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Conference and School on Predictability of Natural Disasters for our Planet in Danger. A System View; Theory, Models, Data Analysis

25 June - 6 July, 2007

Dynamical Systems
Chaos in Deterministic Systems

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Dynamical Systems

Lecture 1
Chaos in Deterministic Systems

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A Systems View: Theory, Models and Data Analysis
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy
25 June – 6 July 2007

Dynamical Systems

1. Chaos in deterministic systems

2. Low-order dynamical systems and predictability

3. Attractors and bifurcations

Theory of Error Growth

1. Linear stability analysis

2. Structure of errors

3. Growth of random errors

Lecture Notes

George Mason University Course on Predictability

ICTP SMR 1849 Lectures on
Dynamical Systems (Krishnamurthy)
Theory of Error Growth (Krishnamurthy)
Predictability of Large Weather Models (Straus)
Climate Predictability (Straus)

are based on the course "Predictability of Weather and Climate" (CLIM 751) taught at the Department of Climate Dynamics, George Mason University, VA, USA.

More detailed notes of the George Mason course can be found at

http://mason.gmu.edu/~vkrishna/CLIM751.htm

ICTP Lectures

ICTP Lectures of Krishnamurthy (Dynamical Systems and Theory of Error Growth) can be found temporarily at

ftp://grads.iges.org/pub/krishna/ictp/

Prediction and Predictability

Lorenz, E. N., 1963: The predictability of hydrodynamic flows. *Trans. New York Acad. Sci.*, Ser II, **25**, 409-423.

Lorenz, E. N., 1984: Some aspects of atmospheric predictability, *Problems and prospects in long and medium range weather forecasting*, D. M. Burridge and E. Kallen, Eds., Springer-Verlag, 1-20.

Predicting the future state of the system consisting of atmosphere, ocean, land, etc. is the main goal of the field of weather and climate.

For both weather and climate, distinguished by their time scale, the present knowledge of this subject is based not only on studies of complex models of atmosphere, ocean, land etc. but also on studies of simple mathematical models.

It is also quite common knowledge that there is a lack of perfection in predicting weather and climate even with the use of complex models and high speed computers.

How predictable is the system?

The main concern will be the predictability of weather and climate systems.

Prediction

The state of the atmosphere and its surroundings (weather and climate system) is continually evolving in accordance with a set of physical laws. There are two methods of prediction.

In *dynamical prediction*, the process of predicting future states of the atmosphere consists of extrapolating forward from the present state according to these laws.

Alternately, in *statistical prediction*, the rules for extrapolation are established empirically, based on a sequence of past states

No system of prediction regularly produces perfect forecasts. The reasons are

- Imperfect knowledge of the state of the atmosphere from which one extrapolates
- Inadequacy of the methods by which one extrapolates because of incomplete knowledge of the physical laws and imperfect numerical prediction schemes.

A measure of the accuracy of the forecast can be devised.
For example, root-mean-square error can serve as a measure of the accuracy.

Predictability of a system

Predictability is the degree of accuracy with which it is possible to predict the state of the system in the future.

For physical systems, the predictions are based on imperfect knowledge of the system's past and present states.

For example, the instruments may not be capable of measuring to the desired accuracy. The network of measurements may not resolve the structure of the physical and dynamical features.

How well can one predict the state of the atmosphere, or some aspect of it, at a given time range, using a given system of observations and a given procedure for extrapolation? (*Compare with reality.*)

How well can one predict if the observations and the extrapolation are brought arbitrarily close to perfection? (*It depends on determinism.*)

For a quantitative answer, we require a measure of the goodness of a prediction, or of a prediction procedure.

Determinism and randomness

Time series is a function of continuous or discrete time.
Range: $-\infty$ to ∞ , or truncated
e.g., sequence of weather elements in whose predictability we are interested

A *process* is an ensemble of time series.
Each series is a *realization* of the process.
Separate members of the ensemble are realizations. Global weather is a single realization.

A process is *deterministic* if the present state of a realization completely determines the state at any specified future time.
i.e., if two realizations of a process are identical at one time, they must be identical at all future times.

A process is *random* or *stochastic* if the present state of a realization merely determines the probability distribution of states at a specified future time.
i.e., two realizations of the process may be identical in the past, but different at a future time.

A process is completely random if both past and future realizations are different. The knowledge of the present state tells us nothing about the future.

Determinism and randomness …

It may appear that realizations of a deterministic process is perfectly predictable whereas those of a random process is less predictable.

Is weather a realization of a random process or a deterministic process since its behavior depends to some extent on human activity?

Most mathematical models of the atmosphere are deterministic. However, the present state of the realization is imperfect.

The predictability study will be confined to deterministic systems.

Statistical prediction

Statistical prediction is based on empirical formulas derived from the observed past states of the atmosphere.

Sample of observations of the past behavior is finite in size.

An infinite number of formulas which the sample of data fits can be found, provided the formulas are allowed to become complicated enough.

The sample itself provides no basis for selection among these formulas.

There is no “best” formula from the point of view of a single sample.

The simplest and most widely used method is the linear formula.

Linear regression

Predictand: y

Predictors: $x_1, x_2, \cdots, x_M$

Prediction formula: $\hat{y} = a_1 x_1 + a_2 x_2 + \cdots + a_M x_M$

Predicted value: $y = \sum_{i=1}^{M} a_i x_i + e,$ $\qquad e =$ error

Choose $a_1, \cdots, a_M$ to minimize $\langle e^2 \rangle$

A sample of data consisting of N values of y and corresponding values of $x_1, x_2, \cdots, x_M$ are chosen to establish the required formula.

Linear procedures have been used extensively for predicting various weather elements at various time ranges.

e.g., India uses 8-10 parameter formulas for the prediction of seasonal monsoon rainfall.

Failure of Linear prediction and Discovery of Deterministic Chaos

Lorenz, E. N., 1962: The statistical prediction of solutions of dynamic equations. *Proc. Internat. Symp. Numerical Weather Prediction*, Tokyo, Japan, November 1960, 629-635.

Statistical prediction of solutions of dynamic equations

Linear regression methods are not successful in yielding nearly perfect weather forecasts. These statistical predictions are better than guess but are not considered satisfactory by the average forecaster.

Dynamical methods also did not provide satisfactory forecasts.

What are the reasons for failure?

- Is weather basically unpredictable from the available initial data?

- Are linear regression methods inadequate?

If Wiener's theorem on statistical prediction is correctly interpreted, the linear formula may require an infinite amount of past data and many terms may be needed for a good approximation.

To test whether linear regression methods are adequate for weather prediction:

1. Generate data by obtaining numerical solutions to a set of deterministic equations
2. Apply linear regression method to investigate the predictability

i.e., determine whether a hydrodynamical flow which is intrinsically predictable by means of its own governing equations is also predictable by linear statistical means.

To obtain the most suitable equations for such a test:

- The deterministic equations must be nonlinear.

- Exclude equations whose solutions do not vary with time (stationary) or vary in a regular manner.

- Statistics of the solutions should be independent of the initial conditions (i.e., exclude conservative systems that have invariant energy integral).

- Representative weather prediction model providing the appropriate nonlinearity

Lorenz's two-layer model: Nonperiodic solutions

Two-layer quasi-geostrophic model was selected for generating data from a set of deterministic nonlinear equations. The model is one of the simplest systems to represent the baroclinic flow in the atmosphere.

Lorenz, E. N., 1960: Energy and numerical weather prediction. *Tellus*, 12, 364-373.
Lorenz, E. N., 1963: The Mechanics of vacillation, *J. Atmos. Sci.*, 20, 448-464.

For the nondivergent part of the horizontal flow:
Stream functions in lower and upper layers: $\psi-\tau, \quad \psi+\tau$
(Potential) Temperature in lower and upper layers: $\theta-\sigma$, $\theta+\sigma$

For the irrotational part of the horizontal flow:
Velocity potential in lower and upper layers: χ, $-\chi$

0mb	---------------------	
250mb	- - - - - - - - - - - - -	$\psi+\tau, \quad \theta+\sigma, \quad -\chi$
500mb	---------------------	$\psi, \quad \theta$
750mb	- - - - - - - - - - - - -	$\psi-\tau, \quad \theta-\sigma, \quad \chi$
1000mb	---------------------	

Let f and static stability be constant.

Vorticity and Temperature equations:

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi,\nabla^2\psi) - J(\tau,\nabla^2\tau) - h'\nabla^2(\psi-\tau)$$

$$\frac{\partial}{\partial t}\nabla^2\tau = -J(\psi,\nabla^2\tau) - J(\tau,\nabla^2\psi) + f\nabla^2\chi + h'\nabla^2(\psi-\tau) - 2h''\nabla^2\tau$$

$$\frac{\partial\theta}{\partial t} = -J(\psi,\theta) + \sigma\nabla^2\chi - h_N(\theta-\theta^*)$$

$$\nabla^2\theta = A\nabla^2\tau \quad \text{(Thermal wind equation)}$$

A is a constant that depends on the properties of the fluid.

The forcing and dissipation are thermal and mechanical in nature.

Friction at the lower surface is proportional to the flow in the lower layer, and the friction at the surface separating the two layers is proportional to the vertical shear. The coefficients of friction are h' and h''.

The forcing is in the form of heating which is proportional to the difference between the existing temperature field and a standard temperature field θ^*. The coefficient of heating is h_N.

Most Important features of the model

- Nonlinearity
- Forcing
- Dissipation

Simplification of the model

Restrict the flow to an infinite strip bounded by $y = 0$ and $y = \pi/l$ (channel).

Expand ψ and τ using a truncated set of orthogonal functions that are appropriate for the geometry and boundary conditions of the flow.

Orthogonal functions:

$\cos mly$
$\sqrt{2} \sin mly \cos nkx$
$\sqrt{2} \sin mly \sin nkx$

for integer values of m and n

Truncated spectral expansion:

$$\psi = \psi_A \cos ly + \psi_C \cos 2ly$$
$$+ \sqrt{2}(\psi_K \sin ly + \psi_M \sin 2ly) \cos kx$$
$$+ \sqrt{2}(\psi_L \sin ly + \psi_N \sin 2ly) \sin kx$$

Similar expansion for τ

Only the largest scale motion has been retained.

Substitute these expansion in the model equations and obtain 12 ordinary differential equations in 12 variables

$$\psi_A, \psi_K, \psi_L, \psi_C, \psi_M, \psi_N$$
$$\tau_A, \tau_K, \tau_L, \tau_C, \tau_M, \tau_N$$

Denote these variables by $X_1, X_2, \ldots, X_{12}$

Simplified spectral form of the equations (12 ODEs):

$$\frac{dX_i}{dt} = \sum_{m,n} a_{imn} X_m X_n + \sum_m b_{im} X_m + c_i, \qquad i = 1, \cdots, 12$$

Deterministic forced dissipative nonlinear system

(for spectral expansion, see Lorenz, 1963: Mechanics of Vacillation. *J. Atmos. Sci.*, 20, 448-464)

12-variable Lorenz Model

Assign proper values for all the constants.

Integrate the model with some initial conditions.

The values of frictional coefficients, heating coefficient and the static stability were fixed. The model's behavior was studied by varying the forcing term:

$$\tau^* = \tau_A{}^* \cos y + \tau_C{}^* \cos 2y + \sqrt{2}\, \tau_M{}^* \sin 2y \cos 2x$$

The model exhibits weather features such as cyclones and anticyclones.

Model's behavior:

$\tau_A{}^* = 0.1, \quad \tau_C{}^* = 0.0, \quad \tau_M{}^* = 0.0\ ;$
$\tau_A{}^* = 0.1, \quad \tau_C{}^* = -0.018 \text{ to } -0.026, \quad \tau_M{}^* = 0.0$

Vacillation (regular behavior with quasi-periodic variation)

Statistical prediction is trivial for these regular variations.

Irregular behavior

$\tau_A^* = 0.1, \quad \tau_C^* = -0.025, \quad \tau_M^* = -0.025$

Integrations of more than 20 years showed no repetition of previous states.

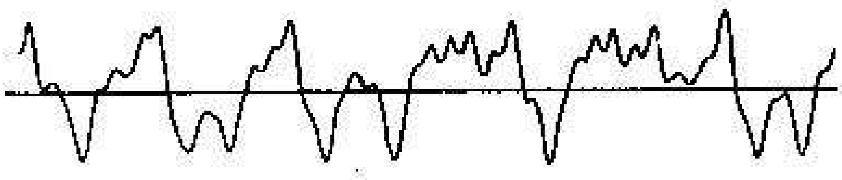


Fig. 2. Graph of $-\psi_\sigma$ against time for a particular eight-month period. Horizontal line is zero line.

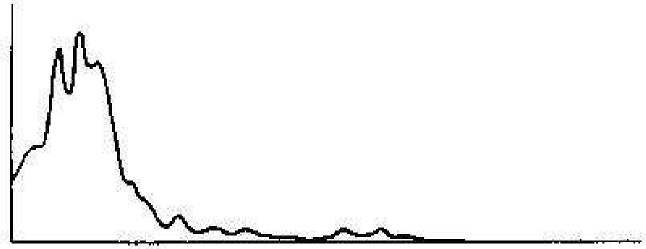


Fig. 3. Power spectrum of ψ_c for 2800-day period, computed from autocorrelations up to 200 days' lag.

(From Lorenz, E. N., 1962: The statistical prediction of solutions of dynamic equations. *Proc. Internat. Symp. Numerical Weather Prediction*, Tokyo, Japan, November 1960, 629-635)

The time series seems to have randomness, but actually is nonperiodic.
The spectrum of the time series is continuous rather than a line spectrum, and resembles spectra of actual atmospheric variables except for the absence of variance in higher frequencies.

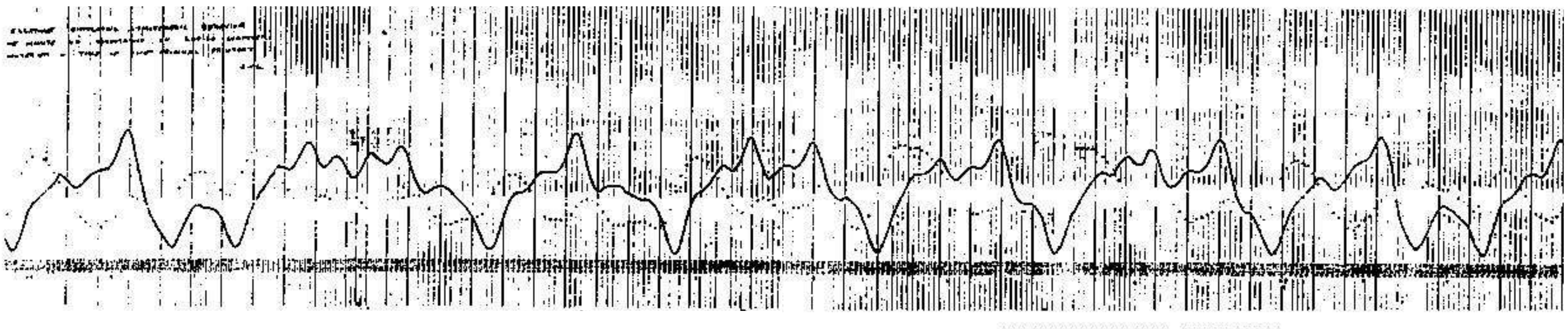
This result marks the discovery of nonperiodic behavior in a deterministic system.

Chaos

The variation of two variables of Lorenz's 12-variable model for a ten-month period shows a succession of "episodes" consisting of values rising abruptly, remaining high for about a month or so and then decreasing equally abruptly. The episodes are not identical and are not equal in duration. The behavior is definitely nonperiodic. Such variation is now commonly called chaos.

Lorenz, E. N., 1993: *Essence of Chaos*, University of Washington Press

Lorenz's original printout of symbols representing two variables of the twelve-variable model for a ten-month period. A solid curve has been drawn through the symbols for one variable, while a dotted curve (faintly visible) has been drawn for the other.



Linear statistical prediction

Predict one variable from the present and past of all twelve variables.

Reduction of total variance in linear prediction of twelve variables, with indicated times of predictors and predictands are shown in the table (from Lorenz 1962).

Prediction one day in advance is nearly perfect when the predictors occur at one-day intervals.
Predictions two days ahead are good but not perfect.
Predictions four days in advance fall far short of perfection.

Predictor days	Predictand day	Reduction of variance
0	+1	.972
-1, 0	+1	.997
-2, -1, 0	+1	.999
0	+2	.912
-1, 0	+2	.977
-2, -1, 0	+2	.990
0	+2	.912
-2, 0	+2	.964
-4, -2, 0	+2	.976
0	+4	.715
-2, 0	+4	.841
-4, -2, 0	+4	.883
0	+4	.715
-4, 0	+4	.811
-8, -4, 0	+4	.854

Solutions of deterministic dynamic equations cannot in general be reproduced by linear formulas.
In predicting the atmosphere as much as 24 hours ahead, better results should eventually be obtained by some nonlinear statistical procedure, or by the methods of dynamical weather prediction.

Quadratic map: Nonperiodic behavior in a simple model

(Lorenz, E. N., 1964: The problem of deducing the climate from the governing equations. *Tellus*, 16, 1-11)

Discrete deterministic nonlinear system

$$X_{n+1} = f(X_n)$$
$$X_{n+1} = rX_n(1 - X_n)$$

This system is called *Quadratic map or Quadratic difference equation* and can be considered as a very simple climate model. e.g., X_n is the maximum temperature at Trieste on day n.
(Also called *Logistic map* in other fields)

Using a linear transformation $Y_n = -X_n/r + 1/2$ and $c = r^2/4 - r/2$,

we obtain an alternate form of the quadratic map:

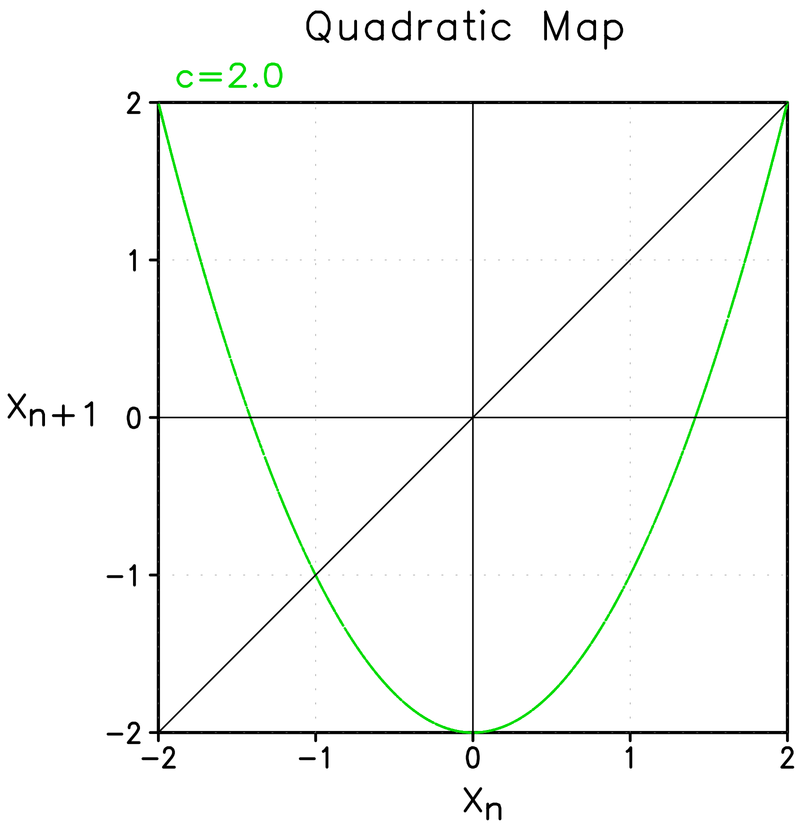
$$X_{n+1} = X_n^{\ 2} - c$$

For specified values of c and X_0, we get a sequence
$X_0,\ X_1,\ X_2,\ X_3,\ X_4, \ldots..$ at times $t_0,\ t_1,\ t_2,\ t_3,\ t_4, \ldots..$

Quadratic Map

$$If \quad 0 \le c \le 2 \quad and \quad -c \le X_0 \le c,$$

$$then \quad -c \le X_n \le c$$

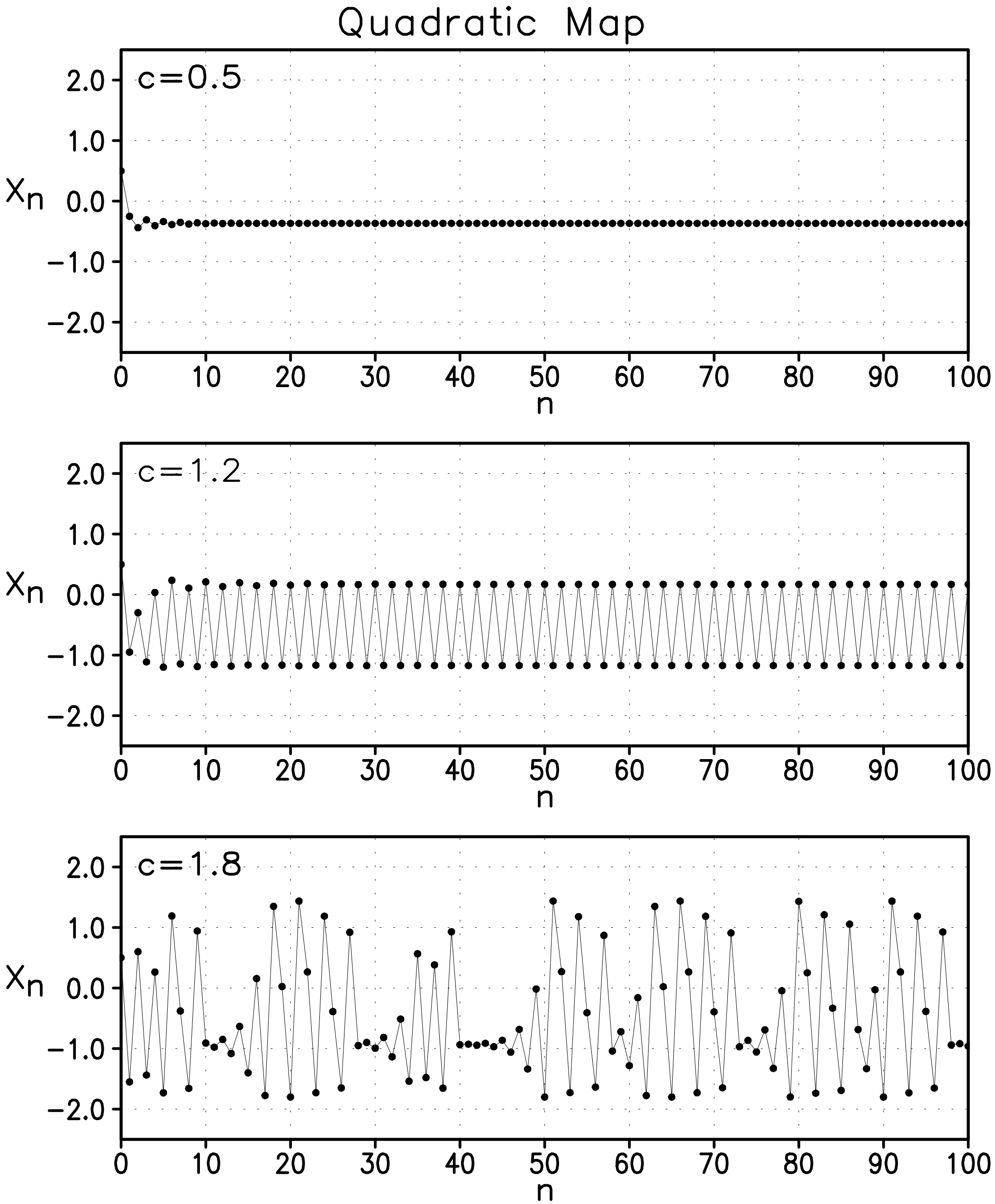


Solutions of Quadratic map $X_{n+1} = X_n^2 - c$

n	c = 0.5 X_n	c = 1.2 X_n	c = 1.8 X_n
0	0.50000	0.50000	0.50000
1	-0.25000	-0.95000	-1.55000
2	-0.43750	-0.29750	0.60250
3	-0.30859	-1.11149	-1.43699
4	-0.40477	0.03542	0.26495
5	-0.33616	-1.19875	-1.72980
6	-0.38700	0.23699	1.19221
7	-0.35023	-1.14384	-0.37863
8	-0.37734	0.10836	-1.65664
9	-0.35762	-1.18826	0.94445
10	-0.37211	0.21196	-0.90802
11	-0.36153	-1.15507	-0.97550
12	-0.36929	0.13420	-0.84839
13	-0.36362	-1.18199	-1.08023
14	-0.36778	0.19710	-0.63310
15	-0.36474	-1.16115	-1.39919
.	.	.	.
.	.	.	.

Solutions of Quadratic map

n	c = 0.5 X_n	c = 1.2 X_n	c = 1.8 X_n
.	.	.	.
.	.	.	.
85	-0.36603	-1.17082	-1.69006
86	-0.36603	0.17083	1.05630
87	-0.36603	-1.17082	-0.68423
88	-0.36603	0.17081	-1.33182
89	-0.36603	-1.17082	-0.02624
90	-0.36603	0.17083	-1.79931
91	-0.36603	-1.17082	1.43752
92	-0.36603	0.17082	0.26647
93	-0.36603	-1.17082	-1.72900
94	-0.36603	0.17082	1.18943
95	-0.36603	-1.17082	-0.38527
96	-0.36603	0.17082	-1.65157
97	-0.36603	-1.17082	0.92769
98	-0.36603	0.17082	-0.93940
99	-0.36603	-1.17082	-0.91753
100	-0.36603	0.17082	-0.95814
	Steady	**Periodic**	**Chaotic**


Quadratic Map
c=0.5
c=1.2
c=1.8
Xn
n
2.0
1.0
0.0
−1.0
−2.0
0
10
20
30
40
50
60
70
80
90
100

Discussion at the end of the presentation of

Lorenz, E. N., 1962: The statistical prediction of solutions of dynamic equations. *Proc. Internat. Symp. Numerical Weather Prediction*, Tokyo, Japan, November 1960, 629-635.

DISCUSSION

Bolin: Did you change the initial condition just slightly and see how much different results were in the forecasting in this way?

A: As a matter of fact, we tried out that once with the same equation to see what could happen. We changed one of the 12 variables by a factor of a small fraction of 1%, a change which would be considered to be smaller than observational error. We found that this error grew and continued to grow at a slow exponential rate. After 1 or 2 months, it is still pretty small so the map looked about right but it is comparable with the observational error. But after 6 months there is no resembrance at all between the 2 maps, which implied that at least for this particular set of equations there is a limit to how far you can forecast. Thus by these dynamic methods if you assume you have any observational error whatever to begin with, eventually the error predominates. Each of the series has the same statistics.
But they were simply different series after about 6 months.

Dynamics - A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle &Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

From Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press