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Conference and School on Predictability of Natural Disasters for our Planet in Danger. A System View; Theory, Models, Data Analysis

25 June - 6 July, 2007

Dynamical Systems
Attractors & Bifurcations

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Dynamical Systems

Lecture 3
Attractors and Bifurcations

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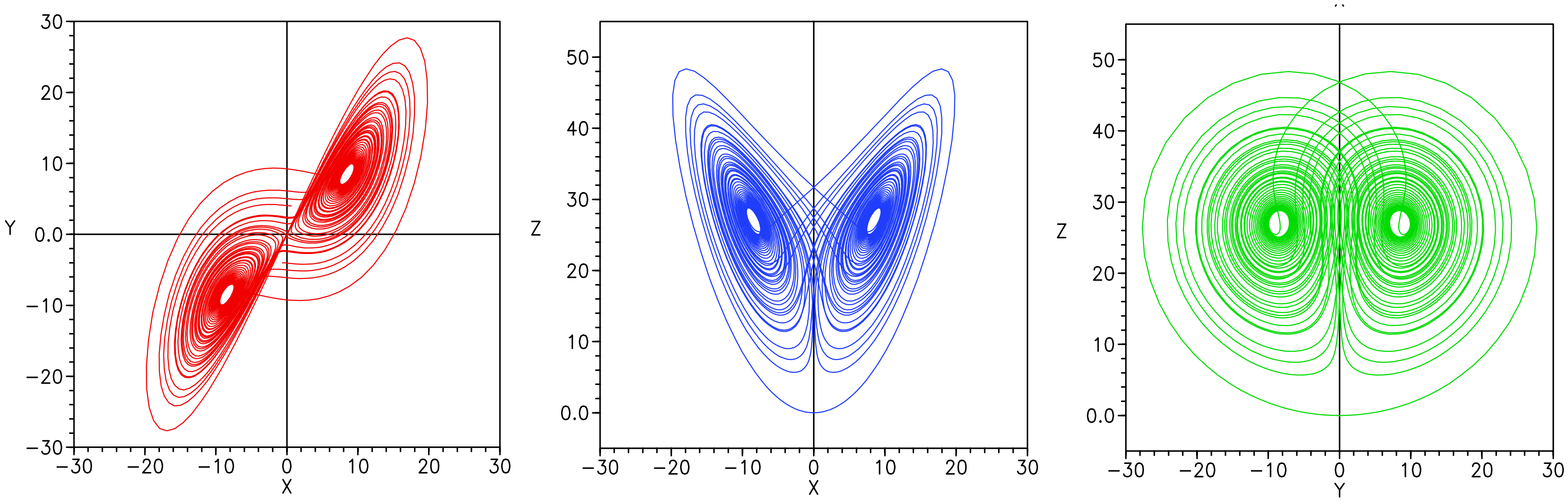
Structure of Chaos

Boundary

The nonperiodic or chaotic solutions are confined within a boundary. The approximate boundary has been drawn by finding the approximate unstable manifold of the steady state O. Two trajectories emanate from O, i.e., approach it as $t \to -\infty$.

The boundary does not include the unstable steady states O, C and C'. Any solution that enters the region within the boundary will be trapped there for all future times.



Fractal Structure

The chaotic solutions are shown by Lorenz as contours of X in the Y-Z plane. Thus X is represented as one single-valued function of Y and Z over much of range of Y and Z, and as two single-valued functions over the reminder. It seems as though the trajectories are confined to a pair of surfaces which appear to merge in the lower part. The spiral about C is in the upper surface while the spiral about C' is in the lower surface.

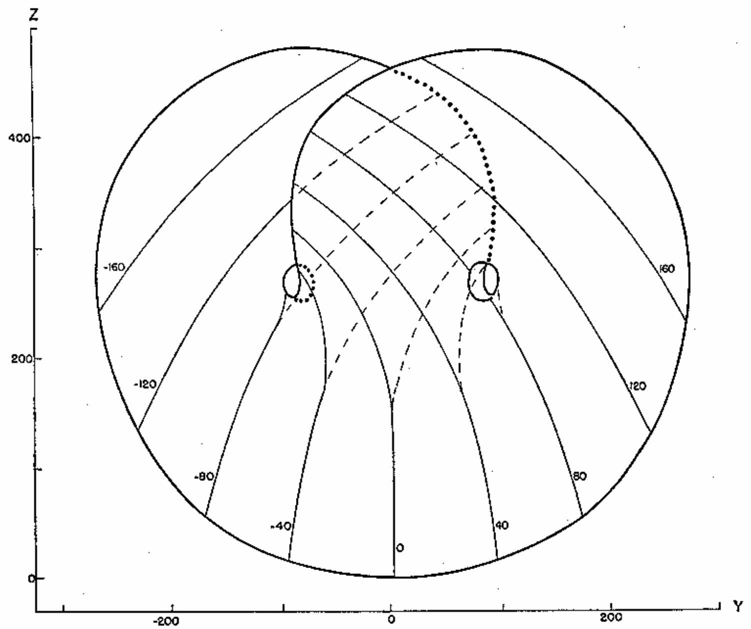
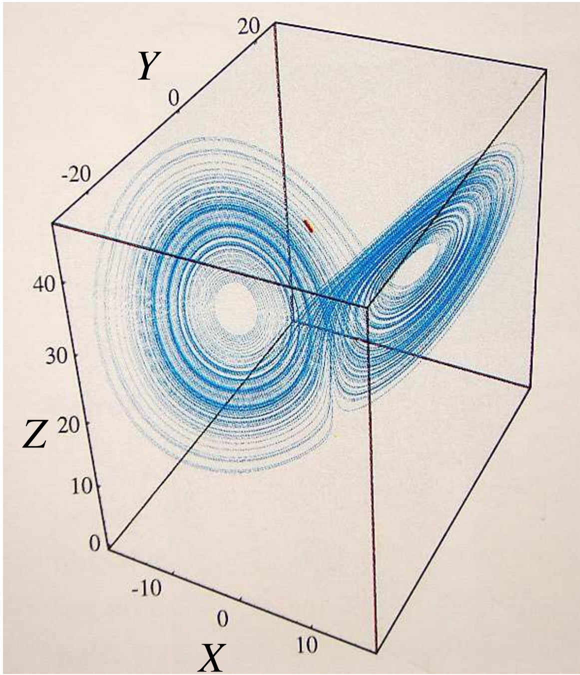


FIG. 3. Isopleths of X as a function of Y and Z (thin solid curves), and isopleths of the lower of two values of X, where two values occur (dashed curves), for approximate surfaces formed by all points on limiting trajectories. Heavy solid curve, and extensions as dotted curves, indicate natural boundaries of surfaces.

Lorenz, E. N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130-141



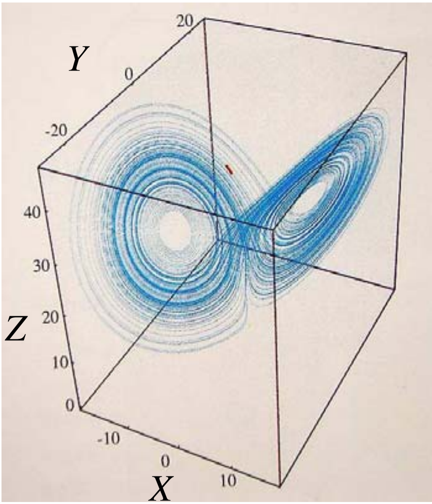
Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press

The two surfaces merely appear to merge. However, there are actually an infinite number of surfaces. They are close to one or the other of two seemingly merging surfaces.

For example, the two $X=0$ cross-sections may appear to merge, but they are shown not to merge when magnified.

Further magnification of the surfaces show that the cross-sections have thickness. i.e., there are many (infinite) points in the Y-Z plane where $X=0$ intersects.

There are also gaps in between the intersections. The cross-sections clearly reveal *fractal* structure and self-similarity.



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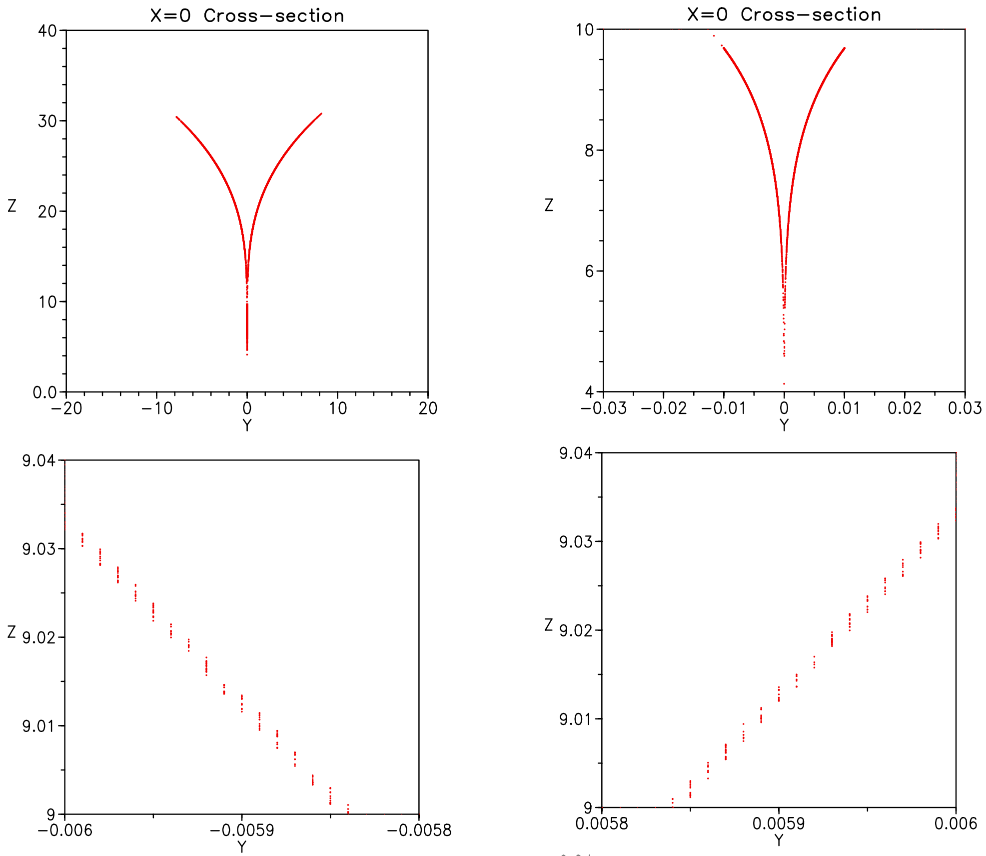
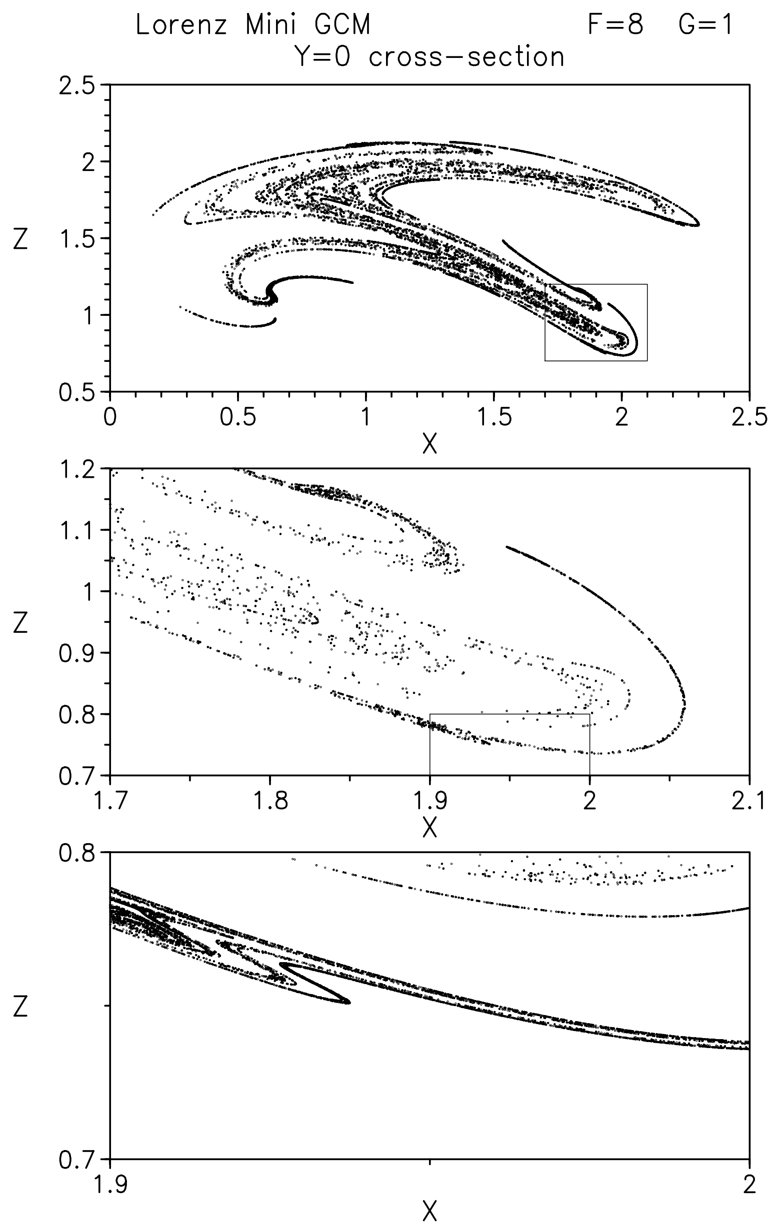


Chaos in Lorenz's "Mini GCM"

Lorenz, E. N., 1984: Irregularity: a fundamental property of the atmosphere. *Tellus*, 36A, 98-110.

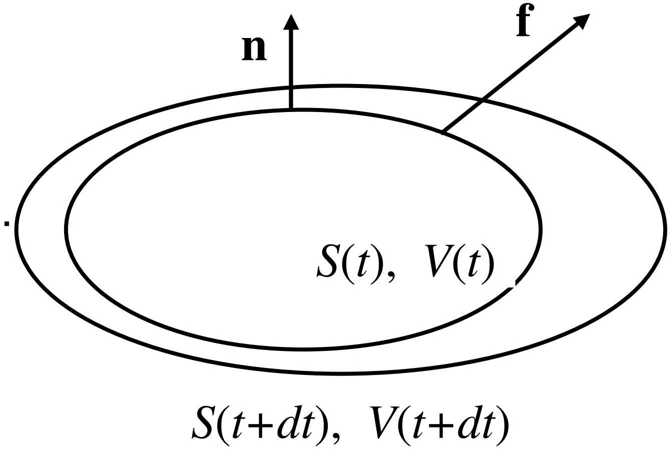
Such fractal and self-similar structures of chaotic attractors are clearly evident in other models also. Another 3-Variable model from Lorenz (1984) shows more intricate structure of a chaotic attractor for a particular choice of parameter values.



Volume contraction

Why are the solutions confined within a small boundary?

Consider a dynamical system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$



Choose an arbitrary closed surface $S(t)$ of volume $V(t)$.
Let the initial points of the trajectories be on S.
For an infinitesimal time dt, let $S(t)$ evolves into a new surface $S(t+dt)$ with a volume $V(t+dt)$.

A small area dA sweeps out a volume (f•ndt)dA in time dt.

$$V(t+dt) = V(t) + \int_S (\mathbf{f} \bullet \mathbf{n}dt)dA$$

$$\frac{dV}{dt} = \int_S \mathbf{f} \bullet \mathbf{n}dA$$

Using the divergence theorem, we obtain

$$\frac{dV}{dt} = \int_V \nabla \bullet \mathbf{f}dV$$

Volume contraction in Lorenz model

$$\nabla \bullet \mathbf{f} = \frac{\partial}{\partial X}(-\sigma X + \sigma Y) + \frac{\partial}{\partial Y}(-XZ + rX - Y) + \frac{\partial}{\partial Z}(XY - bZ)$$
$$= -\sigma - 1 - b < 0$$

Rate of change of volume is $\dfrac{dV}{dt} = -(\sigma + b + 1)V$

Volume evolves as $V(t_1) = e^{-(\sigma+b+1)(t_1-t_0)}V(t_0)$

Thus the volume is shrinking at an exponential rate.

For the parameter values used by Lorenz, if $t_1 = t_0 + 0.7$, $V(t_1) = 0.00007V(t_0)$ showing that the volume has decreased by several orders of magnitude in a short period of time.

Two states separated from each other in a suitable direction come together very rapidly and appear to merge. Thus two surfaces which appear to merge remain as distinct surfaces.

The reason for the exponential shrinking of the volume is the dissipation in the system.

Trapping of the solutions

For conservative systems for which some positive definite quantity Q (such as energy) is constant with time, each trajectory is confined to a particular surface of constant Q.

In systems with forcing and dissipation, whenever Q equals or exceeds a certain value Q_1, if the dissipation acts to decrease Q more rapidly than the forcing can increase Q, then $(-dQ/dt)$ has a positive lower bound where $Q \geq Q_1$. Each trajectory must ultimately become trapped in the region where $Q < Q_1$.

For Lorenz model, $Q = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right)$

Q is conserved in the absence of forcing and dissipation.

With forcing and dissipation,

$$\frac{dQ}{dt} = -\sigma X^2 - Y^2 - bZ^2 + b(r+\sigma)Z = -\left[\sigma X^2 + Y^2 + b\left(Z - \frac{r+\sigma}{2}\right)^2\right] + b\left(\frac{r+\sigma}{2}\right)^2$$

RHS $= 0$ on the surface of an ellipsoid E.
RHS > 0 in the interior of E.
Every orbit ultimately enters the sphere and is trapped within the sphere.

As t progresses,

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdot\cdot\cdot\cdot \rightarrow S_\infty$

$V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \cdot\cdot\cdot\cdot \rightarrow V_\infty$

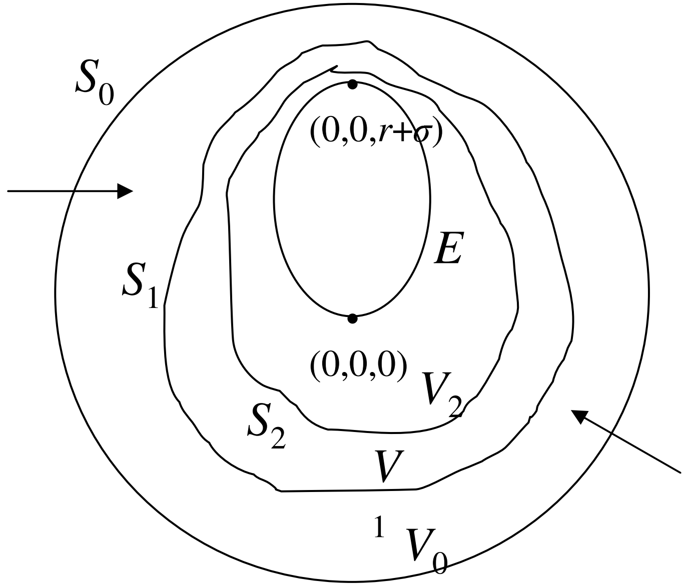
The volume decays exponentially

$$\frac{dV}{dt} = -(\sigma + b + 1)V$$

Let the intersection be

$V_\infty = V_0 \cap V_1 \cap V_2 \cdot\cdot\cdot\cdot$

All the orbits are ultimately contained in V_∞ which has zero volume. This does not imply that V_∞ shrinks to a point; it may simply become flattened into a surface.



Attractor

Attractor is a set of phase space points to which all neighboring trajectories converge asymptotically.

Attractor is a closed invariant set. Any trajectory that starts in the attractor stays in the attractor.

The set of all initial conditions that reaches the attractor is the *basin of attraction*.

Attractor is minimal. There is no subset that satisfies the above conditions.

Attractor is relevant only for dissipative systems.

Types of attractors

Steady state attractor is the stable steady state solution of the system. It is just a point in the phase space.
It is also called *fixed point* or *equilibrium point*.

Periodic attractor is the stable periodic solution forming a closed loop or cycle in the phase space.
It is also called *limit cycle*.

Quasi-periodic attractor is the trajectory on a torus that almost (but not quite) repeats the same route periodically. Each trajectory winds around endlessly on the torus, never intersecting itself and yet never quite closing.

Chaotic attractor is the set of all nonperiodic solutions confined to a bounded region of phase space with volume zero. The cross-sections have fractal structure.
It is also called *strange attractor*.

Steady state attractors in Lorenz model

$\underline{r < 1}$

$$X = 0, \quad Y = 0, \quad Z = 0$$

State of no convection (O)
There is only one steady state attractor.

$\underline{1 < r < 24.74}$

$$X = Y = \pm[b(r-1)]^{1/2}, \quad Z = r-1$$

Steady convection (C, C')
For all values of r in this range, there are two steady state attractors (C, C'). The basins of attraction for these two attractors are different. It is difficult to predict which initial state will reach which attractor (either C or C'). When there are two or more steady state attractors, they are also referred to as multiple equilibria.

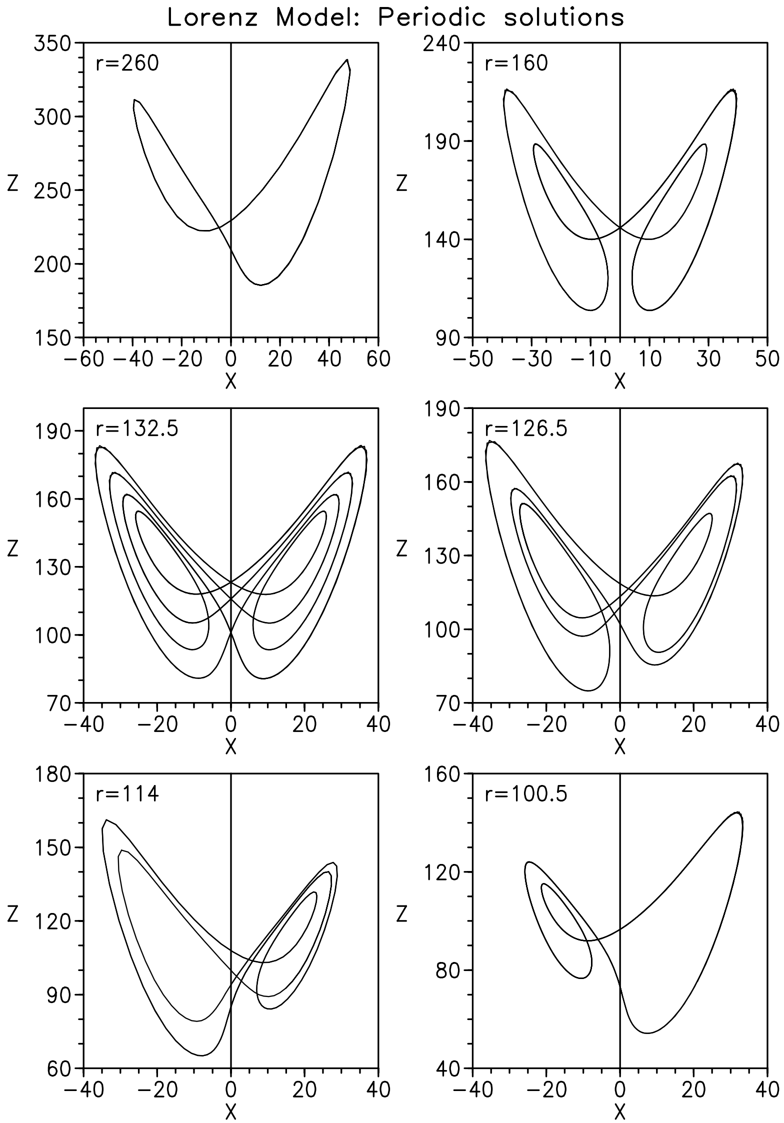
$\underline{r > 24.74}$

There are no steady state attractors.

Periodic attractors in Lorenz model

As *r* is varied, the behavior of the Lorenz model also varies from having chaotic attractors to periodic attractors at irregular intervals of *r*. Windows of periodic solutions and windows of nonperiodic solutions alternate as *r* varies.

$r = 260.0$	xy	Nonsymmetric
$r = 160.0$	x^2y^2	Symmetric
$r = 132.5$	$x^2y\, xy^2\, xy$	Symmetric
$r = 126.5$	$xy\, x^2y^2$	Nonsymmetric
$r = 114.0$	$x^2y\, xy$	Nonsymmetric
$r = 100.5$	xy^2	Nonsymmetric

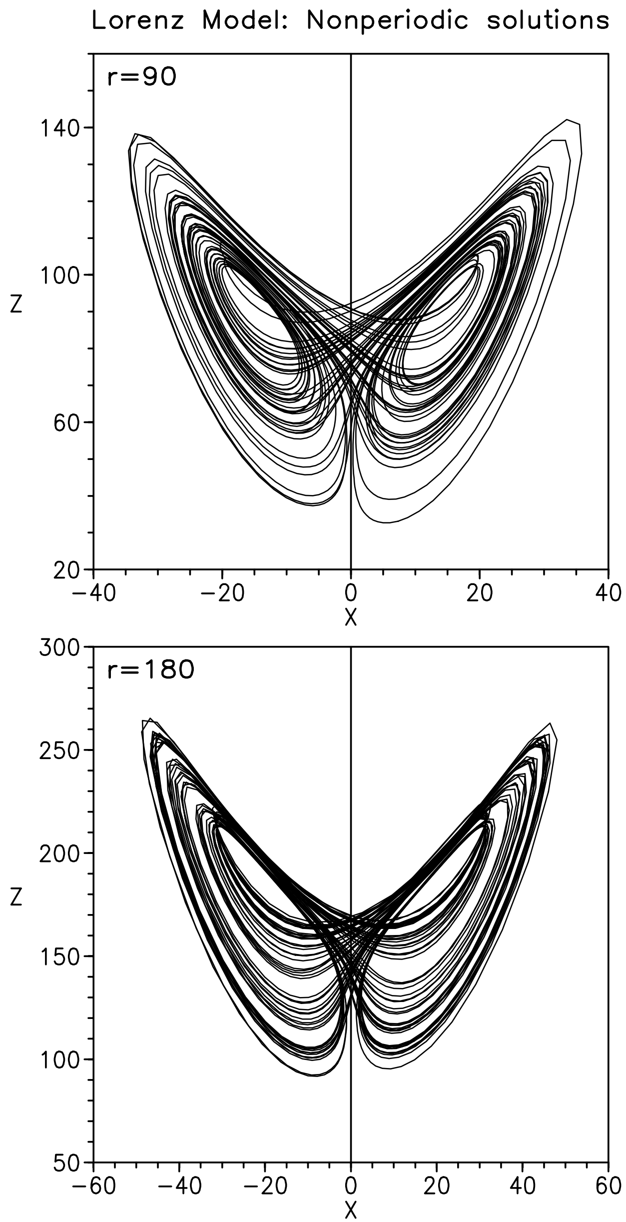


Chaotic attractors in Lorenz model

Chaotic attractors exist at different values of r.

The attractors in different intervals of r have different characteristics.

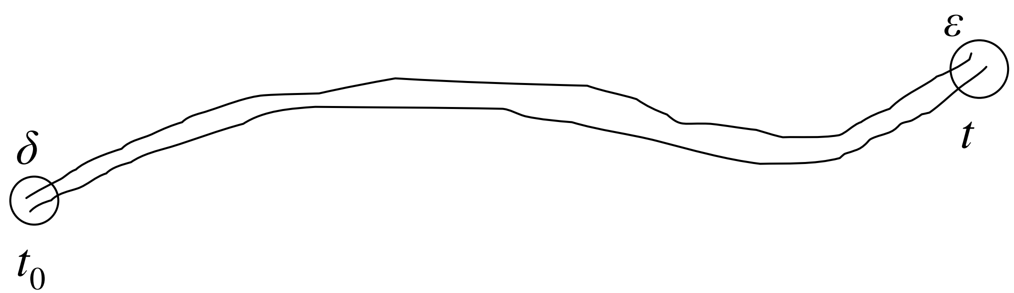
The chaotic attractors at r=90 and r=180 are located in different parts of the phase space.



Stability

An orbit is called ***stable*** at a point $X(t_0)$ if any other orbit passing sufficiently close to $X(t_0)$ at time t_0 remains close to $X(t)$ as $t \to \infty$.

An orbit $X(t)$ is ***stable*** at $t = t_0$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|Y(t_0) - X(t_0)| < \delta$ and $t > t_0$, $|Y(t) - X(t)| < \varepsilon$.



This is called ***Liapunov stable***: "start near stay near"

Otherwise, $X(t)$ is un***stable***.

If $X(t)$ is stable at $t = t_0$, it is stable for all $t > t_0$ (and also at $t < t_0$ if the system is defined by differential equations).

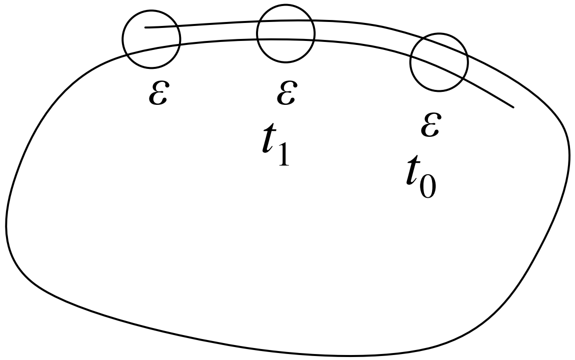
If $X(t)$ is Liapunov stable and if $|Y(t) - X(t)| \to 0$ as $t \to \infty$ (i.e., attracting), then $X(t)$ is ***asymptotically stable***.

Periodicity

Since each point lies on a unique orbit, any orbit passing through a point through which it has previously passed must continue to repeat its past behavior and must be *periodic*.

An orbit $X(t)$ is *quasi-periodic* if for some arbitrary large time interval τ, $X(t+\tau)$ ultimately remains arbitrarily close to $X(t)$.

$X(t)$ is *quasi-periodic* at if, for any $\varepsilon > 0$ and τ_0, there exists a $\tau > \tau_0$ such that $|X(t+\tau) - X(t)| < \varepsilon$ if $t > t_0$.



Periodic orbits are special cases of quasi-periodic orbits.

An orbit with a stable limiting orbit is quasi-periodic (includes periodic orbits). These orbits are the periodic or quasi-periodic attractors.

Nonperiodic orbits

An orbit that is not quasi-periodic is called *nonperiodic*.

If $X(t)$ is nonperiodic, $X(t_1+ \tau)$ may be arbitrarily close to $X(t_1)$ for some time t_1 and some arbitrarily large time interval τ, but if this is so, $X(t_1+ \tau)$ cannot remain arbitrarily close to $X(t)$ as $t\rightarrow\infty$.

A nonperiodic orbit is *unstable*. It implies that two states differing by imperceptible amounts may eventually evolve into two considerably different states.

If there is any error in observing the present state, an acceptable prediction in the distant future may well be impossible.

Instability places a limit on the predictability of the system if the observations are less than perfect. The deciding factor in predictability is stability versus instability.

Attractors and Stability

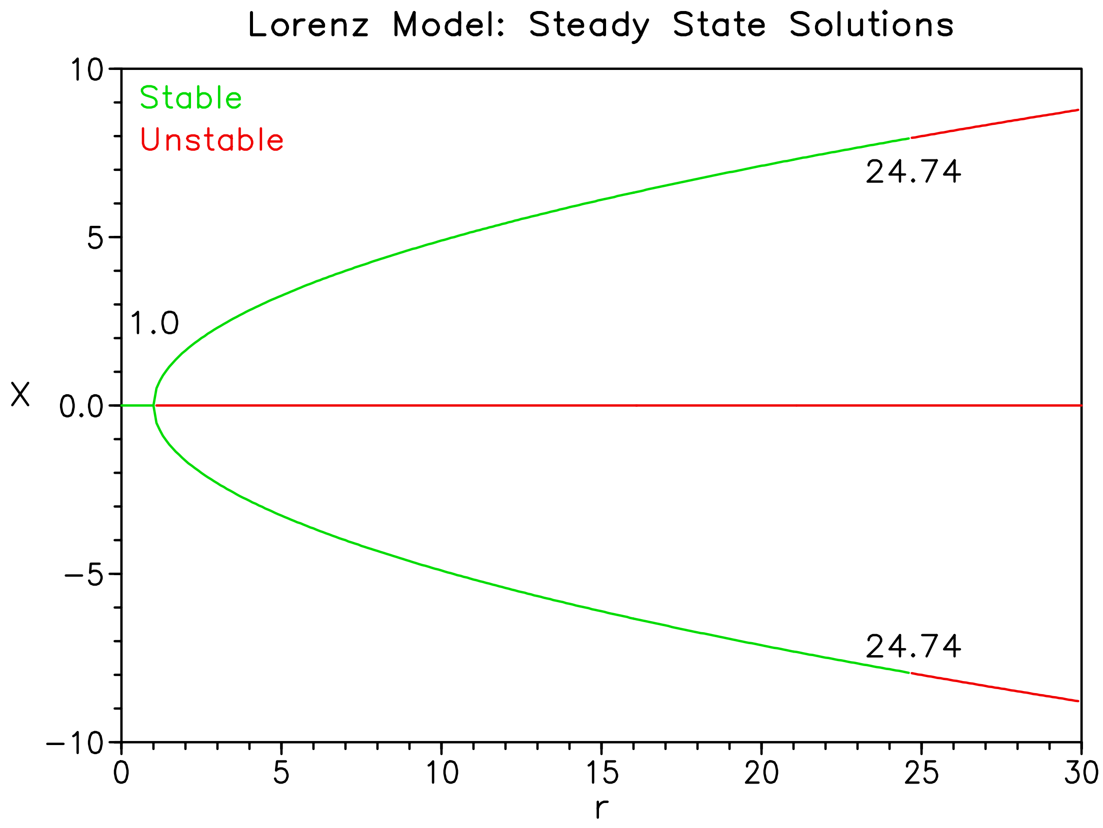
Steady state attractors (fixed points) are stable.
Unstable steady states are not attractors.

Periodic attractors (limit cycles) are stable.
Unstable periodic solutions are not attractors.

Nonperiodic (chaotic) attractors consist of points that are only unstable.

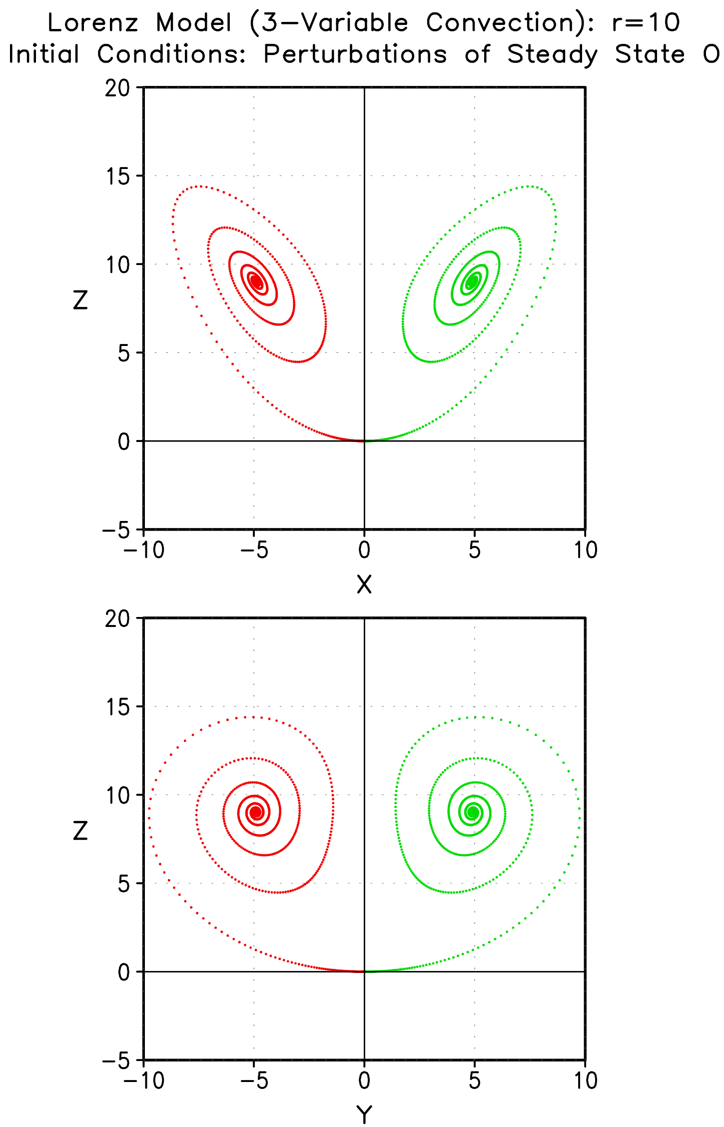
Stability of Steady States in Lorenz Model

$\sigma = 10$ and $b = 8/3$



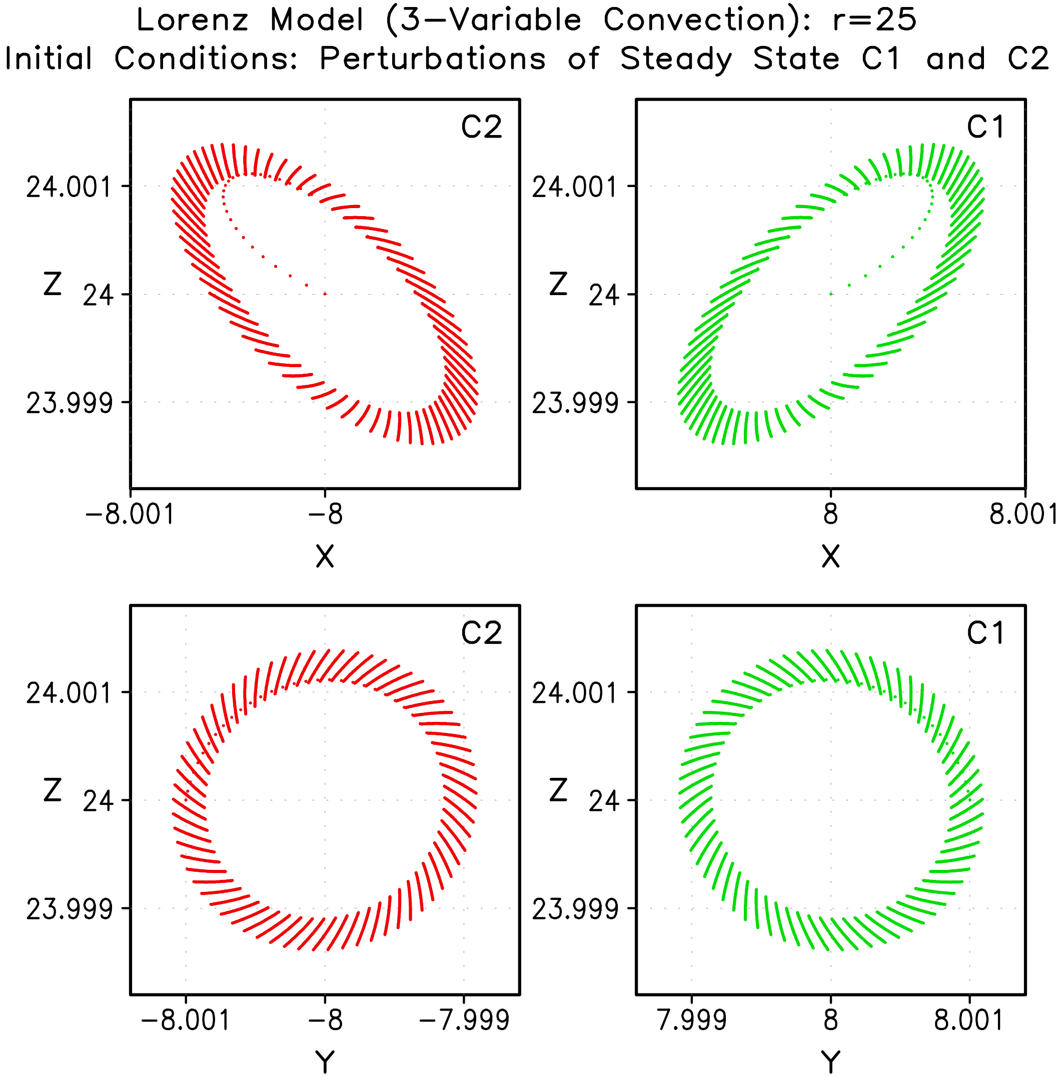
Instability of steady state O

Initial states starting with small perturbations over the steady state O will reach either steady state C or C' in the range $1 < r < 24.74$. The orbit spirals into either C or C' depending on the initial state.



Instability of C and C'

An orbit starting from an initial state with a small perturbation over the steady state C or C' moves around in an unstable periodic orbit when r is just above r_c.



Bifurcation

Whenever the solution changes qualitatively at a fixed value, called the critical value, of a parameter, it is called a *bifurcation*.

A point in the parameter space where such an event occurs is defined as the *bifurcation point*.

Several (two or more) solution branches, either *stable or unstable*, emerge from a bifurcation point.

The representation of any characteristic property of the solutions as a function of the bifurcation parameter constitutes a *bifurcation diagram*.

Bifurcation of steady states (fixed points)

Pitchfork bifurcation

One stable solution becomes unstable while two new stable solutions come into existence. This bifurcation is common in systems with symmetry, e.g., Lorenz model at $r = 1$.

Saddle-node bifurcation

At the saddle-node bifurcation point, fixed points are created and destroyed. As the parameter varied, a stable fixed point and an unstable fixed meet and mutually annihilate.

Transcritical bifurcation

A stable fixed point and an unstable fixed point meet at the bifurcation point and exchange symmetry. After the bifurcation, the stable fixed becomes unstable and the unstable fixed point becomes stable.

The above three bifurcations involve collision of two or more fixed points.

Hopf bifurcation

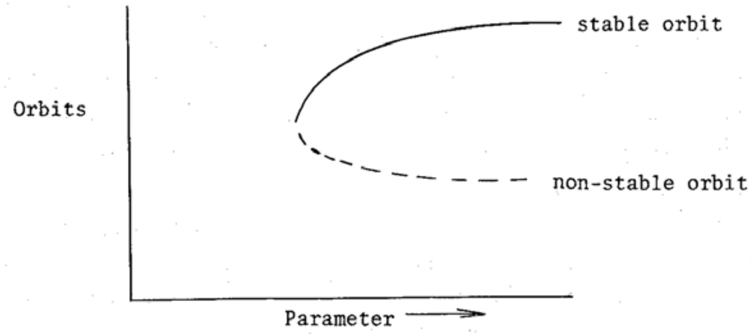
The steady state becomes unstable at the critical value and the resulting motion is a small-amplitude limit cycle (periodic orbit).

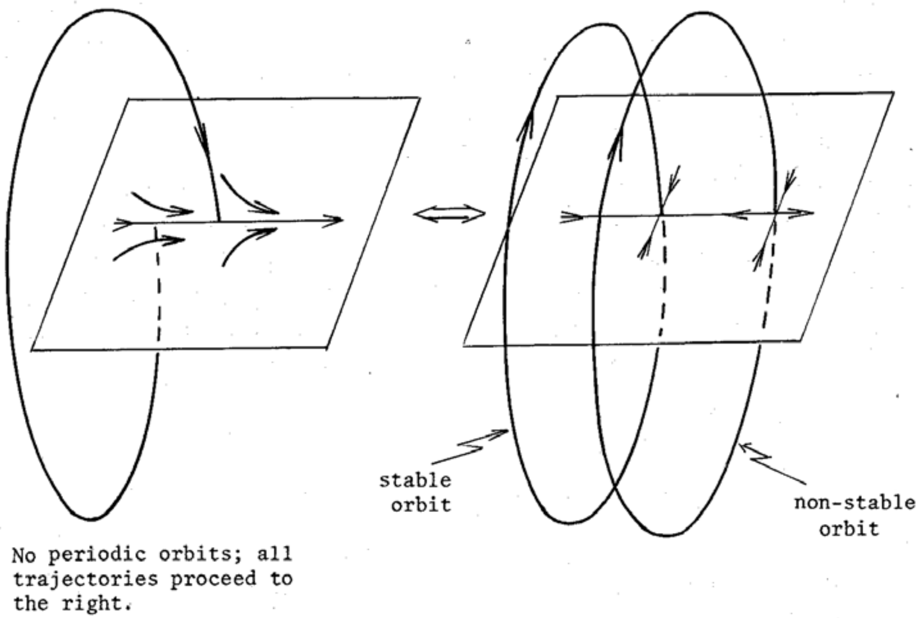
At $r = 24.74$, the Lorenz model undergoes a Hopf bifurcation, but it is subcritical and results in the creation of an unstable periodic orbit.

Bifurcations of periodic solutions

Saddle-node bifurcation

As the parameter value is changed, a stable periodic solution and an unstable periodic solution come together and annihilate. After annihilation, type I intermittent chaos occurs.





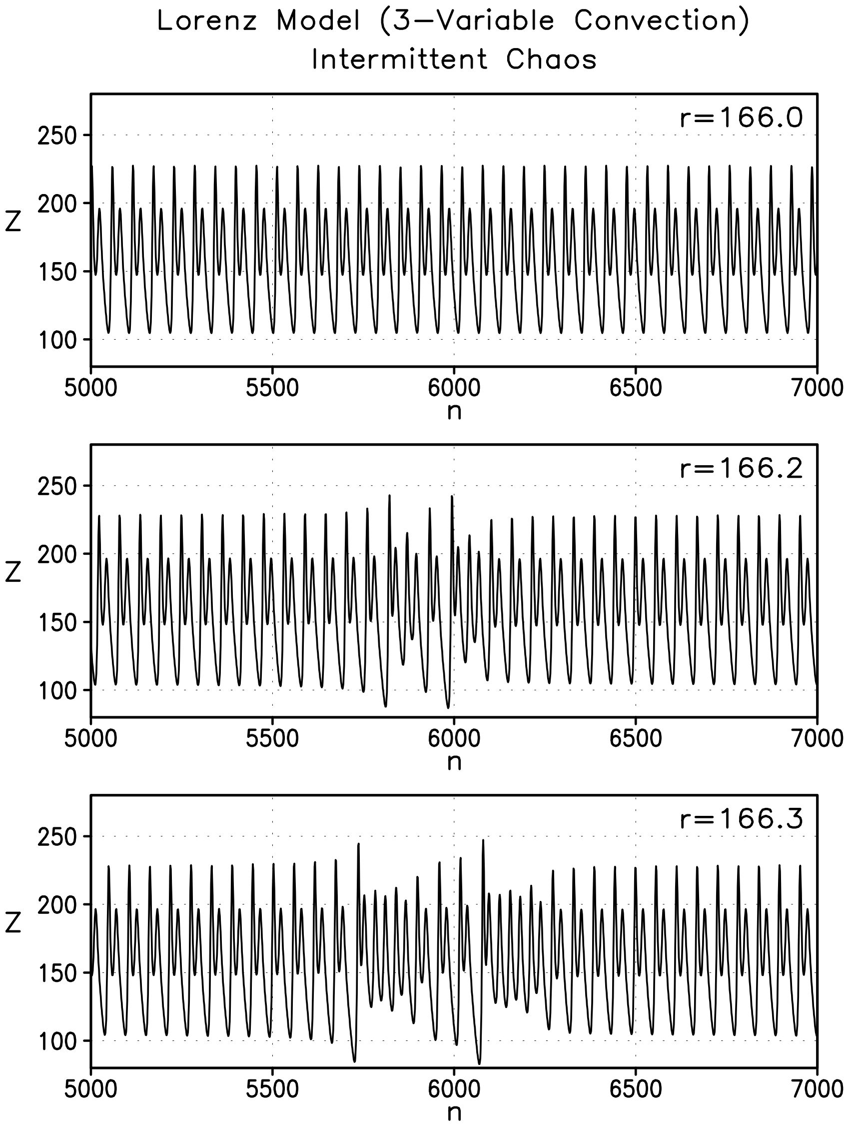
Sparrow, C., 1983: *The Lorenz equations: Birfucations, chaos and strange attractors*, Springer.

Intermittent chaos in Lorenz model

After a saddle node bifurcation, intermittent chaos occurs in the Lorenz model. The periodic orbit at $r = 166$ is in the stable branch before the bifurcation.

At $r = 166.2$, the intermittent chaos occurs with bursts of nonperiodic variation at irregular intervals between nearly periodic behavior.

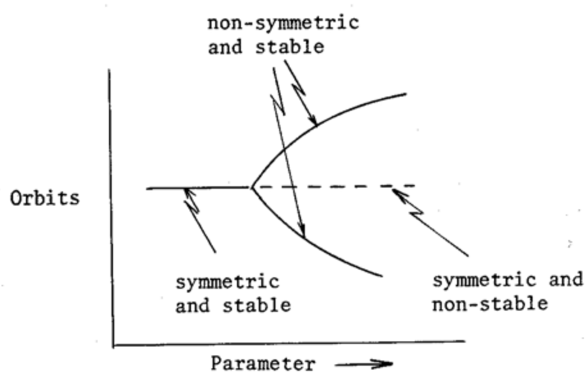
The intermittent bursts of chaos increase in frequency and duration as r changes, such as in $r = 166.3$.

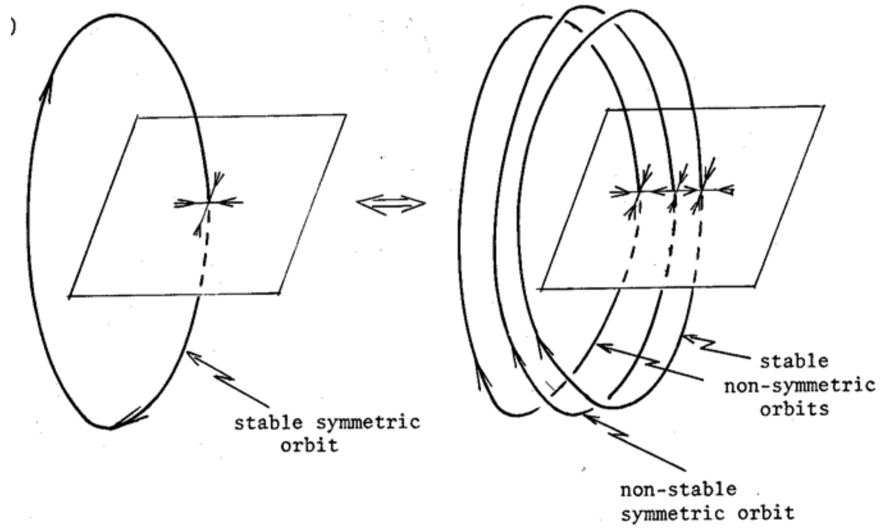


Symmetric saddle-node bifurcation

In a symmetric saddle-node bifurcation, one stable symmetric orbit becomes unstable while two stable nonsymmetric orbits come into existence after the bifurcation point. This bifurcation is a result of some symmetry in the model and is similar to the pitchforck bifurcation of the fixed point.

Example in Lorenz model: symmetric xy orbit at $r = 350$ and nonsymmetric xy orbits at $r = 260$.

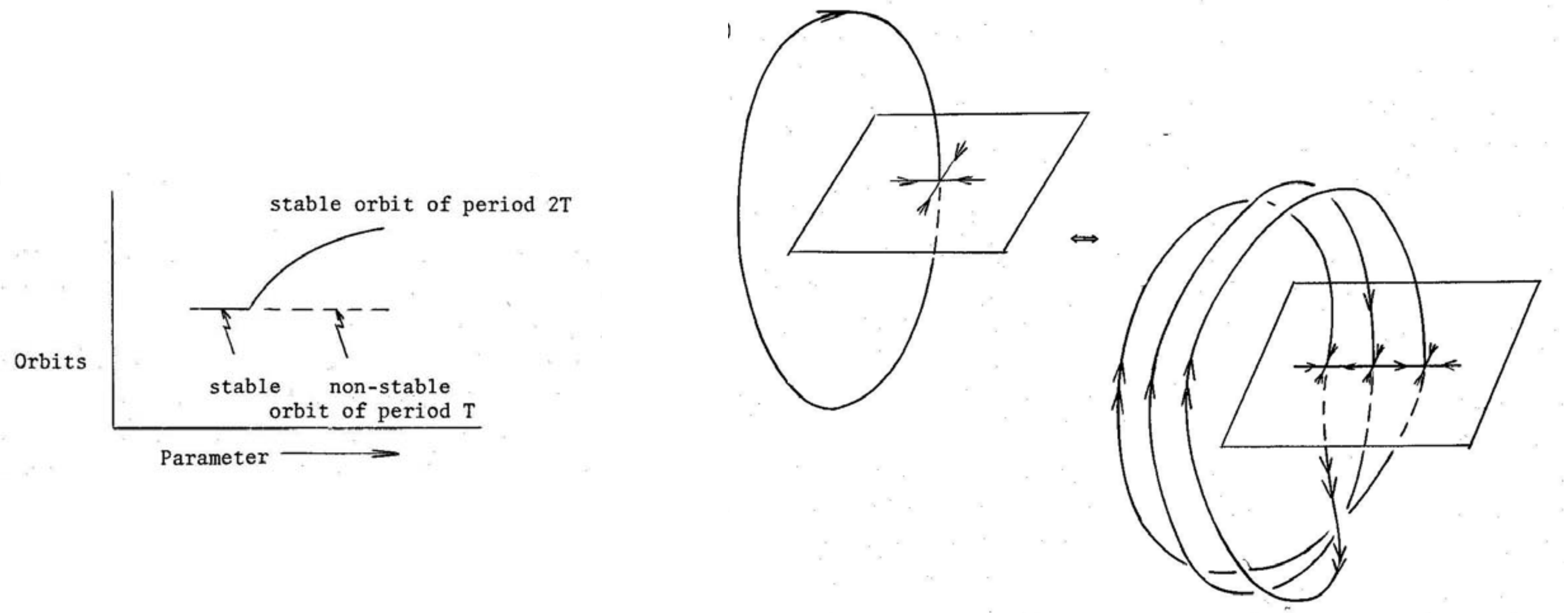




Sparrow, C., 1983: *The Lorenz equations: Birfucations, chaos and strange attractors*, Springer.

Period-doubling bifurcation

As the parameter value is changed, a periodic orbit of period T becomes unstable while a stable periodic orbit of period $2T$ comes into existence. As the parameter is further changed, period $2T$ orbit becomes unstable and a stable $4T$ orbit is born. The period doubling sequence continues until the transition to chaos occurs.



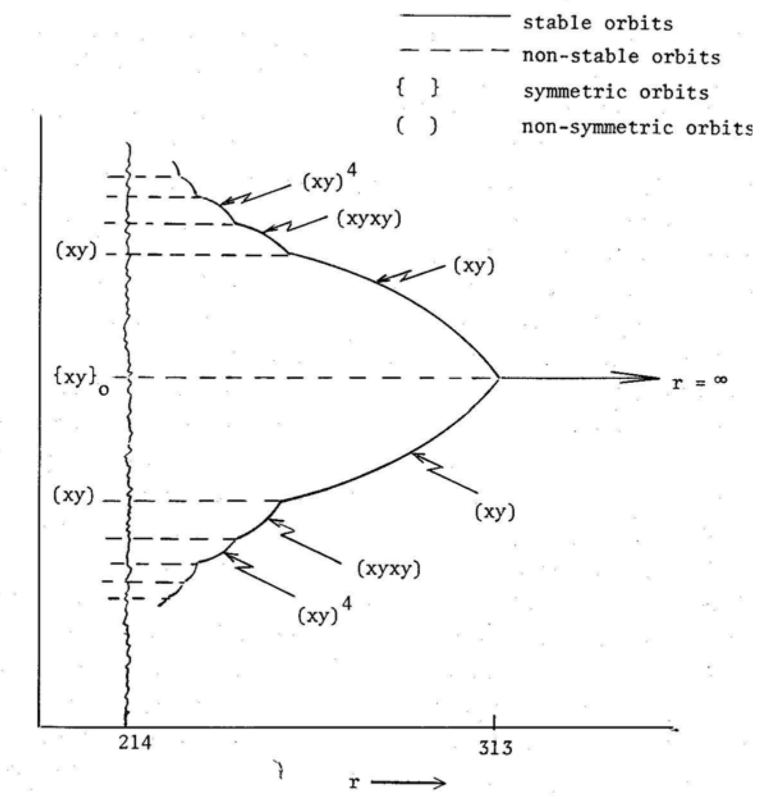
Sparrow, C., 1983: *The Lorenz equations: Birfucations, chaos and strange attractors*, Springer.

Let r_n be the value of the parameter at which a period-doubling bifurcation occurs. Then r_{n-1} and r_{n+1} are the values at which previous and subsequent period-doubling bifurcations take place.

The length (in parameter value) of each successive periodic window reduces according to *Feigenbaum constant* given by

$$\delta = \lim_{n\to\infty} \frac{(r_{n-1} - r_n)}{(r_n - r_{n+1})} = 4.6692016..... \qquad (9.1)$$

The Feigenbaum constant is universal.



Period doubling in Lorenz model

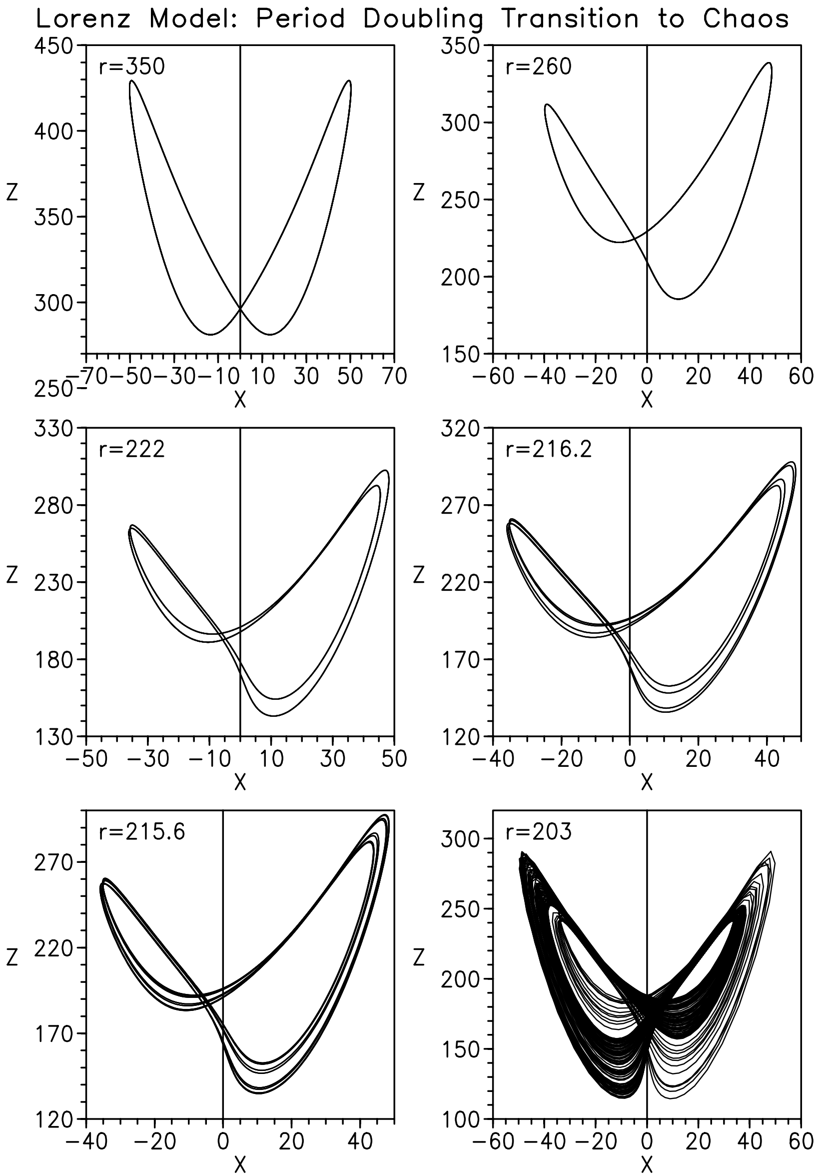
Sparrow, C., 1983: *The Lorenz equations: Birfucations, chaos and strange attractors*, Springer.

Period doubling in Lorenz Model

At $r = 350$, the model has a symmetric xy orbit. After undergoing a symmetric saddle node bifurcation, there is a nonsymmetric xy orbit at $r = 260$.

When r is further decreased, the model undergoes period doubling bifurcation. There is a period-2 orbit $(xy)^2$ at $r = 222$, a period-4 orbit $(xy)^4$ at $r = 216.2$, a period-8 orbit $(xy)^8$ at r = 215.6.

This sequence ends in chaos as shown for $r = 203$.



Hopf bifurcation of the periodic solution

This is also known as the Ruelle-Takens transition to chaos
In this scenario, three successive Hopf bifurcations occur before the transition to chaos.
Steady state → Limit cycle → T^2 Torus → T^3 Torus → Chaos
The three independent frequencies are f_1, f_2 and f_3.

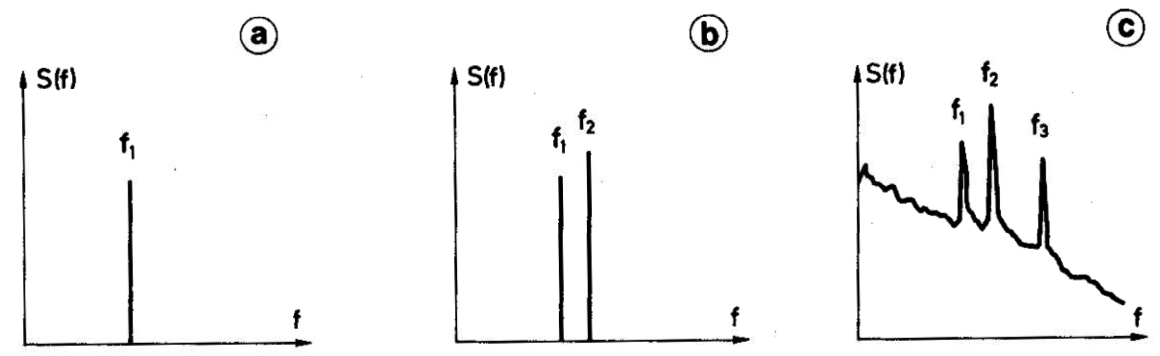
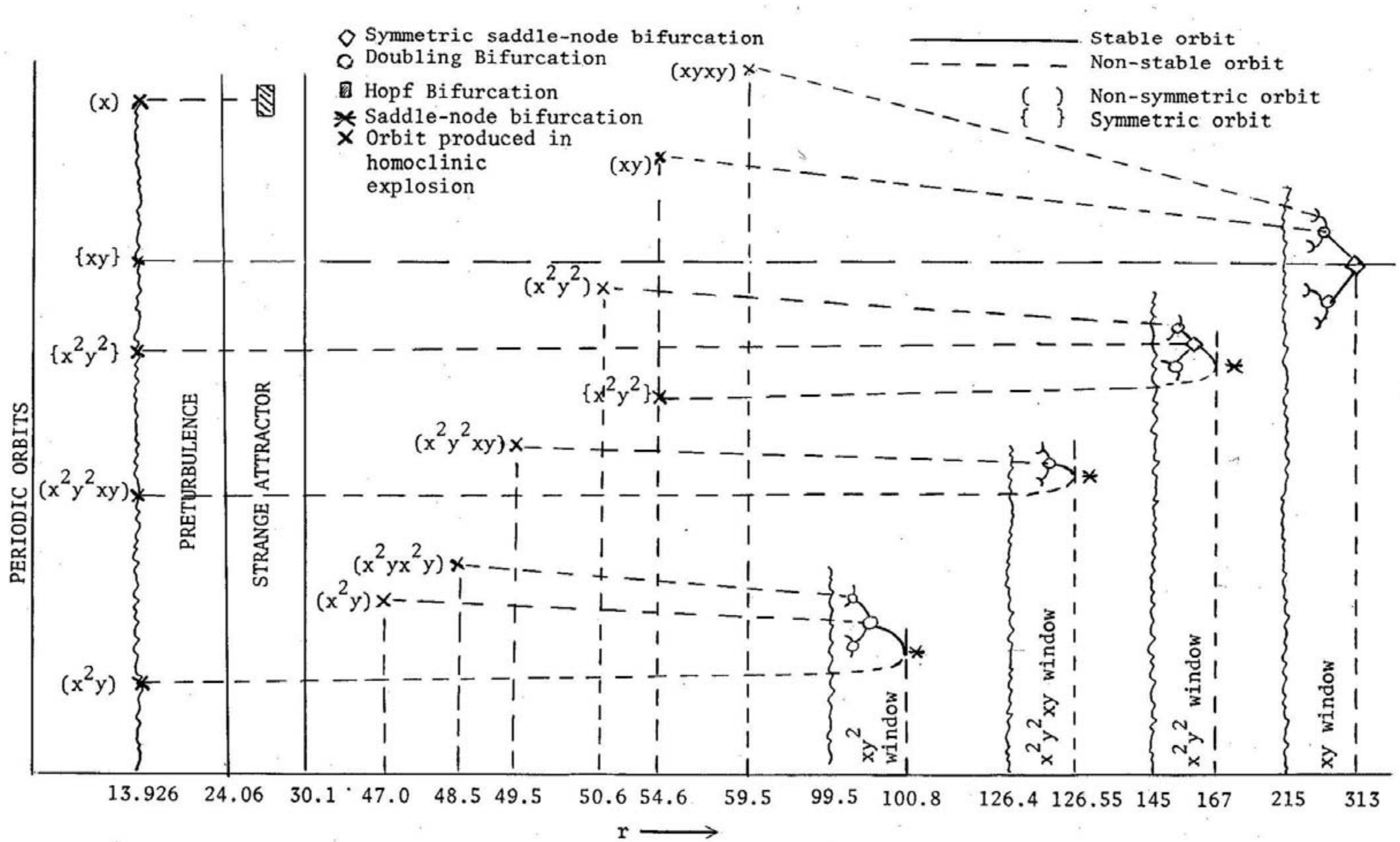


Figure VII.5 Schematic evolution of the power spectrum according to the Ruelle-Takens theory.
***a*) periodic regime**
***b*) quasiperiodic regime with two frequencies**
***c*) chaotic regime**
$S(f)$ is the power spectrum defined in chapter III (designated there by $|\hat{x}_k|^2$).

Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons

Bifurcations in the Lorenz model



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