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# Theory of Error Growth
# Linear Stability Analysis

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# Theory of Error Growth

## Lecture 1
## Linear Stability Analysis

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A Systems View: Theory, Models and Data Analysis
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## Dynamical Systems

1. Chaos in deterministic systems

2. Low-order dynamical systems and predictability

3. Attractors and bifurcations

## Theory of Error Growth

1. Linear stability analysis

2. Structure of errors

3. Growth of random errors

## Linear Stability Analysis

Study the stability of solutions with respect to small perturbations (or errors).
Consider a dynamical system

$$\frac{dX_i}{dt} = F_i(X_1, \cdots, X_M), \quad i = 1, \cdots, M$$

In compact notation,

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

where $\mathbf{X} = (X_1, \cdots, X_M)$ and $\mathbf{F} = (F_1, \cdots, F_M)$

Consider two solutions $\mathbf{X}$ and $\mathbf{X}+\mathbf{x}$, where $\mathbf{x} = (x_1,\ldots,x_M)$ is small.

$$\frac{d}{dt}(\mathbf{X}+\mathbf{x}) = \mathbf{F}(\mathbf{X}+\mathbf{x}) = \mathbf{F}(\mathbf{X}) + \frac{\partial \mathbf{F}(\mathbf{X})}{\partial \mathbf{X}}\mathbf{x} + \cdots \text{higher order terms}$$

Neglecting higher order terms (because $\mathbf{x}$ is small), we obtain a linear equation for $\mathbf{x}$.

Error equation: $\dfrac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x}$

where

$$\mathbf{H} = \begin{pmatrix} \dfrac{\partial F_1}{\partial X_1} & \cdots & \dfrac{\partial F_1}{\partial X_M} \\ \vdots & & \vdots \\ \dfrac{\partial F_M}{\partial X_1} & \cdots & \dfrac{\partial F_M}{\partial X_M} \end{pmatrix}$$

If $\mathbf{X}$ is a steady state, $\mathbf{H}$ is constant,

If $\mathbf{X}$ is periodic or chaotic, $\mathbf{H}$ is time-dependent.

Because $\mathbf{H}$ is linear, the error equation can be integrated from time $t_0$ to $t_1$ to obtain

$$\mathbf{x}(t_1) = \mathbf{A}(t_1,t_0)\ \mathbf{x}(t_0)$$

$$x_i(t_1) = \sum a_{ij}(t_1,t_0)\ x_j(t_0), \qquad i = 1,\ \ldots,\ M$$

$\mathbf{A}$ is a square matrix which depends on the behavior of $\mathbf{X}$ between $t_0$ and $t_1$.

$\mathbf{A}$ is multiplicative:

If $t_0 < t_1 < t_2$,

$$\mathbf{x}(t_2) = \mathbf{A}(t_2, t_0)\mathbf{x}(t_0)$$
$$\mathbf{x}(t_2) = \mathbf{A}(t_2, t_1)\mathbf{x}(t_1) = \mathbf{A}(t_2, t_1)\mathbf{A}(t_1, t_0)\mathbf{x}(t_0)$$

Equating the RHS of the two equations**,**

$\mathbf{A}(t_2,t_0) = \mathbf{A}(t_2,t_1)\ \mathbf{A}(t_1,t_0)$

Simple solution:

If $M = 1$ or
if $M > 1$ and $\mathbf{H}$ is constant,

$$\mathbf{x}(t_1) = \mathbf{x}(t_0)\exp\left(\int_{t_0}^{t_1} \mathbf{H}\,dt\right)$$

## Linear Stability Analysis of Lorenz Model

Let the state of the system at time $t$ be $(X, Y, Z)$ and
let a state with a small perturbation be $(X+x, Y+y, Z+z)$ where $x$, $y$, $z$ are small.

The linear perturbation equation for the Lorenz model is

$$\frac{d\mathbf{x}}{dt} = \mathbf{Hx}$$

$$\begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-Z & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The stability equation is linear and should be integrated numerically when $(X,Y,Z)$ is time-dependent.

When $(X,Y,Z)$ is time-independent, the stability equation can be solved by assuming some form of the solution for $(x,y,z)$.

## Stability of steady states

Denote the steady states by $(X_0,Y_0,Z_0)$.
Lorenz model has three steady states $O$, $C$ and $C'$. $\mathrm{H}$ is constant.

Solve the linear stability equation by assuming
$x = x_0 \exp(\lambda t), \quad y = y_0 \exp(\lambda t), \quad z = z_0 \exp(\lambda t)$

Solve the characteristic equation

$$\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r-Z_0 & -1-\lambda & -X_0 \\ Y_0 & X_0 & -b-\lambda \end{vmatrix} = 0$$

or perform an eigenanalysis of $\mathbf{H}$ for each steady state.
Solve $\mathbf{Hv}=\lambda\mathbf{v}$, where $\mathrm{v}$ is an eigenvector with a corresponding eigenvalue $\lambda$.
For distinct eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ with eigenvectors $(\mathbf{v_1},\mathbf{v_2},\mathbf{v_3})$, the general solution is

$$\mathbf{x}(t) = c_1\exp(\lambda_1 t)\mathbf{v_1} + c_2\exp(\lambda_2 t)\mathbf{v_2} + c_3\exp(\lambda_3 t)\mathbf{v_3} \qquad (8.14)$$

where $c_1$, $c_2$ and $c_3$ depend on the initial perturbation $(x_0,y_0,z_0)$.
The stability of the steady state depends on $\lambda$.

### *Stability of $O$*

$X_0 = 0, \; Y_0 = 0, \; Z_0 = 0$

Solve $(\lambda+b)[\lambda^2 + (\sigma+1)\lambda + \sigma(1-r)] = 0$

When $r > 0$, the characteristic equation has three real roots.

$$\lambda = -b$$

$$\lambda = -\frac{1}{2}(\sigma+1) \pm [(\sigma+1) - 4\sigma(1-r)]^{1/2}$$

When $0 < r < 1$, all three roots are negative. This means that the perturbation decays at an exponential rate. The steady state $O$ is stable in this case.

When $r > 1$, one root is positive indicating that the perturbation grows at an exponential rate. The steady state $O$ is now unstable.

## *Stability of $C$ and $C'$*

When $r > 1$, there are two more steady states

$$C: \qquad X_0 = Y_0 = \sqrt{b(r-1)}, \qquad Z_0 = r-1$$

$$C': \qquad X_0 = Y_0 = -\sqrt{b(r-1)}, \qquad Z_0 = r-1$$

For stability analysis, solve the characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r+\sigma)b\lambda + 2\sigma b(r-1) = 0$$

When $r > 1$, the equation has one real root and two complex conjugate roots.
The complex roots become pure imaginary if

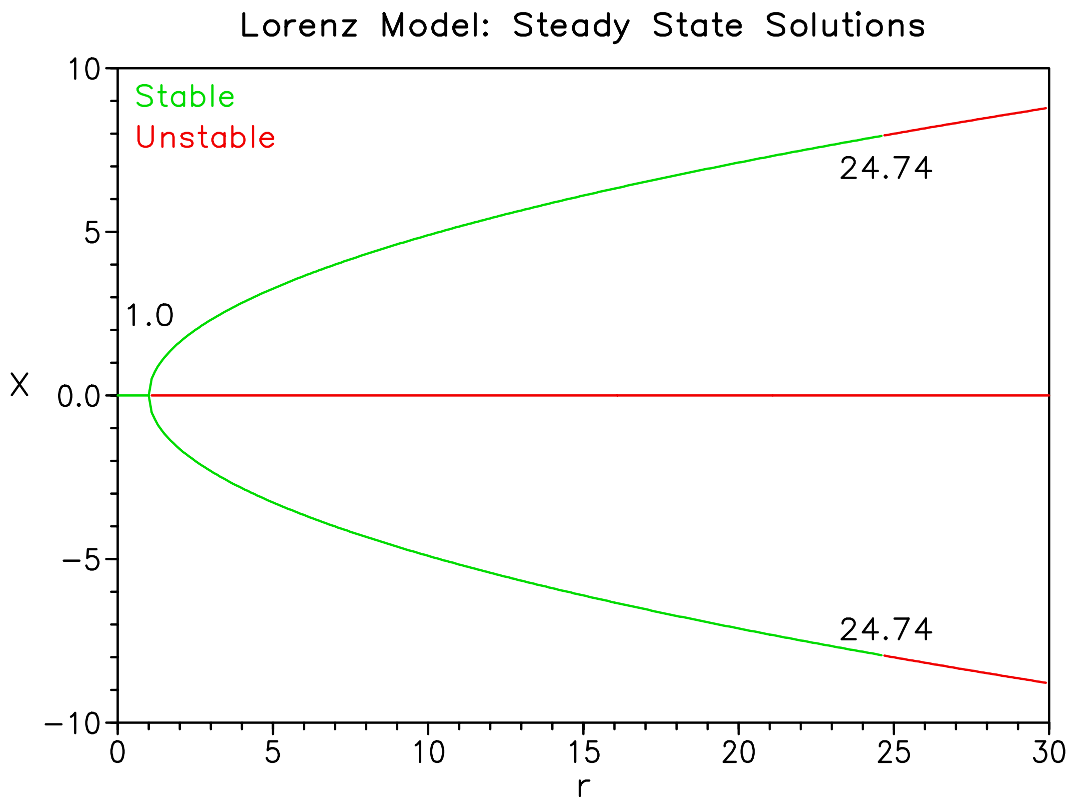
$$r = r_c = \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}$$

i.e., if the complex root is $\lambda = \lambda_r + i\lambda_i$, $\quad \lambda_r < 0$ for $1 < r < r_c$ and $\lambda_r > 0$ for $r > r_c$.
This is the critical value for the instability of $C$ and $C'$.

If $\sigma > b+1$, the steady states $C$ and $C'$ will become unstable for sufficiently high Rayleigh numbers.
For $\sigma = 10$ and $b = 8/3$, the instability occurs at the critical value of $r_c = 24.74$.

## Stability of Steady States in Lorenz Model

$\sigma = 10$ and $b = 8/3$



## Stability of Periodic Solutions and Transition to Chaos

### Floquet theory

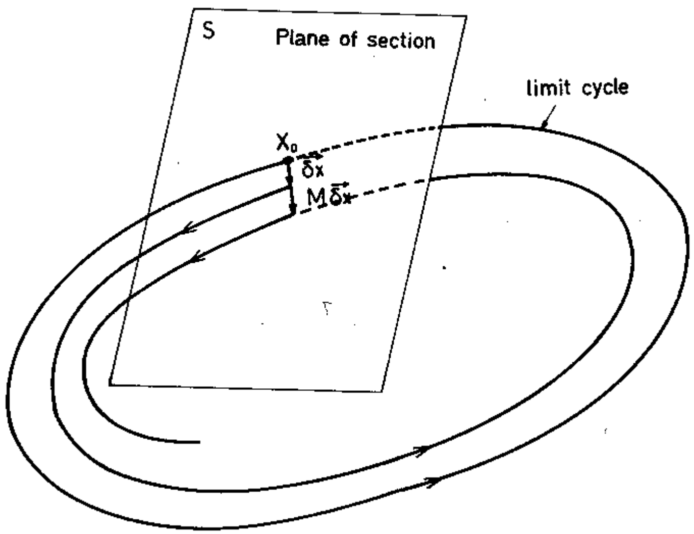
Consider a periodic solution
$\mathbf{X}(t+T) = \mathbf{X}(t)$, where $T$ = period.

A periodic solution corresponds to a fixed point $\mathbf{X_0}$ on a Poincaré cross-section $S$.
The stability of the periodic solution is the same as the stability of the fixed point on the Poincaré cross-section $S$.

Let $\delta\mathbf{X}$ be a small perturbation such that $\mathbf{X_0}+\delta\mathbf{X}$ is in $S$.

Linearizing the flow about the periodic orbit, the initial condition $\mathbf{X_0} + \delta\mathbf{X}$ is mapped to $\mathbf{X_0} + \mathbf{M}\,\delta\mathbf{X}$ at the end of the period $T$.

$$\delta\mathbf{X}(T) = \mathbf{M}\,\delta\mathbf{X}(0)$$




Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons

$\mathbf{M}$ is a square matrix called the *Floquet matrix* and determines the stability of the periodic orbit.

$\mathbf{M}$ can be computed by numerically integrating

$$\frac{d(\delta\mathbf{X})}{dt} = \mathbf{H}(\delta\mathbf{X})$$

for exactly one period on the Poincaré cross-section.

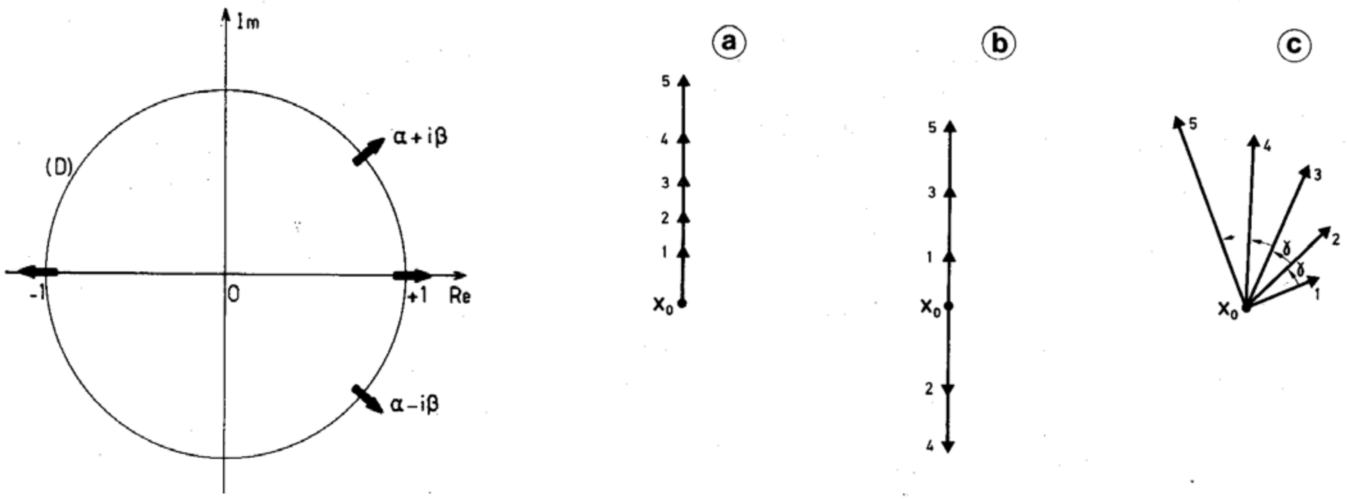
The stability of the periodic solution is determined by the eigenvalues $\lambda_i$ of $\mathbf{M}$.
One of the eigenvalues will always be equal to one, corresponding to the direction of the flow.
If all other eigenvalues are located inside the unit circle complex plane, the periodic solution is *stable*.
i.e., the closed orbit is stable if $|\lambda_i| < 1$ all $i = 1, \ldots, M-1$.

If at least one of the eigenvalues is outside the unit circle, the periodic solution is *unstable*.
i.e., the modulus of the eigenvalue is greater than one.

The $\lambda_i$ are called the *Floquet multipliers*.

## Instabilities of periodic solutions

There are three possibilities for an eigenvalue $\lambda_i$ to cross the unit circle and cause instability.




(From Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons)

(a) $\lambda_i > 1$:  $\delta X$ for each cycle is amplified in the same direction.  This is ***saddle-node*** bifurcation.

(b) $\lambda_i < -1$:  $\delta X$ is amplified in the opposite direction alternately after each cycle.  This is ***subharmonic*** or ***period-doubling*** bifurcation.

(c) $\lambda_i = \alpha + i\beta$ with $|\lambda_i| > 1$: $\delta X$ rotates by an angle $\gamma$ after each cycle, while their lengths increase. This is ***Hopf*** bifurcation.

## Evolution of small errors in chaotic systems

Lorenz, E. N., 1965: A study of the predictability of a 28-variable atmospheric model. *Tellus*, 17, 321-333.

## Linear tangent equation

As discussed earlier, the evolution of small perturbations in an $M$-dimensional dynamical system represented by

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

is given by the *linear tangent equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x}$$

where $\mathbf{X}$ and $\mathbf{X}+\mathbf{x}$ are basic and perturbed states of the system and

$$\mathbf{H} = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_M} \\ \vdots & & \vdots \\ \frac{\partial F_M}{\partial X_1} & \cdots & \frac{\partial F_M}{\partial X_M} \end{pmatrix}$$

When $\mathbf{X}$ is a state of a chaotic attractor, $\mathbf{H}$ is time-dependent.

Because $\mathbf{H}$ is linear, the tangent equation can be integrated from time $t_1$ to $t_2$ to obtain

$$\mathbf{x}(t_2) = \mathbf{A}(t_2,t_1)\ \mathbf{x}(t_1)$$

$$x_i(t_2) = \sum a_{ij}(t_2,t_1)\ x_j(t_1), \qquad i = 1, \ldots., M$$

$\mathbf{A}$ is an $M \times M$ square matrix which depends on the behavior of $\mathbf{X}$ between $t_1$ and $t_2$.
The matrix $\mathbf{A}$ controls the growth of small errors during the interval $t_1$ to $t_2$, and is called the *error matrix*. It is also known as the *resolvent* or *propagator* of the tangent equation.

Note that

if $t_1 < t_2 < t_3$, $\quad \mathbf{A}(t_3,t_1) = \mathbf{A}(t_3,t_2)\ \mathbf{A}(t_2,t_1)$

$$\mathbf{A}(t_1,t_2) = \mathbf{A}(t_2,t_1)^{-1}$$

## Growth of errors

An individual set of errors $\mathbf{x}$ can be treated as a point in the $M$-dimensional phase space. The amplitude of the error is defined as the distance of this point from the origin.

The squared-amplitude of the error at time $t_1$ is

$$\mathbf{x}(t_1)^T \mathbf{x}(t_1) = \sum_{i=1}^{M} x_i^2(t_1)$$

where the superscript $T$ denotes the transpose of a matrix.

The squared-amplitude of the error at time $t_2$ is

$$\mathbf{x}(t_2)^T \mathbf{x}(t_2) = \left[\mathbf{A}(t_2, t_1)\mathbf{x}(t_1)\right]^T \left[\mathbf{A}(t_2, t_1)\mathbf{x}(t_1)\right]$$

$$\mathbf{x}(t_2)^T \mathbf{x}(t_2) = \mathbf{x}(t_1)^T \mathbf{A}(t_2, t_1)^T \mathbf{A}(t_2, t_1)\mathbf{x}(t_1)$$

The matrix $\mathbf{A}^T\mathbf{A}$ is symmetric positive-definite and possesses $M$ real positive eigenvalues.

## Growth of an initial sphere of errors

Consider an ensemble of random initial errors, each of amplitude $\varepsilon$ at time $t_1$, occupying the surface of an $M$-dimensional sphere

$$\sum_{i=1}^{M} x_i^2(t_1) = \varepsilon^2 \qquad \text{or}$$

$$\mathbf{x}(t_1)^T\mathbf{x}(t_1) = \varepsilon^2$$

If each error in the ensemble evolves according to the propagator of the tangent equation, the sphere will be deformed into an ellipsoid

$$\mathbf{x}(t_2)^T\mathbf{x}(t_2) = \mathbf{S}^2\varepsilon^2$$

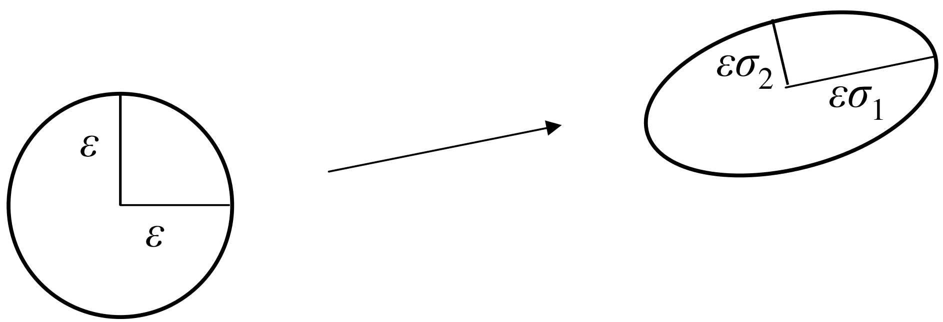
at time $t_2$. Here, the matrix $\mathbf{S}^2$ is diagonal with diagonal elements consisting of the $M$ real positive eigenvalues of $\mathbf{A}^T\mathbf{A}$.

If $\sigma_1{}^2,\ldots,\sigma_M{}^2$ are the the eigenvalues of $\mathbf{A}^T\mathbf{A}$, then

$\varepsilon\sigma_1,\ldots,\varepsilon\sigma_M$ are the lengths of the semiaxes of the ellipsoid.

$\sigma_1,\ldots, \sigma_M$ are the singular values of $\mathbf{A}$ and depend on $t_1$ and $t_2$ and

let $\sigma_1 > \sigma_2 > \ldots > \sigma_M$.




Whether or not any small errors grow between $t_1$ and $t_2$ depends on whether any semi-axis of the ellipsoid is greater than the radius $\varepsilon$ of the sphere.

The error growth, therefore, depends on whether the singular value $\sigma_1$, or the eigenvalue $\sigma_1{}^2$ is greater than one.