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Conference and School on Predictability of Natural Disasters for our Planet in Danger. A System View; Theory, Models, Data Analysis

25 June - 6 July, 2007

Dynamical Systems
Low Order Dynamical Systems & Predictability

V. Krishnamurthy

Center for Ocean-Land-Atmosphere Studies, Calverton, USA & George Mason University, Fairfax, USA

Dynamical Systems

Lecture 2
Low-order Dynamical Systems and Predictability

V. Krishnamurthy

Center for Ocean-Land-Atmosphere Studies
Institute of Global Environment and Society
Calverton, MD 20705, USA
and
Department of Climate Dynamics
George Mason University
Fairfax, VA 22030, USA

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Dynamical Systems

A system that evolves in time according to some governing laws is a *dynamical system*. Each successive state is a function of the preceding state.

Examples: oscillators, planets in gravitation, atmospheric system

Types of dynamical systems

Conservative: The system possesses a conserved quantity, such as energy, a constant of motion.

Dissipative: Energy is not conserved because of dissipation and forcing.

Linear: The system contains only first power in the dynamical equations.

Nonlinear: The system contains nonlinear terms in the dynamical equations.

Representation of dynamical systems

Differential equations

Differential equations describe the evolution of the dynamical system in continuous time.

A coupled set of ordinary differential equations, such as Lorenz model, can be written as

$$\frac{dX_i}{dt} = F_i(X_1, \cdots, X_M), \quad i = 1, \cdots, M$$

e.g., $$\frac{dX_i}{dt} = \sum_{j,k} a_{ijk} X_j X_k - \sum_j b_{ij} X_j + c_i$$

where $\sum_{j,k} a_{ijk} X_j X_k$ vanishes identically, $\sum_j b_{ij} X_j$ is positive definite and

$c_1, c_2, \cdots, c_M$ are constants.

The set of differential equations ensures that the solutions exist and that they are unique and continuous. Because of the existence and uniqueness, different trajectories never interesect.

Phase Space is an M- dimensional Euclidean space whose coordinates are $X_1,\ldots X_M$.

Each point $(X_1,\ldots X_M)$ in the phase space represents an instantaneous state of the system.

A state varying according to the dynamical equations is represented by a *trajectory* or *orbit* in the phase space.

Uniqueness: Through each point, there is a unique orbit

Difference equations

Difference equations describe the evolution in discrete time
e.g., Quadratic map (logistic map)

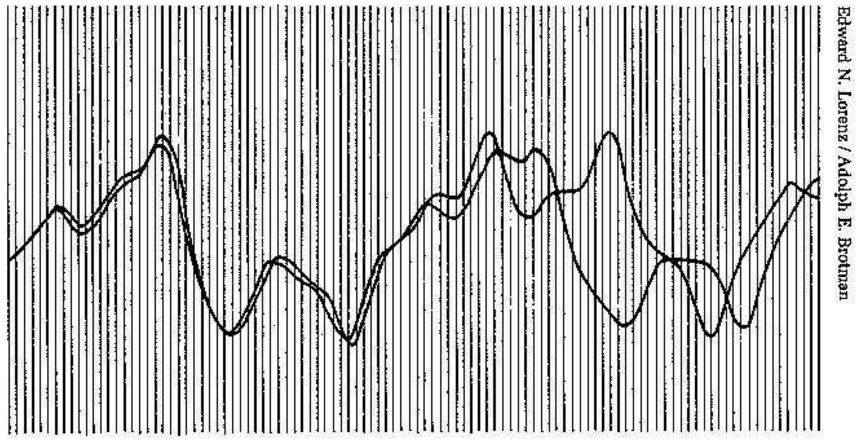
$$X_{i,n+1} = G_i(X_{1,n},\cdots,X_{M,n}), \quad i = 1,\cdots M$$

Chaos: Sensitive Dependence on Initial Conditions

Two solutions starting from nearly identical states (differing only in third decimal place) in Lorenz's numerical integration soon begin to diverge. The differences steadily double in size every four days or so until the two solutions lose resemblance to one another somewhere during the second month.

Such sensitive dependence on initial conditions is characteristic of chaos in all systems. If the real atmosphere behaves like the simple Lorenz model, it would be impossible to make long-range weather forecasting.

The Butterfly Effect 17



From Gleick, J, 1987: *Chaos*, Penguin Books

HOW TWO WEATHER PATTERNS DIVERGE. From nearly the same starting point, Edward Lorenz saw his computer weather produce patterns that grew farther and farther apart until all resemblance disappeared. (From Lorenz's 1961 printouts.)

Simple Predictability Experiments

"Identical twin" experiment

Find the basic solutions with certain initial condition. Introduce a small error in the initial condition and find the perturbed solutions. Compare the two solutions and study the evolution of the error.

Example 1

$$X_{n+1} = X_n^2 - c\,, \qquad c = 0.5$$

X_n = basic ("true") solution

X_0' = "observed" value

X_n' = perturbed ("predicted") solution

$X_0' - X_0$ = "observed error"

$X_n' - X_n$ = error at time n

n	X_n	X_n'	$X_n' - X_n$
0	0.40000	0.40100	0.0010000
1	-0.34000	-0.33920	0.0008010
2	-0.38440	-0.38494	-0.0005440
3	-0.35224	-0.35182	0.0004186
4	-0.37593	-0.37622	-0.0002947
5	-0.35868	-0.35846	0.0002216
6	-0.37135	-0.37151	-0.0001590
7	-0.36210	-0.36198	0.0001181
8	-0.36888	-0.36897	-0.0000855
9	-0.36392	-0.36386	0.0000631
10	-0.36756	-0.36761	-0.0000459
11	-0.36490	-0.36487	0.0000338
12	-0.36685	-0.36687	-0.0000246
13	-0.36542	-0.36540	0.0000181
14	-0.36647	-0.36648	-0.0000132
15	-0.36570	-0.36569	0.0000097
16	-0.36626	-0.36627	-0.0000071
17	-0.36585	-0.36585	0.0000052
18	-0.36615	-0.36616	-0.0000038
19	-0.36593	-0.36593	0.0000028
20	-0.36609	-0.36610	-0.0000020
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.			

Simple Predictability Experiments

Example 1 (contd.)

$$X_{n+1} = X_n^2 - c\,, \qquad c = 0.5$$

The error steadily decreases and decays to zero. The steady state solution is stable.

Lorenz, E. N., 1985: The growth of errors in prediction. *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, M. Ghil and R. Benzi, Eds., LXXXVIII Corso Soc. Itiliana di Fisica, Bologna, Italy, 243-265.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	-0.36603	-0.36603	0.0000000
82	-0.36603	-0.36603	0.0000000
83	-0.36603	-0.36603	0.0000000
84	-0.36603	-0.36603	0.0000000
85	-0.36603	-0.36603	0.0000000
86	-0.36603	-0.36603	0.0000000
87	-0.36603	-0.36603	0.0000000
88	-0.36603	-0.36603	0.0000000
89	-0.36603	-0.36603	0.0000000
90	-0.36603	-0.36603	0.0000000
91	-0.36603	-0.36603	0.0000000
92	-0.36603	-0.36603	0.0000000
93	-0.36603	-0.36603	0.0000000
94	-0.36603	-0.36603	0.0000000
95	-0.36603	-0.36603	0.0000000
96	-0.36603	-0.36603	0.0000000
97	-0.36603	-0.36603	0.0000000
98	-0.36603	-0.36603	0.0000000
99	-0.36603	-0.36603	0.0000000
100	-0.36603	-0.36603	0.0000000

Simple Predictability Experiments

Example 2

$X_{n+1} = X_n^2 - c$, c = 1.2

X_n = basic ("true") solution
X_0' = "observed"
X_n' = perturbed ("predicted") solution
$X_0' - X_0$ = "observed error"
$X_n' - X_n$ = error at time n

The error amplifies during the first few time steps and then undergoes damped oscillations.

n	X_n	X_n'	$X_n' - X_n$
0	0.50000	0.50100	0.0010000
1	-0.95000	-0.94900	0.0010010
2	-0.29750	-0.29940	-0.0019009
3	-1.11149	-1.11036	0.0011346
4	0.03542	0.03290	-0.0025210
5	-1.19875	-1.19892	-0.0001722
6	0.23699	0.23740	0.0004129
7	-1.14384	-1.14364	0.0001959
8	0.10836	0.10791	-0.0004481
9	-1.18826	-1.18836	-0.0000969
10	0.21196	0.21219	0.0002303
11	-1.15507	-1.15498	0.0000977
12	0.13420	0.13397	-0.0002257
13	-1.18199	-1.18205	-0.0000605
14	0.19710	0.19725	0.0001431
15	-1.16115	-1.16109	0.0000564
16	0.14827	0.14814	-0.0001310
17	-1.17802	-1.17806	-0.0000388
18	0.18772	0.18781	0.0000915
19	-1.16476	-1.16473	0.0000344
20	0.15667	0.15659	-0.0000800
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.			

Simple Predictability Experiments

Example 2 (contd.)

$$X_{n+1} = X_n^2 - c\,, \qquad c = 1.2$$

By about time step 90, the two solutions are identical upto five decimal places. The error has decayed. The periodic solution is stable.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	-1.17083	-1.17083	0.0000000
82	0.17083	0.17083	0.0000001
83	-1.17082	-1.17082	0.0000000
84	0.17081	0.17081	-0.0000001
85	-1.17082	-1.17082	0.0000000
86	0.17083	0.17083	0.0000000
87	-1.17082	-1.17082	0.0000000
88	0.17081	0.17081	0.0000000
89	-1.17082	-1.17082	0.0000000
90	0.17083	0.17083	0.0000000
91	-1.17082	-1.17082	0.0000000
92	0.17082	0.17082	0.0000000
93	-1.17082	-1.17082	0.0000000
94	0.17082	0.17082	0.0000000
95	-1.17082	-1.17082	0.0000000
96	0.17082	0.17082	0.0000000
97	-1.17082	-1.17082	0.0000000
98	0.17082	0.17082	0.0000000
99	-1.17082	-1.17082	0.0000000
100	0.17082	0.17082	0.0000000

Simple Predictability Experiments

Example 3

$X_{n+1} = X_n^2 - c\,,\qquad c$ = **1.8**

X_n = basic ("true") solution
$X_0{}'$ = "observed"
$X_n{}'$ = perturbed ("predicted") solution
$X_0{}' - X_0$ = "observed error"
$X_n{}' - X_n$ = error at time n

The error grows irregularly, gaining an order of magnitude in about five time steps and becomes comparable to X_n itself.

n	X_n	$X_n{}'$	$X_n{}' - X_n$
0	0.50000	0.50100	0.0010000
1	-1.55000	-1.54900	0.0010010
2	0.60250	0.59940	-0.0031021
3	-1.43699	-1.44072	-0.0037284
4	0.26495	0.27568	0.0107293
5	-1.72980	-1.72400	0.0058006
6	1.19221	1.17218	-0.0200341
7	-0.37863	-0.42600	-0.0473684
8	-1.65664	-1.61852	0.0381141
9	0.94445	0.81962	-0.1248300
10	-0.90802	-1.12823	-0.2202084
11	-0.97550	-0.52711	0.4483977
12	-0.84839	-1.52216	-0.6737677
13	-1.08023	0.51697	1.5971992
14	-0.63310	-1.53275	-0.8996509
15	-1.39919	0.54931	1.9485018
16	0.15773	-1.49826	-1.6559891
17	-1.77512	0.44477	2.2198931
18	1.35105	-1.60218	-2.9532298
19	0.02534	0.76697	0.7416291
20	-1.79936	-1.21175	0.5876038
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.			

Simple Predictability Experiments

Example 3 (contd.)

$X_{n+1} = X_n^2 - c$, c = **1.8**

The error varies irregularly, but does not amplify forever because both X_n and X_n' are bounded.

When the error becomes comparable to X_n itself, the error has reached saturation. At this point, the prediction X_n' has become worthless.

The nonperiodic solution is unstable.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	0.25407	1.09554	0.8414776
82	-1.73545	-0.59978	1.1356682
83	1.21179	-1.44026	-2.6520482
84	-0.33157	0.27435	0.6059283
85	-1.69006	-1.72473	-0.0346710
86	1.05630	1.17469	0.1183943
87	-0.68423	-0.42010	0.2641366
88	-1.33182	-1.62352	-0.2916942
89	-0.02624	0.83581	0.8620564
90	-1.79931	-1.10142	0.6978925
91	1.43752	-0.58688	-2.0243978
92	0.26647	-1.45558	-1.7220418
93	-1.72900	0.31870	2.0476958
94	1.18943	-1.69843	-2.8878565
95	-0.38527	1.08467	1.4699306
96	-1.65157	-0.62350	1.0280698
97	0.92769	-1.41125	-2.3389325
98	-0.93940	0.19162	1.1310159
99	-0.91753	-1.76328	-0.8457533
100	-0.95814	1.30917	2.2673060

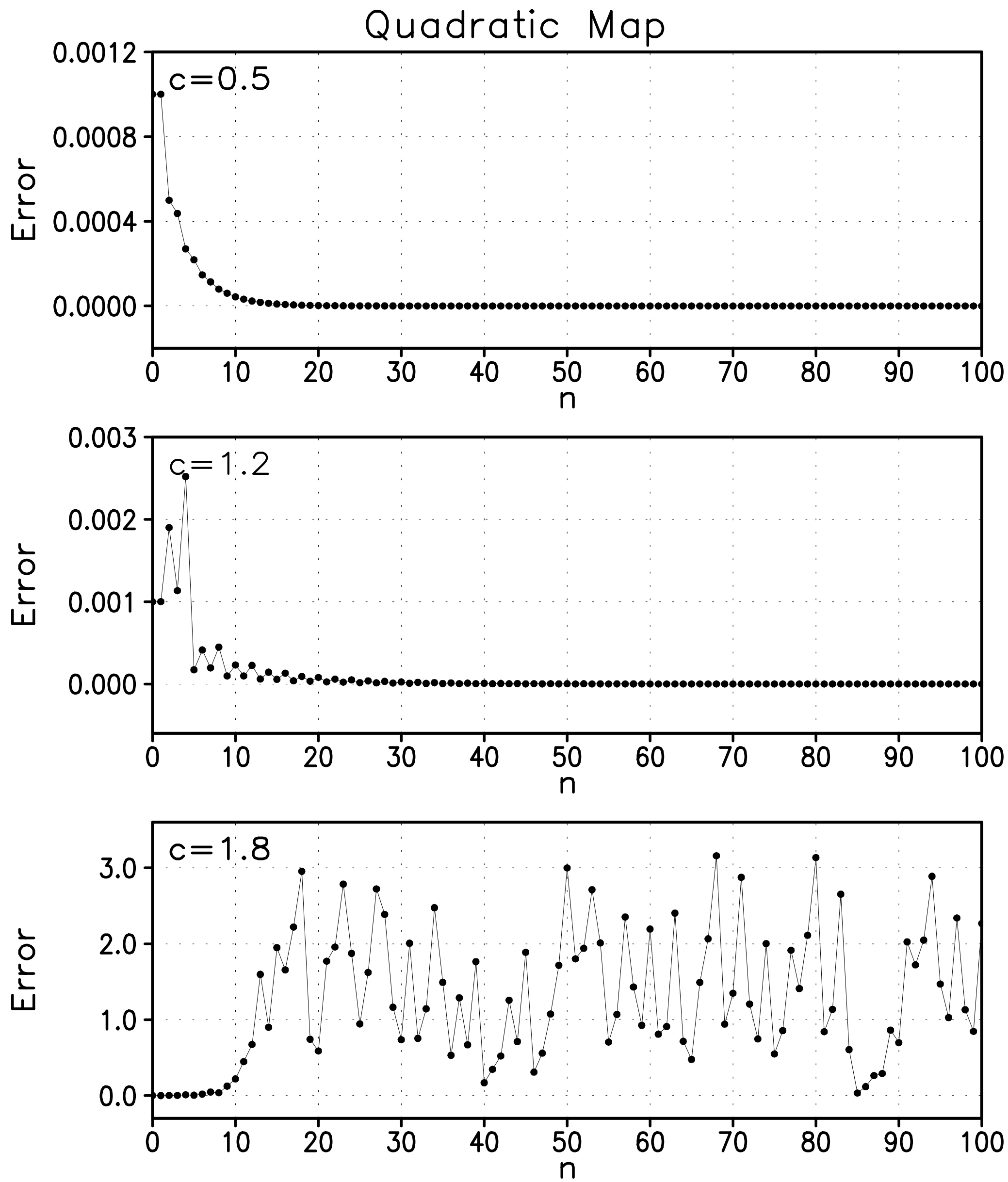
Error Growth in Quadratic Map

Absolute value of $(X_n' - X_n)$ is plotted as error.

Steady State:
The error decays to zero and the system is stable.

Periodic:
The error decays to zero and the system is stable.

Nonperiodic:
The error grows and becomes as large as the difference between two randomly selected states of the system. The system is unstable.



Error Growth in Nonlinear Systems

Non-periodic or chaotic solutions of deterministic nonlinear systems exhibit an important property:

Sensitive dependence on initial conditions

A small initial error will grow and ultimately become as large as the difference between two randomly chosen solutions of the system.

Consequences

It is impossible to make perfect weather predictions, or even mediocre predictions sufficiently far into the future.

The main reason for this unpredictability is the errors made in observations. A small error in the initial observation will grow and make the forecast unreliable and ultimately worthless, even if the forecasting model is perfect.

Nonperiodic solutions are unstable. Errors grow because of the instability of the system.

Periodic solutions are stable and predictable.

Stability

The decay or growth of errors influences the accuracy of predictions.

A solution is *stable* if any other sufficiently close solution remains arbitrarily close (a).

Otherwise, the solution is *unstable* and the nearby solution diverges (b).

If the solution is *stable*, it is *periodic* because when an approximate repetition of a previous state occurs, future states must remain arbitrarily close to the previous history (c).

If the solution is *nonperiodic*, it is necessarily *unstable* (d).

The deciding factor in predictability is stability versus instability.

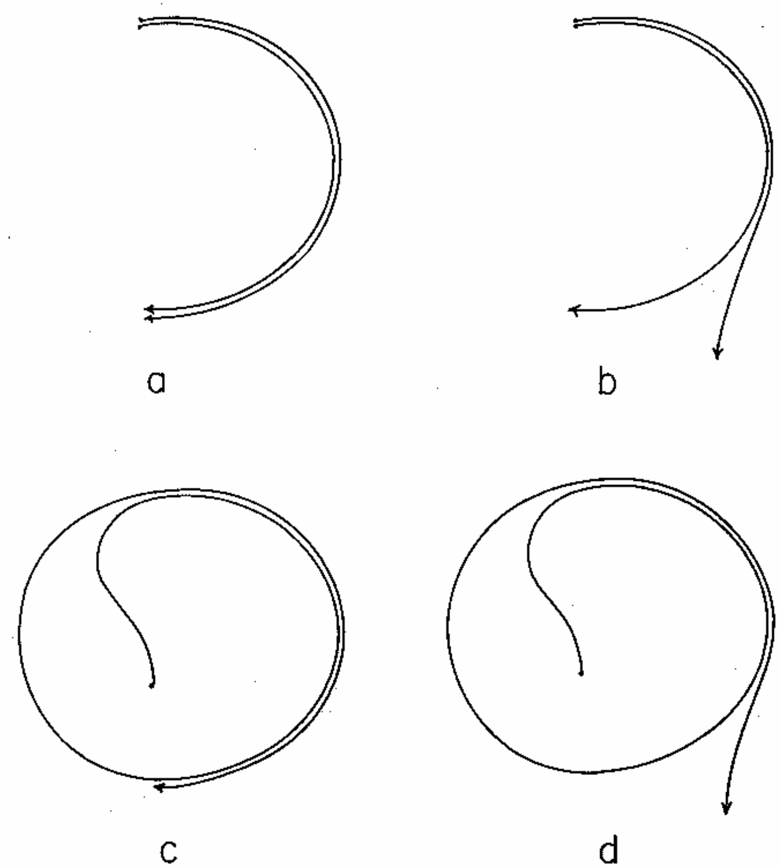


FIGURE 3. Schematic trajectories in phase space; (a) neighboring stable trajectories; (b) neighboring unstable trajectories; (c) stability implying periodicity (after the transient flow has died out); (d) nonperiodicity implying instability.

Lorenz, E. N., 1963: The predictability of hydrodynamic flows. *Trans. New York Acad. Sci.*, Ser II, **25**, 409-423.

The Butterfly Effect

Lorenz (1963): *"… one flap of a sea gull's wings would be enough to alter the course of the weather forever."*

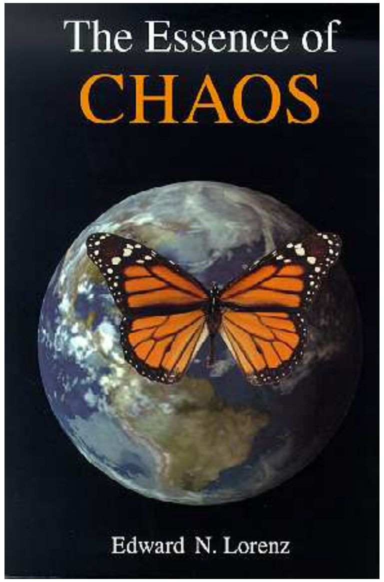
Lorenz (1972): *"Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" (Amer. Assoc. Adv. Sci., 139th meeting)*

If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.

Over the years miniscule disturbances neither increase nor decrease the frequency of occurrence of various weather events such as tornados; the most that they may do is to modify the sequence in which these events occur.

Can two particular weather situations differing by as little as the immediate influence of a single butterfly will generally after sufficient time evolve into two situations differing by as much as the presence of a tornado?

Is the behavior of the atmosphere unstable with respect to perturbations of small amplitude?



Lorenz Model
(3-variable convection model)

Lorenz, E. N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130-141.

The behavior of a simple continuous nonlinear system is discussed.

The system is the famous Lorenz model which is a very low-order model of Rayleigh-Bénard convection.

When the motion of the fluid is caused by a difference in density created by a temperature difference, it is called convection. The fluid absorbs heat at one place, moves to another place and dissipates heat by mixing with the colder fluid. Atmospheric motions are mostly convective in nature because of the thermal inequalities set up by solar heating.

In a well-known laboratory experiment of convection, called Rayleigh-Bénard convection, a fluid is subjected to controlled temperature difference. The motion of the fluid is studied by varying the temperature difference.

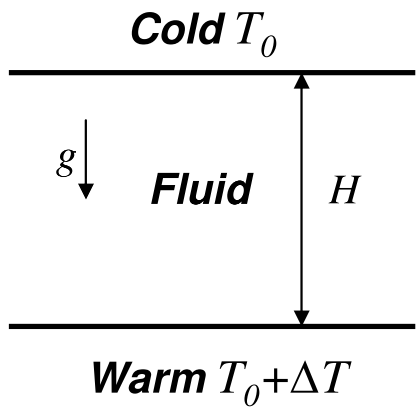
Experiment

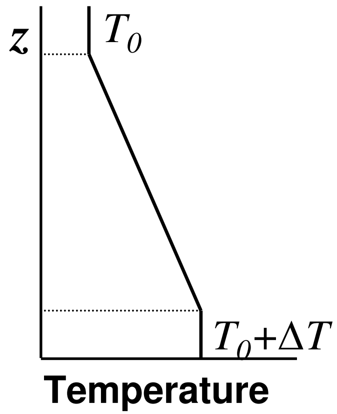
(Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons, Section V.3)

A layer of fluid of uniform thickness H is confined between two horizontal plates. The upper plate is at a temperature T_0 and the lower plate at temperature $T_0+\Delta T$.

With such a temperature difference, the fluid in the upper part of the layer is cold and dense whereas the fluid in the lower part is warmer and less dense. The warm fluid tends to rise while the colder fluid tends to fall.

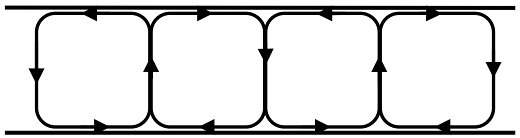
When ΔT is small, there is no convective motion because of the stabilizing effect of friction. The heat transfer is through conduction. In this steady state of the fluid in which there is no motion, the temperature varies linearly with height.





Convection

When ΔT is increased above a certain critical value, the steady state of no motion becomes unstable, and sustained convection begins. Above the convection threshold, a regular structure of rolls with parallel horizontal axes is formed.



The structure consists of alternating rising and descending currents.
The currents are equidistant from one another.
Two adjacent rolls rotate in opposite directions.
The convection is stationary (steady state).

Convective instabilities were first observed experimentally by Bénard in 1900. The theoretical explanation was first given by Rayleigh in 1916. Hence this phenomenon is called Rayleigh-Bénard convection.

When ΔT is increased, the convection pattern first becomes more complicated but retains certain regularity.

When ΔT is further increased, the pattern is completely destroyed and replaced by a disordered configuration in perpetual motion. The fluid motion is turbulent.

Governing Equations

(Saltzman, B., 1962: Finite amplitude free convection as an initial value problem. *J. Atmos. Sci.*, **19**, 329-341)

There is no variation along the y-axis.

All motions are parallel to the x-z plane.

In Boussinesq approximation, the equations governing convection are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \nu\nabla^2 w + g\alpha T$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\nabla^2 T$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0$$

(u,w) = velocity along (x,z)

P = pressure

T = temperature departure
(convection – no convection)

α = coefficient of thermal expansion

κ = thermal conductivity

ν = kinematic viscosity

g = acceleration due to gravity

Stream function and temperature departure:

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}$$

$$T(x,z,t) = \left(\overline{T}(0,t) - \frac{\Delta T}{H} z \right) + \theta(x,z,t)$$

T is expressed as a sum of linear variation between upper and lower boundary and a departure θ from the linear variation.

Vorticity and temperature equations:

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,z)} + \nu \nabla^4 \psi + g\alpha \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \theta}{\partial t} = -\frac{\partial(\psi, \theta)}{\partial(x,z)} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial x} + \kappa \nabla^2 \theta$$

Boundary conditions assuming free (no-stress) boundaries:

$$\psi = 0, \qquad \nabla^2 \psi = 0$$

Rayleigh's finding

If Rayleigh *number* $R_a = \dfrac{g\alpha H^3 \Delta T}{\kappa\nu}$ exceeds a critical value $R_c = \dfrac{\pi^4 (1+a^2)^3}{a^2}$,

motion of the following form develops.

$$\psi = \psi_0 \sin\left(\pi a H^{-1} x\right)\sin\left(\pi H^{-1} z\right)$$
$$\theta = \theta_0 \cos\left(\pi a H^{-1} x\right)\sin\left(\pi H^{-1} z\right)$$

Here a is the aspect ratio of vertical to horizontal length scales. The minimum value of R_c = $27\pi^4/4$ occurs when a^2 = ½.

Spectral expansion

Introduce the following highly truncated spectral expansions in vorticity and temperature equations (4.3)

$$\psi = \frac{\sqrt{2}\,\kappa\left(1+a^2\right)}{a} X \sin\left(\pi a H^{-1} x\right)\sin\left(\pi H^{-1} z\right)$$
$$\theta = \frac{\sqrt{2} R_c \Delta T}{\pi R_a} Y \cos\left(\pi a H^{-1} x\right)\sin\left(\pi H^{-1} z\right) - Z \sin\left(2\pi H^{-1} z\right)$$

Simplify to obtain a set of three ordinary differential equations.

Lorenz Model (3-variable)

The simplification leads to

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$

$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

X: Intensity of convective motion
Y: Temperature difference between ascending and descending currents
Z: Distortion of vertical temperature profile from linearity

$\sigma = \nu/\kappa$ Prandtl number (determines hydrodynamic versus thermal instability)
$b = 4/(1+a2)$
$r = R_a/R_c$ (Forcing proportional to Rayleigh number)

Lorenz model is a low-order convection model described by just three ordinary differential equations. It is the simplest forced dissipative nonlinear system.

Solutions of Lorenz Model

Symmetry of the model

The equations are invariant under the transformation

$(X, Y, Z) \rightarrow (-X, -Y, Z)$ for all values of r.

i.e., if (X, Y, Z) is a solution, then $(-X, -Y, Z)$ is also a solution.

Solutions

To determine the solutions of the model, it is necessary to numerically integrate the model for a given set of parameters.

However, since the model is simple, the steady state solutions can be determined analytically. When the steady states are stable, such solutions can also be found by numerical integration starting with some arbitrary initial conditions.

Steady States

Steady states are easily found by solving

$$\frac{dX}{dt} = 0, \quad \frac{dY}{dt} = 0, \quad \frac{dZ}{dt} = 0$$

Solve

$$\begin{aligned} -\sigma X + \sigma Y &= 0 \\ -XZ + rX - Y &= 0 \\ XY - bZ &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X &= Y \\ Z &= Y^2 / b \\ Y^3 - b(r-1)Y &= 0 \end{aligned}$$

Steady state solutions:

(1) $X = Y = Z = 0$ Steady State O

(2) $X = Y = [b(r-1)]^{1/2}, \quad Z = r-1$ Steady State C

(3) $X = Y = -[b(r-1)]^{1/2}, \quad Z = r-1$ Steady State C'

When $\underline{r < 1}$, there is only one steady state:

$X = 0, \quad Y = 0, \quad Z = 0$ State of no convection (O)

When $\underline{r > 1}$, there are three steady states:

$X = 0, \quad Y = 0, \quad Z = 0$ No convection (O)

$X = Y = \pm[b(r-1)]1/2, \quad Z = r-1$ Steady convection (C, C')

Numerical integrations

Integrate the model for the following parameter values

$$\sigma = 10, \quad a^2 = 1/2, \quad b = 8/3$$
$$r = 0.5, 10.0$$

The numerical integration is carried out with a time increment of

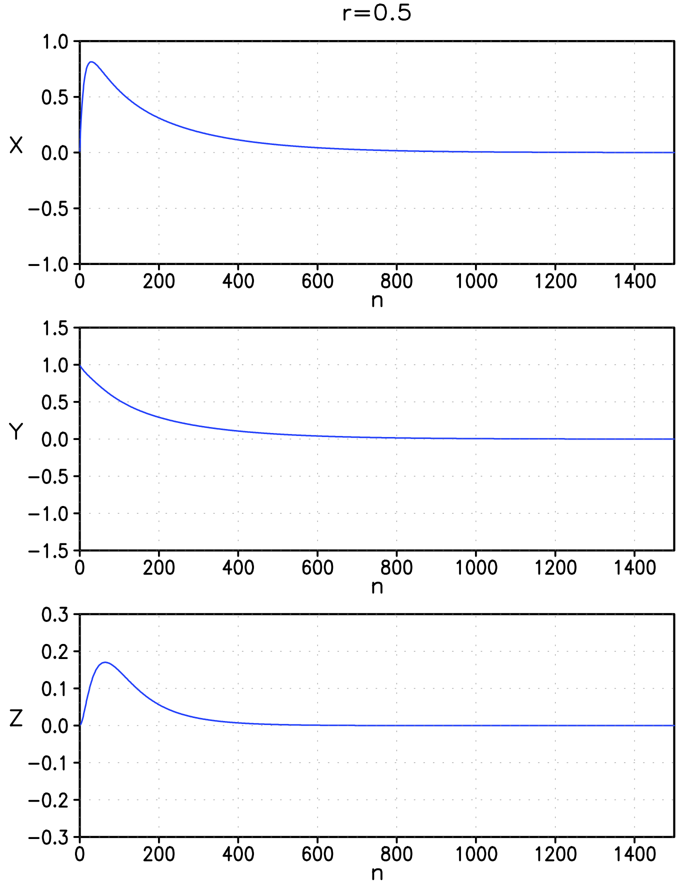
$$\Delta\tau = 0.01$$

Steady State Solutions

$r = 0.5$

Rest, No Convection

$O\!: X = 0, \quad Y = 0, \quad Z = 0$

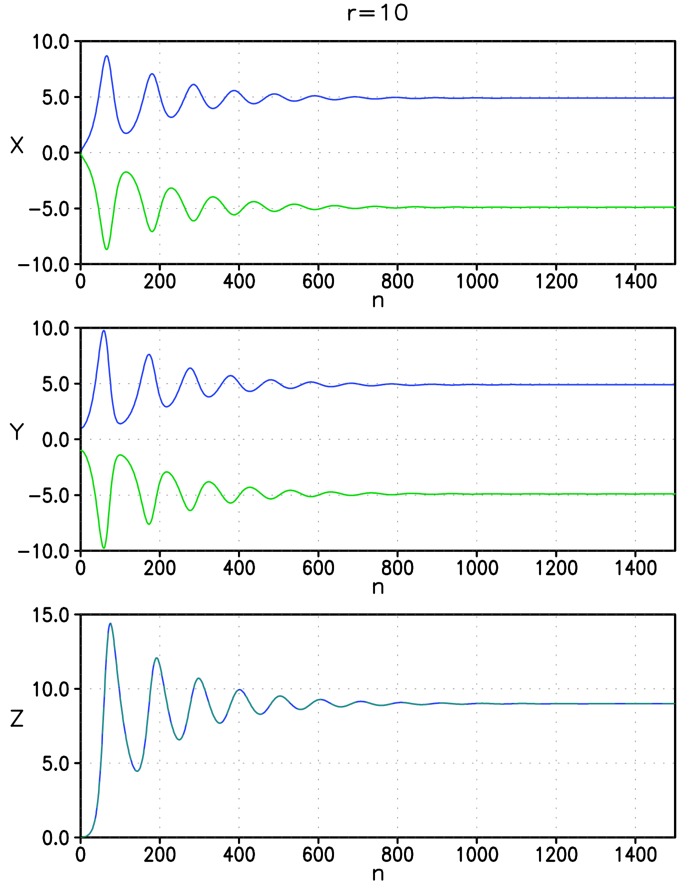


$r = 10.0$

Steady Convection

$C\!: X = 2\sqrt{6}, \quad Y = 2\sqrt{6}, \quad Z = 9$

$C'\!: X = -2\sqrt{6}, \quad Y = -2\sqrt{6}, \quad Z = 9$



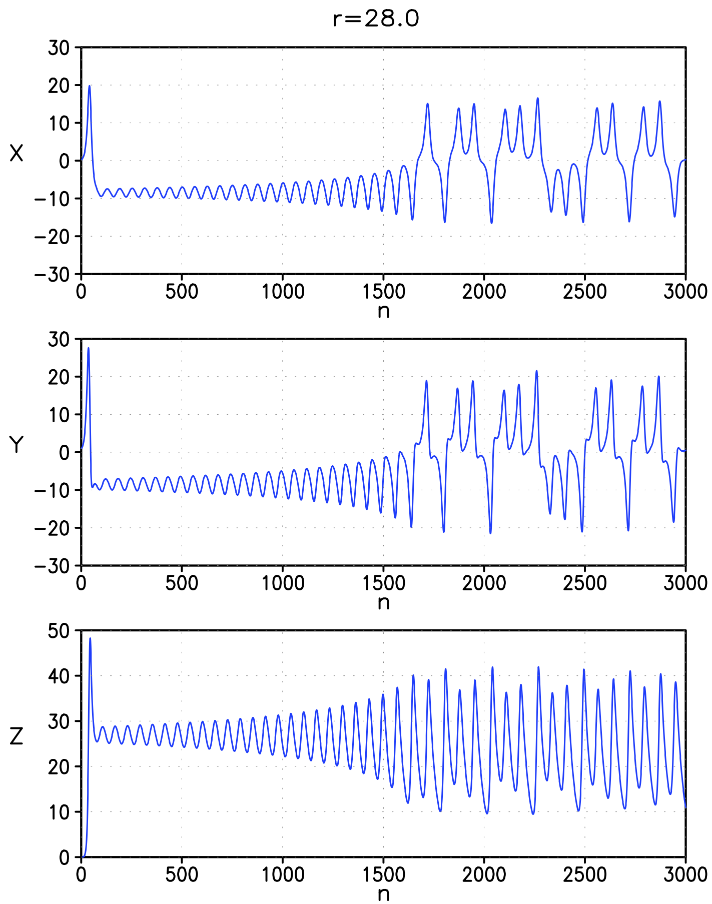
Time-dependent Solutions

As r is increased, the steady convection becomes unstable at a critical value of r. The critical Rayleigh number for instability of steady convection $(C,\ C')$ occurs when $r = 24.74$.

The solutions of Lorenz model at a slightly supercritical value $r = 28$ is studied.

The model is numerically integrated starting with a small perturbation over the state of no convection.

The state of rest is clearly unstable with all three variables growing rapidly. In less than 50 steps, the strength of convection exceeds that of steady convection and the system reaches a state close to the steady convection. The motion then undergoes systematic amplified oscillation until about step 1650. The subsequent behavior of the system is irregular or nonperiodic.



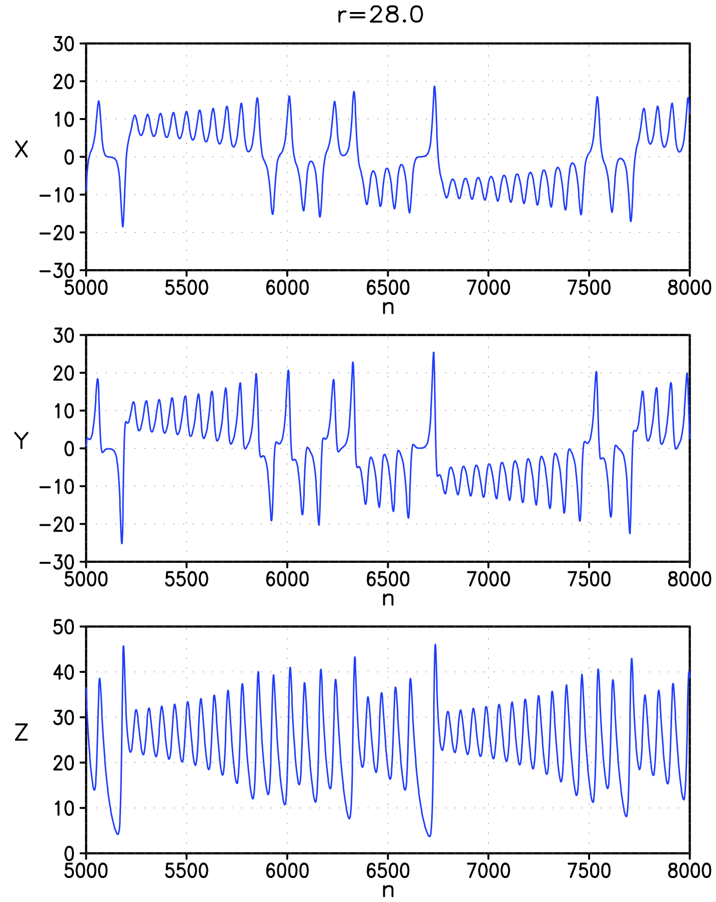
Time-dependent Solutions

$r = 28$

The irregular motion of the system reached at about step 1650 continues for subsequent time steps. The variables X and Y change sign at irregular intervals, reaching sometimes one or more extremes of one sign before changing sign again.

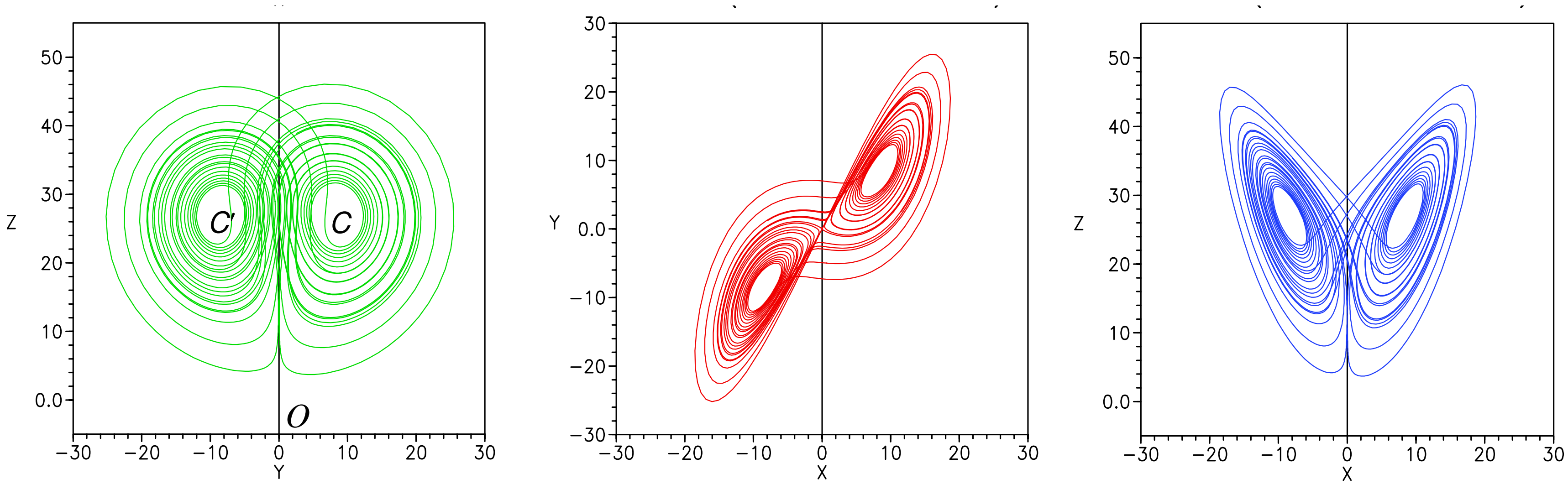
When X and Y have same sign, warm fluid is ascending and cold fluid is descending. When X and Y are of opposite signs, the warm fluid is descending and cold fluid is ascending.

The time variation of (X,Y,Z) is nonperiodic. The fluid motion is turbulent or chaotic.



Projection on two-dimension

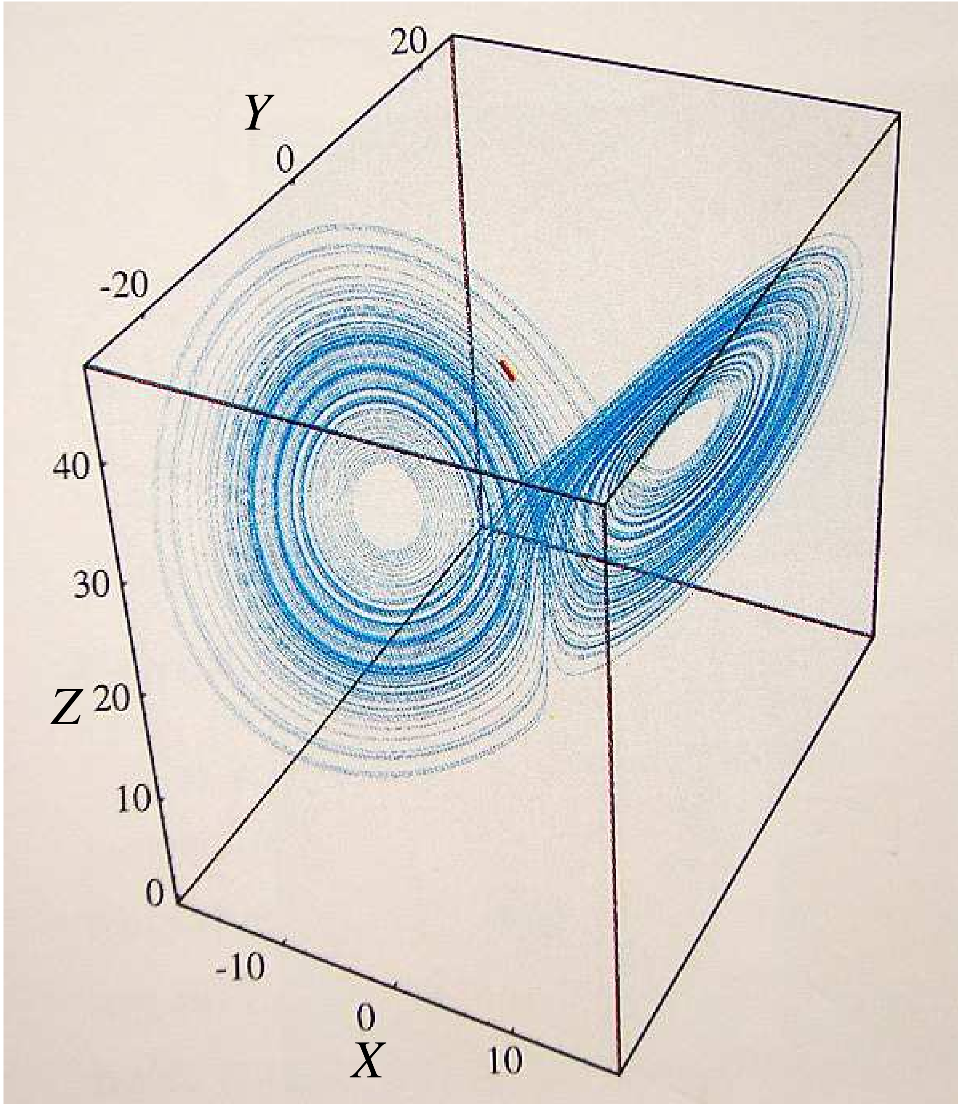
The trajectory of the solutions are projected on Y-Z, X-Z and X-Y planes. The trajectory moves around the steady state C, crosses a plane and moves around C' and returns to the neighborhood of C. This process continues at irregular intervals. The trajectory never passes through C, C' or O as all three steady states are unstable.



Three-dimensional structure

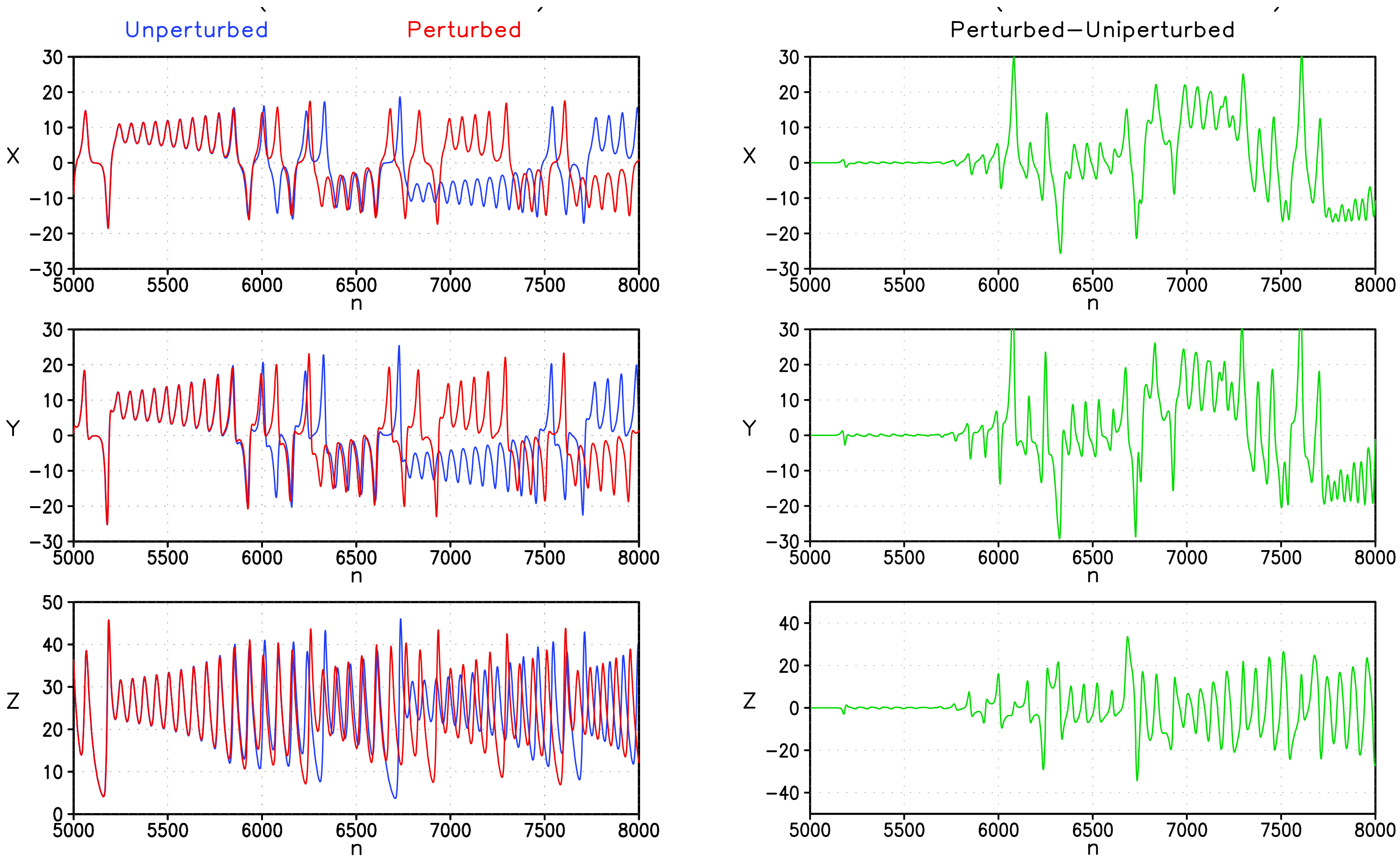
When the trajectory of the solutions is shown in 3-dimension, it is possible to imagine how a trajectory does not intersect itself. This is necessary for nonperiodic solutions. A trajectory may come arbitrarily close to a point that it has visited in the past but will soon diverge.

From Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press.



Instability of solutions: “Identical twin” experiment

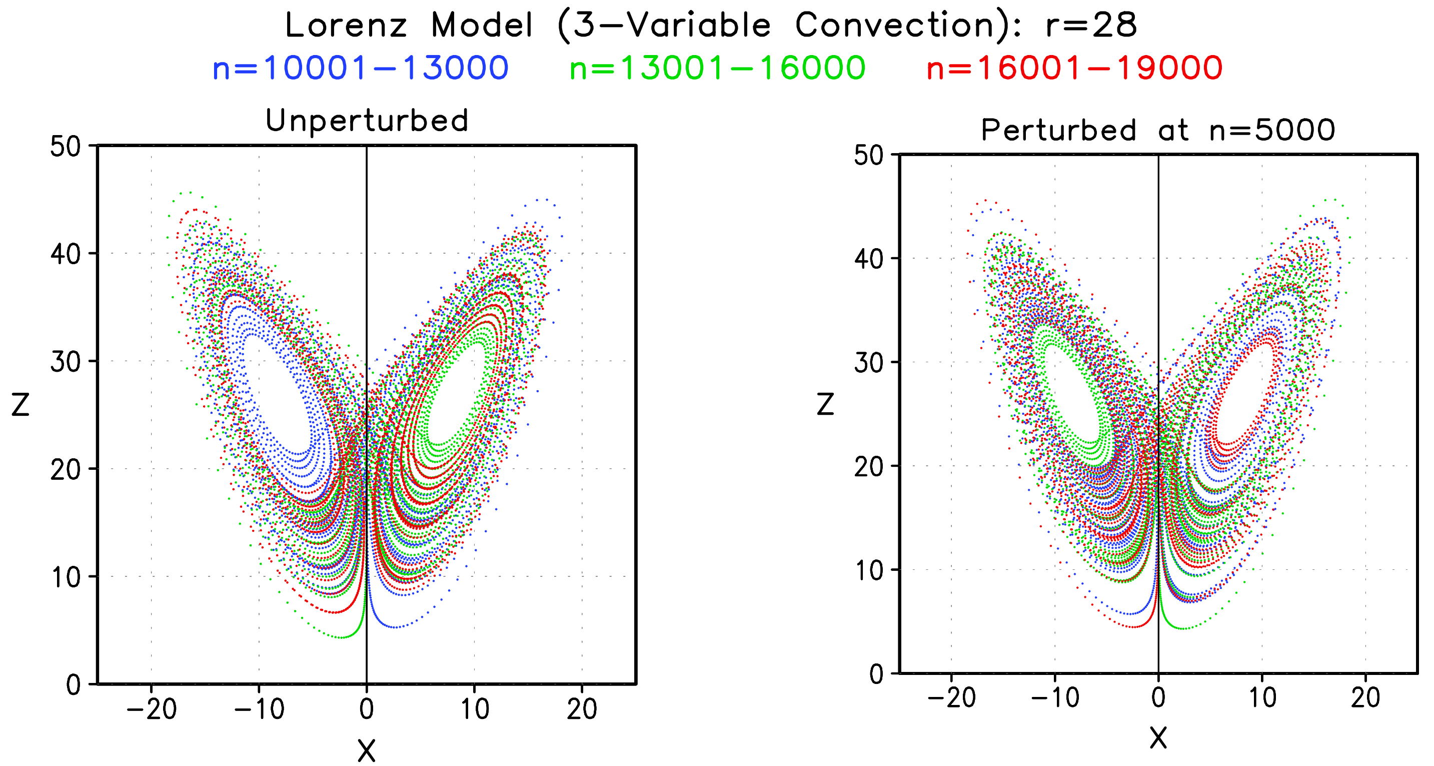
A new integration with a small perturbation added to the original solution at time 5000 is carried out. The two solutions stay close for a while and then diverge. The difference between the two solutions become as large as the variables themselves by step 6000. The nonperiodic solution at $r = 28$ is unstable.



Projection of “identical twin” trajectories

The evolutions of the unperturbed and perturbed trajectories are shown as projections on two-dimensional space.

The projections of unperturbed and perturbed trajectories are shown in different colors for different segments of time and the divergence of trajectories is clearly evident.



Predictability experiment

An ensemble of 10000 nearby points at an initial $t = 0$ around a basic state is allowed to evolve.
Blue points are from unperturbed integration.
Red points show the evolution of the perturbed initial states.

"As each point moves according to Lorenz equations, the blob is stretched into a thin filament… Ultimately, the points spread over … showing that the final state could be almost anywhere, even though the initial conditions were almost identical."

From Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press