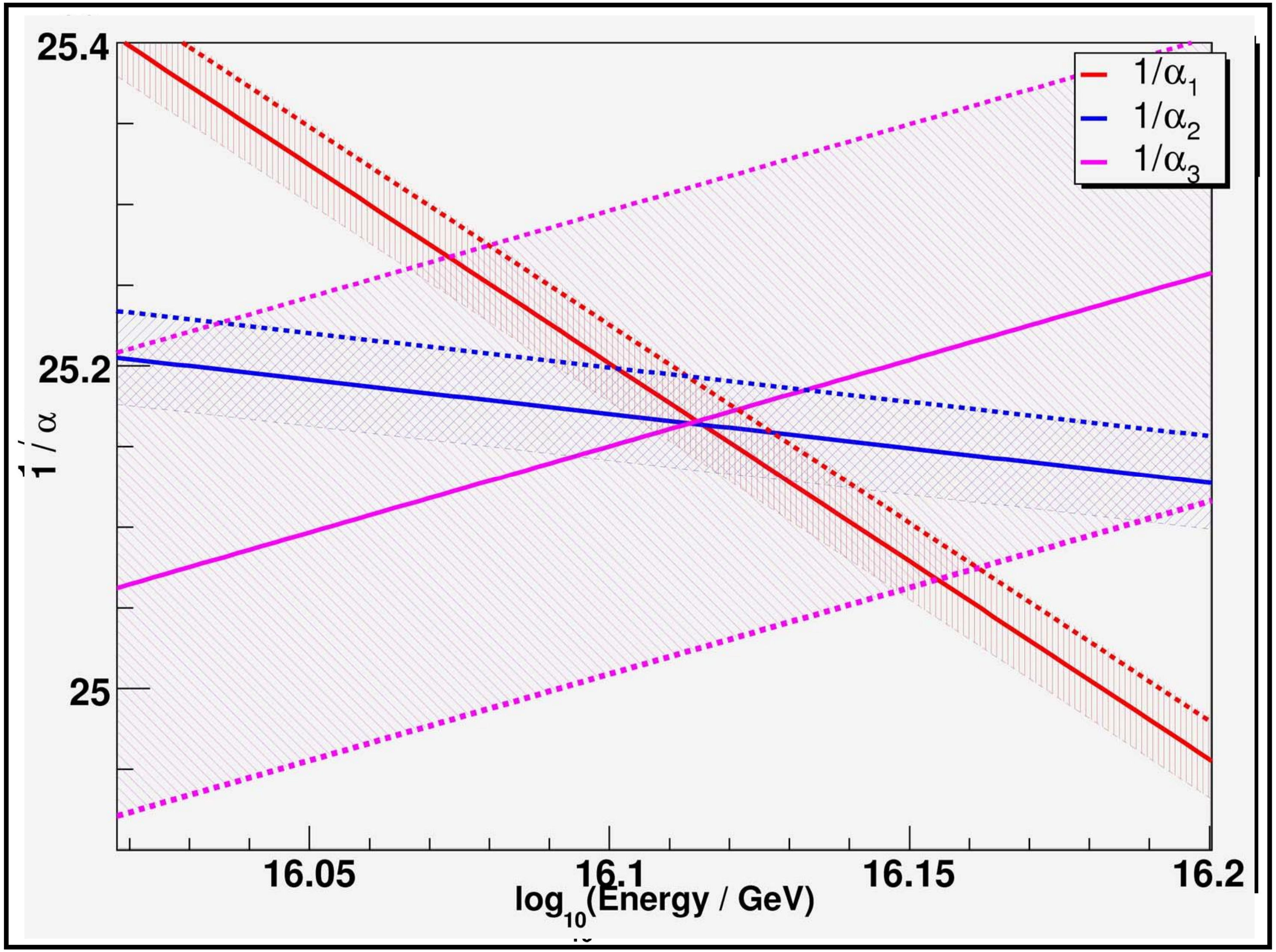
MSSM

Two-loop RGEs for the MSSM with 1 TeV effective SUSY threshold:



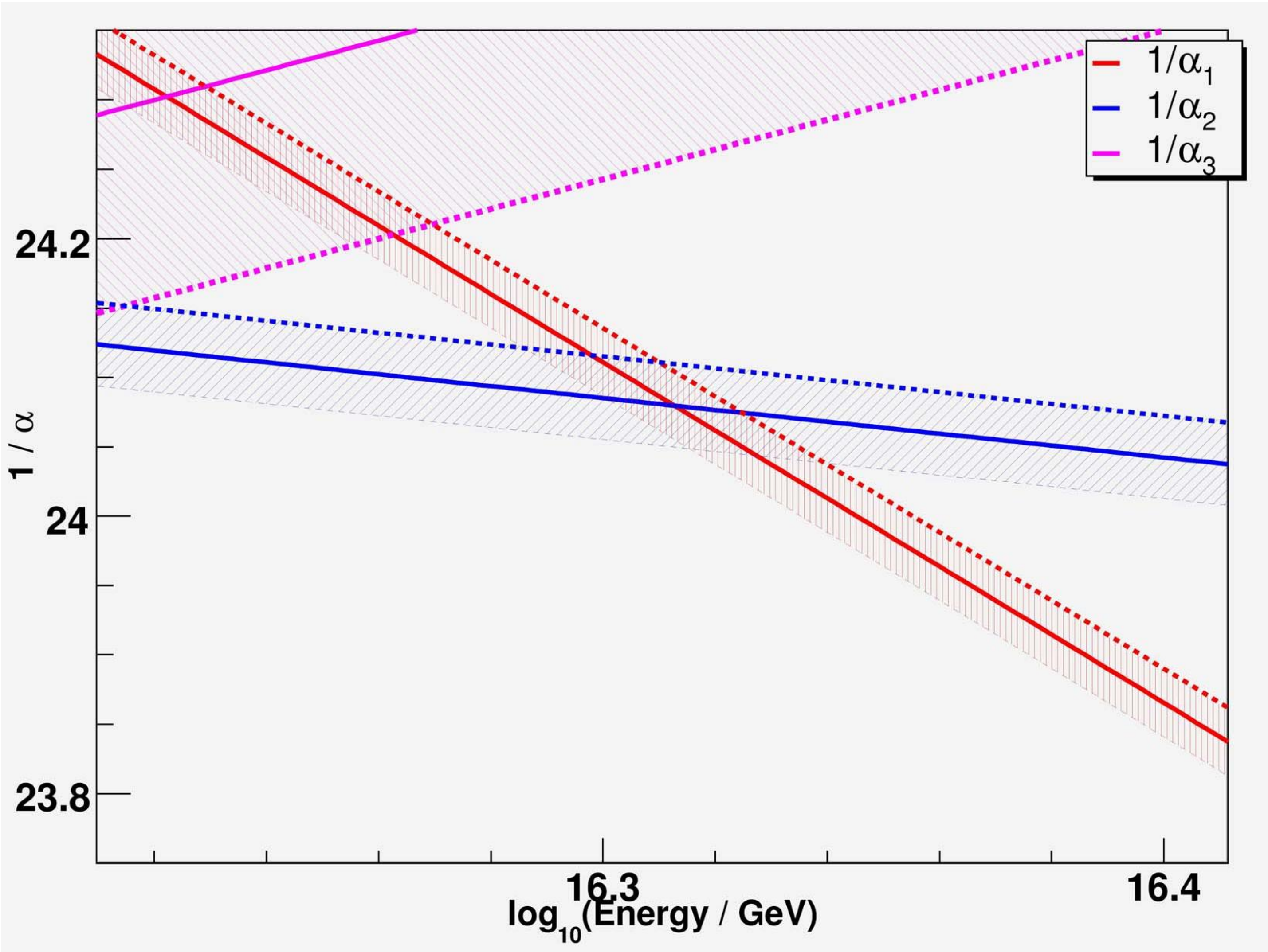
MSSM

Two-loop RGEs for the MSSM with 1 TeV effective SUSY threshold:



MSSM

Two-loop RGEs for the MSSM with 250 GeV effective SUSY threshold:



RGEs for t,b,τ in the MSSM

$$Y_u \approx \begin{pmatrix} 0 & & \\ & 0 & \\ & & Y_t \end{pmatrix}, \quad Y_d \approx \begin{pmatrix} 0 & & \\ & 0 & \\ & & Y_b \end{pmatrix}, \quad Y_e \approx \begin{pmatrix} 0 & & \\ & 0 & \\ & & Y_\tau \end{pmatrix}$$

$$\begin{aligned}
\frac{dY_t}{dt} &= \frac{1}{16\pi^2} Y_t [6|Y_t|^2 + |Y_b|^2 - (\frac{16}{3} g_3^2 + 3g_2^2 + \frac{13}{15} g_1^2)] \\
\frac{dY_b}{dt} &= \frac{1}{16\pi^2} Y_b [6|Y_b|^2 + |Y_t|^2 + |Y_\tau|^2 - (\frac{16}{3} g_3^2 + 3g_2^2 + \frac{7}{15} g_1^2)] \\
\frac{dY_\tau}{dt} &= \frac{1}{16\pi^2} Y_\tau [4|Y_\tau|^2 + 3|Y_b|^2 - (3g_2^2 + \frac{9}{5} g_1^2)],
\end{aligned}$$

RGEs for Yukawa matrices in MSSM

RGEs (one-loop accuracy)

$$\frac{dY_u}{dt} = \frac{1}{16\pi^2}[N_q.Y_u + Y_u.N_u + (N_{H_u})Y_u]$$
$$\frac{dY_d}{dt} = \frac{1}{16\pi^2}[N_q.Y_d + Y_d.N_d + (N_{H_d})Y_d]$$
$$\frac{dY_\nu}{dt} = \frac{1}{16\pi^2}[N_l.Y_\nu + Y_\nu.N_\nu + (N_{H_u})Y_\nu]$$
$$\frac{dY_e}{dt} = \frac{1}{16\pi^2}[N_l.Y_e + Y_e.N_e + (N_{H_d})Y_e]$$

Wavefunction anomalous dimensions

$$N_q = Y_u Y_u{}^\dagger + Y_d Y_d{}^\dagger - (\frac{8}{3}g_3^2 + \frac{3}{2}g_2^2 + \frac{1}{30}g_1^2)\hat{1}$$
$$N_u = 2Y_u{}^\dagger Y_u - (\frac{8}{3}g_3^2 + \frac{8}{15}g_1^2)\hat{1}$$
$$N_d = 2Y_d{}^\dagger Y_d - (\frac{8}{3}g_3^2 + \frac{2}{15}g_1^2)\hat{1}$$
$$N_l = Y_e Y_e{}^\dagger + Y_\nu Y_\nu{}^\dagger - (\frac{3}{2}g_2^2 + \frac{3}{10}g_1^2)\hat{1}$$
$$N_e = 2Y_e{}^\dagger Y_e - \frac{6}{5}g_1^2\hat{1}$$
$$N_\nu = 2Y_\nu{}^\dagger Y_\nu$$
$$N_{H_u} = 3\mathrm{Tr}(Y_u{}^\dagger Y_u) + \mathrm{Tr}(Y_\nu{}^\dagger Y_\nu) - (\frac{3}{2}g_2^2 + \frac{3}{10}g_1^2)$$
$$N_{H_d} = 3\mathrm{Tr}(Y_d{}^\dagger Y_d) + \mathrm{Tr}(Y_e{}^\dagger Y_e) - (\frac{3}{2}g_2^2 + \frac{3}{10}g_1^2)$$

Charged fermion data (high energy)

Ross and Serna

SUSY thresholds

Parameters	Input SUSY Parameters					
$\tan\beta$	1.3	10	38	50	38	38
γ_b	0	0	0	0	-0.22	$+0.22$
γ_d	0	0	0	0	-0.21	$+0.21$
γ_t	0	0	0	0	0	-0.44
Parameters	**Corresponding GUT-Scale Parameters with Propagated Uncertainty**					
$y^t(M_X)$	6^{+1}_{-5}	0.48(2)	0.49(2)	0.51(3)	0.51(2)	0.51(2)
$y^b(M_X)$	$0.0113^{+0.0002}_{-0.01}$	0.051(2)	0.23(1)	0.37(2)	0.34(3)	0.34(3)
$y^\tau(M_X)$	0.0114(3)	0.070(3)	0.32(2)	0.51(4)	0.34(2)	0.34(2)
$(m_u/m_c)(M_X)$	0.0027(6)	0.0027(6)	0.0027(6)	0.0027(6)	0.0026(6)	0.0026(6)
$(m_d/m_s)(M_X)$	0.051(7)	0.051(7)	0.051(7)	0.051(7)	0.051(7)	0.051(7)
$(m_e/m_\mu)(M_X)$	0.0048(2)	0.0048(2)	0.0048(2)	0.0048(2)	0.0048(2)	0.0048(2)
$(m_c/m_t)(M_X)$	$0.0009^{+0.001}_{-0.00006}$	0.0025(2)	0.0024(2)	0.0023(2)	0.0023(2)	0.0023(2)
$(m_s/m_b)(M_X)$	0.014(4)	0.019(2)	0.017(2)	0.016(2)	0.018(2)	0.010(2)
$(m_\mu/m_\tau)(M_X)$	0.059(2)	0.059(2)	0.054(2)	0.050(2)	0.054(2)	0.054(2)
$A(M_X)$	$0.56^{+0.34}_{-0.01}$	0.77(2)	0.75(2)	0.72(2)	0.73(3)	0.46(3)
$\lambda(M_X)$	0.227(1)	0.227(1)	0.227(1)	0.227(1)	0.227(1)	0.227(1)
$\bar\rho(M_X)$	0.22(6)	0.22(6)	0.22(6)	0.22(6)	0.22(6)	0.22(6)
$\bar\eta(M_X)$	0.33(4)	0.33(4)	0.33(4)	0.33(4)	0.33(4)	0.33(4)
$J(M_X)\ \times 10^{-5}$	$1.4^{+2.2}_{-0.2}$	2.6(4)	2.5(4)	2.3(4)	2.3(4)	1.0(2)
Parameters	**Comparison with GUT Mass Ratios**					
$(m_b/m_\tau)(M_X)$	$1.00^{+0.04}_{-0.4}$	0.73(3)	0.73(3)	0.73(4)	1.00(4)	1.00(4)
$(3m_s/m_\mu)(M_X)$	$0.70^{+0.8}_{-0.05}$	0.69(8)	0.69(8)	0.69(8)	0.9(1)	0.6(1)
$(m_d/3\,m_e)(M_X)$	0.82(7)	0.83(7)	0.83(7)	0.83(7)	1.05(8)	0.68(6)
$(\frac{\det Y^d}{\det Y^e})(M_X)$	$0.57^{+0.08}_{-0.26}$	0.42(7)	0.42(7)	0.42(7)	0.92(14)	0.39(7)

3. Textures in a basis

Hierarchical Symmetric Textures

Symmetric hierarchical matrices with 11 texture zero motivated by

$$m_{LR} = \begin{pmatrix} 0 & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \longrightarrow |V_{us}| \approx \left| \sqrt{\frac{m_d}{m_s}} \right| \approx \lambda$$

Gatto et al

This motivates the symmetric down texture at GUT scale of form

$$Y_{LR}^{d} \sim \begin{pmatrix} 0 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \qquad |V_{cb}| \approx \left| \frac{(m_{LR}^{D})_{23}}{m_b} \right| \approx \lambda^2 \qquad |V_{ub}| \approx \left| \frac{(m_{LR}^{D})_{13}}{m_b} \right| \approx \lambda^3$$

$\lambda \approx 0.2$ is the Wolfenstein Parameter

Up quarks are more hierarchical than down quarks

This suggests different expansion parameters for up and down

$$\frac{m_{LR}^{D}}{m_b} \sim \begin{pmatrix} 0 & \bar{\varepsilon}^3 & \bar{\varepsilon}^3 \\ \bar{\varepsilon}^3 & \bar{\varepsilon}^2 & \bar{\varepsilon}^2 \\ \bar{\varepsilon}^3 & \bar{\varepsilon}^2 & 1 \end{pmatrix} \quad \bar{\varepsilon} \sim 0.15$$

$$\frac{m_{LR}^{U}}{m_t} = \begin{pmatrix} 0 & \varepsilon^3 & \varepsilon^3 \\ \varepsilon^3 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix} \quad \varepsilon \sim 0.05$$

$$m_d : m_s : m_b = \bar{\varepsilon}^4 : \bar{\varepsilon}^2 : 1$$

$$m_u : m_c : m_t = \varepsilon^4 : \varepsilon^2 : 1$$

Detailed fits require numerical (order unity) coefficients

$$Y^d(M_X) = y_{33}^d \begin{pmatrix} d\,\epsilon_d^4 & b\,\epsilon_d^3 & c\,\epsilon_d^3 \\ b\,\epsilon_d^3 & f\,\epsilon_d^2 & a\,\epsilon_d^2 \\ c\,\epsilon_d^3 & a\,\epsilon_d^2 & 1 \end{pmatrix} \quad Y^u(M_X) = y_{33}^u \begin{pmatrix} d'\,\epsilon_u^4 & b'\,\epsilon_u^3 & c'\,\epsilon_u^3 \\ b'\,\epsilon_u^3 & f'\,\epsilon_u^2 & a'\,\epsilon_u^2 \\ c'\,\epsilon_u^3 & a'\,\epsilon_u^2 & 1 \end{pmatrix}$$

Detailed fits at the GUT Scale

Ross and Serna

No SUSY thresholds

Parameter	2001 RRRV	Fit A0	Fit B0	Fit A1	Fit B1	Fit A2	Fit B2
$\tan\beta$	Small	1.3	1.3	38	38	38	38
a'	$\mathcal{O}(1)$	0	0	0	0	-2.0	-2.0
ϵ_u	0.05	0.030(1)	0.030(1)	0.0491(16)	0.0491(15)	0.0493(16)	0.0493(14)
ϵ_d	0.15(1)	0.117(4)	0.117(4)	0.134(7)	0.134(7)	0.132(7)	0.132(7)
$\|b'\|$	1.0	1.75(20)	1.75(21)	1.05(12)	1.05(13)	1.04(12)	1.04(13)
$\arg(b')$	90^o	$+93(16)^o$	$-93(13)^o$	$+91(16)^o$	$-91(13)^o$	$+93(16)^o$	$-93(13)^o$
a	1.31(14)	2.05(14)	2.05(14)	2.16(23)	2.16(24)	1.92(21)	1.92(22)
b	1.50(10)	1.92(14)	1.92(15)	1.66(13)	1.66(13)	1.70(13)	1.70(13)
$\|c\|$	0.40(2)	0.85(13)	2.30(20)	0.78(15)	2.12(36)	0.83(17)	2.19(38)
$\arg(c)$	$-24(3)^o$	$-39(18)^o$	$-61(14)^o$	$-43(14)^o$	$-59(13)^o$	$-37(25)^o$	$-60(13)^o$

$$c' = d' = d = 0 \text{ and } f = f' = 1$$

$$Y^u(M_X) = y_{33}^u \begin{pmatrix} d'\epsilon_u^4 & b'\epsilon_u^3 & c'\epsilon_u^3 \\ b'\epsilon_u^3 & f'\epsilon_u^2 & a'\epsilon_u^2 \\ c'\epsilon_u^3 & a'\epsilon_u^2 & 1 \end{pmatrix} \quad Y^d(M_X) = y_{33}^d \begin{pmatrix} d\,\epsilon_d^4 & b\,\epsilon_d^3 & c\,\epsilon_d^3 \\ b\,\epsilon_d^3 & f\,\epsilon_d^2 & a\,\epsilon_d^2 \\ c\,\epsilon_d^3 & a\,\epsilon_d^2 & 1 \end{pmatrix}.$$

Ross and Serna

With SUSY thresholds

Parameter	A	B	C	B2	C2
$\tan\beta$	30	38	38	38	38
γ_b	0.20	-0.22	$+0.22$	-0.22	$+0.22$
γ_t	-0.03	0	-0.44	0	-0.44
γ_d	0.20	-0.21	$+0.21$	-0.21	$+0.21$
a'	0.0	0.0	0.0	-2	-2
ϵ_u	0.0495(17)	0.0483(16)	0.0483(18)	0.0485(17)	0.0485(18)
ϵ_d	0.131(7)	0.128(7)	0.102(9)	0.127(7)	0.101(9)
$\lvert b'\rvert$	1.04(12)	1.07(12)	1.07(11)	1.05(12)	1.06(10)
$\arg(b')$	$90(12)^o$	$91(12)^o$	$93(12)^o$	$95(12)^o$	$95(12)^o$
a	2.17(24)	2.27(26)	2.30(42)	2.03(24)	1.89(35)
b	1.69(13)	1.73(13)	2.21(18)	1.74(10)	2.26(20)
$\lvert c\rvert$	0.80(16)	0.86(17)	1.09(33)	0.81(17)	1.10(35)
$\arg(c)$	$-41(18)^o$	$-42(19)^o$	$-41(14)^o$	$-53(10)^o$	$-41(12)^o$
Y^u_{33}	0.48(2)	0.51(2)	0.51(2)	0.51(2)	0.51(2)
Y^d_{33}	0.15(1)	0.34(3)	0.34(3)	0.34(3)	0.34(3)
Y^e_{33}	0.23(1)	0.34(2)	0.34(2)	0.34(2)	0.34(2)
$(m_b/m_\tau)(M_X)$	0.67(4)	1.00(4)	1.00(4)	1.00(4)	1.00(4)
$(3m_s/m_\mu)(M_X)$	0.60(3)	0.9(1)	0.6(1)	0.9(1)	0.6(1)
$(m_d/3m_e)(M_X)$	0.71(7)	1.04(8)	0.68(6)	1.04(8)	0.68(6)
$\left\lvert \frac{\det Y^d(M_X)}{\det Y^e(M_X)} \right\rvert$	0.3(1)	0.92(14)	0.4(1)	0.92(14)	0.4(1)

Georgi-Jarlskog

$$Y^d(M_X) = y^d_{33}\begin{pmatrix} d\,\varepsilon_d^4 & 1.7\,\varepsilon_d^3 & e^{-i\pi/4}\,\varepsilon_d^3 \\ 1.7\,\varepsilon_d^3 & \varepsilon_d^2 & 2\,\varepsilon_d^2 \\ e^{-i\pi/4}\,\varepsilon_d^3 & 2\,\varepsilon_d^2 & 1 \end{pmatrix}$$

$$\frac{m_b}{m_\tau}(M_{GUT}) = 1, \quad \frac{m_s}{m_\mu}(M_{GUT}) = \frac{1}{3}, \quad \frac{m_d}{m_e}(M_{GUT}) = 3$$

Final remarks on choice of basis

We have considered a particular choice of quark texture in a particular basis

$$\frac{m_{LR}^D}{m_b} \sim \begin{pmatrix} 0 & \bar{\varepsilon}^3 & \bar{\varepsilon}^3 \\ \bar{\varepsilon}^3 & \bar{\varepsilon}^2 & \bar{\varepsilon}^2 \\ \bar{\varepsilon}^3 & \bar{\varepsilon}^2 & 1 \end{pmatrix} \quad \bar{\varepsilon} \sim 0.15 \qquad \frac{m_{LR}^U}{m_t} \sim \begin{pmatrix} 0 & \varepsilon^3 & \varepsilon^3 \\ \varepsilon^3 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix} \quad \varepsilon \sim 0.05$$

But it is shown in the Appendix that all choices of quark mass matrices that lead to the same quark masses and mixing angles may be related to each other under a change of basis.

For example all quark mass matrices are equivalent to the choice

$$m_{LR}^D \sim V_{CKM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad m_{LR}^U \sim \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

However this is only true in the Standard Model, and a given high energy theory of flavour will select a particular preferred basis. Also in the see-saw mechanism all choices of see-saw matrices are NOT equivalent.

Appendix 1 References

W. De Boer hep-ph/9402266

S.Raby ICTP Lectures 1994

G.G.Ross ICTP Lectures 2001

J.C. Pati ICTP Lectures 2001

S. Barr ICTP Lectures 2003

S. Raby hep-ph/0401115

S.Raby PDB 2006

A. Ceccucci et al PDB 2006

G. Ross and M. Serna 0704.1248

D. Chung et al hep-ph/0312378

S.F. King hep-ph/0310204

S.F. King hep-ph/0610239

Appendix 2 Basis Changing

2.1 Quark sector

2.2 Effective Majorana sector

2.3 See-saw sector

2.1 Quark sector

In the quark sector the Dirac mass matrices of the up and down quarks are given by $m^U_{LR} = Y^U_{LR} v_u$, and $m^D_{LR} = Y^D_{LR} v_d$ where $v_u = \langle H^0_u \rangle$ and $v_d = \langle H^0_d \rangle$, and the Lagrangian is of the form $\mathcal{L} = -\bar{\psi}_L Y_{LR} H \psi_R + H.c.$ The change from flavour basis to mass eigenstate basis can be performed with the unitary diagonalization matrices V_{U_L}, V_{U_R} and V_{D_L},V_{D_R} by

$$V_{U_L}\, m^U_{LR}\, V^\dagger_{U_R} = \mathrm{diag}(m_u, m_c, m_t), \qquad V_{D_L}\, m^D_{LR}\, V^\dagger_{D_R} = \mathrm{diag}(m_d, m_s, m_b). \tag{1}$$

The CKM mixing matrix is then obtained from

$$V'_{CKM} = V_{U_L} V^\dagger_{D_L} \tag{2}$$

where quark phase rotations which leave the quark masses real and positive may be used to remove five of the phases leaving one physical phase in the CKM matrix V_{CKM}. The Standard Model quark sector clearly respects the symmetry

$$G_{quark} = U_Q(3) \times U_{U_R}(3) \times U_{D_R}(3) \tag{3}$$

corresponding to quark doublet, right-handed up quark and right-handed down quark rotations, which change the quark basis and the form of the Yukawa matrices, but leave the physics (quark masses and mixings) unchanged. In the quark sector it is well known that the only physical quantities are basis independent invariants formed from the mass matrices, the so-called Jarlkog invariants [6], rather than the mass matrices themselves, since any pair of quark mass matrices which lead to the correct physics may be related to any other pair which lead to the same physics, by a change of basis, up to quark phases, using the symmetry G_{quark}.

This can be proved, for example, by showing that any two pairs of quark mass matrices can be related by a change of basis, using the symmetry G_{quark}, to a common basis in which the up quark mass matrix is diagonal, and the down quark mass matrix is equal, up to quark phases, to the CKM matrix multiplied by a diagonal matrix of down quark masses,

$$m_{LR}^{U}{}' = \mathrm{diag}(m_u, m_c, m_t), \qquad m_{LR}^{D}{}' = V'_{CKM}\mathrm{diag}(m_d, m_s, m_b). \tag{4}$$

Since any two pairs of mass matrices $(m_{LR}^U)_1, (m_{LR}^D)_1$ and $(m_{LR}^U)_2, (m_{LR}^D)_2$ may be related to $m_{LR}^{U}{}', m_{LR}^{D}{}'$ in Eq.4 by a change of basis, it follows that all choices of quark mass matrices which lead to the same physics can be related to each other, up to quark phases, using the symmetry G_{quark}. This implies that the quark mass matrices m_{LR}^U, m_{LR}^D are not physical quantities since they are basis dependent, i.e. not invariant under the symmetry G_{quark}. It is possible to define G_{quark} invariant combinations consisting of determinants and traces of products of the combinations $S_{LL}^U = m_{LR}^U(m_{LR}^U)^\dagger$ and $S_{LL}^D = m_{LR}^D(m_{LR}^D)^\dagger$, for example the determinant of the commutator $\det[S_{LL}^U, S_{LL}^D]$ is an invariant [6].

2.2 Effective lepton sector

From the point of view of low energy neutrino experiments, Majorana neutrino masses arise from the effective operator: $\mathcal{L}^{eff} = -\frac{1}{2} H_u L^T \kappa H_u L + H.c.$ where L are the lepton doublets, H_u are Higgs doublets, and κ is a matrix of effective (dimensional) couplings. In our convention the effective Majorana masses are given by the Lagrangian $\mathcal{L} = -\bar{\nu}_L m^{\nu}_{LL} \nu^c + H.c.$ where $m^{\nu}_{LL} = \kappa^* v_u^2$. The rotation to the mass eigenstate basis can be performed with the unitary diagonalization matrices V_{E_L}, V_{E_R} and V_{ν_L} by

$$V_{E_L}\, m^E_{LR}\, V^{\dagger}_{E_R} = \mathrm{diag}(m_e, m_\mu, m_\tau), \qquad V_{\nu_L}\, m^{\nu}_{LL}\, V^T_{\nu_L} = \mathrm{diag}(m_1, m_2, m_3). \tag{5}$$

The lepton mixing matrix is then obtained from

$$V'_{MNS} = V_{E_L} V^{\dagger}_{\nu_L} \tag{6}$$

where charged lepton phases rotations which leave the charged lepton masses real and positive may be used to remove three of the phases leaving three physical phases in the MNS matrix V_{MNS}.

The effective lepton sector clearly respects the symmetry

$$G^{eff}_{lepton} = U_L(3) \times U_{E_R}(3) \tag{7}$$

corresponding to lepton doublet and right-handed charged lepton rotations, which change the lepton basis and the form of the effective lepton matrices, but leave the physics (lepton masses and mixings) unchanged. The physically measurable low energy lepton parameters are the three charged lepton masses m_e, m_μ, m_τ, the three neutrino masses $m_{1,2,3} > 0$ and the lepton mixing parameters contained in V_{MNS}.

As in the quark sector, any pair of effective lepton matrices m^E_{LR}, m^ν_{LL} which lead to a given low energy physics may be related to any other pair which lead to the same physics, by a change of basis, using the symmetry G^{eff}_{lepton}. This is easily proved (analagous to the quark sector) by tranforming to a common basis in which the charged lepton mass matrix is diagonal, and the effective Majorana neutrino mass matrix is specified in terms of the lepton mixing matrix $V'_{MNS} = V_{E_L} V^\dagger_{\nu_L}$ and the physical neutrino masses m_i,

$$m^E_{LR}{}' = \mathrm{diag}(m_e, m_\mu, m_\tau), \qquad m^\nu_{LL}{}' = V'_{MNS}\mathrm{diag}(m_1, m_2, m_3) V^T_{MNS}{}' \tag{8}$$

where Eq.8, often called the "flavour basis", is analagous to Eq.4. Then, as in the quark case, we can argue that since any two pairs of matrices $(m^E_{LR})_1$, $(m^\nu_{LL})_1$ and $(m^E_{LR})_2$, $(m^\nu_{LL})_2$ can be rotated to the flavour basis then they can therefore be rotated into each other, using the symmetry G^{eff}_{lepton}, analagous to the quark sector result. m^E_{LR}, m^ν_{LL} are clearly basis dependent, but invariants under G^{eff}_{lepton} can be constructed using $S^E_{LL} = m^E_{LR}(m^E_{LR})^\dagger$ and $S^\nu_{LL} = m^\nu_{LL}(m^\nu_{LL})^\dagger$, for example the determinant of the commutator $\det[S^E_{LL}, S^\nu_{LL}]$ is invariant.

2.3 See-saw sector

The starting point of the see-saw mechanism is the Lagrangian,

$$\mathcal{L}_{seesaw} = -Y^E_{LR} H_d \overline{L} E_R - Y^\nu_{LR} H_u \overline{L} \nu_R + \frac{1}{2} \nu^T_R M_{RR} \nu_R + H.c. \tag{9}$$

where all indices have been suppressed, and we have introduced two Higgs doublets H_u, H_d as in the Supersymmetric Standard Model. [2] It is common to call Eq.9 the see-saw Lagrangian. After integrating out the right-handed neutrinos it leads to an effective low energy leptonic Lagrangian of the type discussed in the previous subsection where the effective Majorana mass matrix given by the (type I) see-saw formula:

$$m^\nu_{LL} = v_u^2 Y^\nu_{LR} M^{-1}_{RR} Y^{\nu\,T}_{LR}. \tag{10}$$

The effective low energy matrices are diagonalised by unitary transformations V_{E_L}, V_{E_R} and V_{ν_L} as in Eq.5, and the lepton mixing matrix is as in Eq.6.

The lepton symmetry of the see-saw Lagrangian in Eq.9 is:

$$G_{lepton} = U_L(3) \times U_{E_R}(3) \times U_{\nu_R}(3) \tag{11}$$

corresponding to lepton doublet, right-handed charged lepton and right-handed neutrino rotations, which change the lepton basis and the form of the see-saw matrices, but leave the physics (lepton masses and mixings) unchanged. Using these symmetries we can

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[2] In the case of the Standard Model one of the two Higgs doublets is equal to the charge conjugate of the other, $H_d \equiv H^c_u$.

ask the question whether all sets of see-saw matrices Y^E_{LR}, Y^ν_{LR} and M_{RR} which lead to a given set of low energy physical lepton parameters are equivalent to each other by a change of basis. Analagous to the quark sector, we may attempt to relate all sets of see-saw matrices to a common set of see-saw matrices in which the charged lepton mass matrix is diagonal, and the right-handed neutrino Majorana mass matrix is also diagonal,

$$v_d {Y^E_{LR}}' = \mathrm{diag}(m_e, m_\mu, m_\tau), \quad M'_{RR} = \mathrm{diag}(M_1, M_2, M_3), \quad {Y^\nu_{LR}}' = V_{E_L} Y^\nu_{LR} V^\dagger_{\nu_R} \quad (12)$$

where unitary V_{ν_R} is defined by $V_{\nu_R} M_{RR} V^T_{\nu_R} = M'_{RR}$ and $M_i > 0$.

We refer to the basis of Eq.12 as the "see-saw flavour basis" in analogy to Eq.8. The difference between Eqs.4,8 and Eq.12 is that here ${Y^\nu_{LR}}'$ is not uniquely specified since it is diagonalized by left-handed rotations which are not simply related to the lepton mixing matrix, and in addition its eigenvalues are not simply related to physical neutrino masses. Therefore, unlike the quark sector, or the effective lepton case, there is not a unique common basis. Therefore, we conclude that any two sets of see-saw matrices $(Y^E_{LR})_1$, $(Y^\nu_{LR})_1$, $(M_{RR})_1$ and $(Y^E_{LR})_2$, $(Y^\nu_{LR})_2$, $(M_{RR})_2$ which give the same physical right-handed neutrino masses, light effective neutrino masses, charged lepton masses and lepton mixings, cannot be transformed into each other under the lepton see-saw symmetry G_{lepton} corresponding to basis changes.