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Natural Modes of Variability: Linear and Nonlinear Perspectives I. Linear and Nonlinear Signatures of Modes of Variability

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Natural Modes of Variability:
Linear and Nonlinear Perspectives

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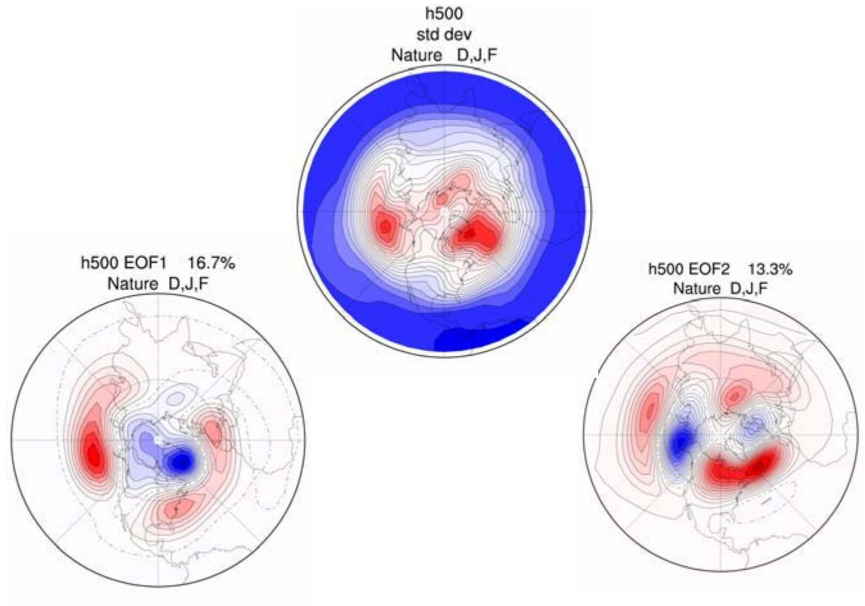
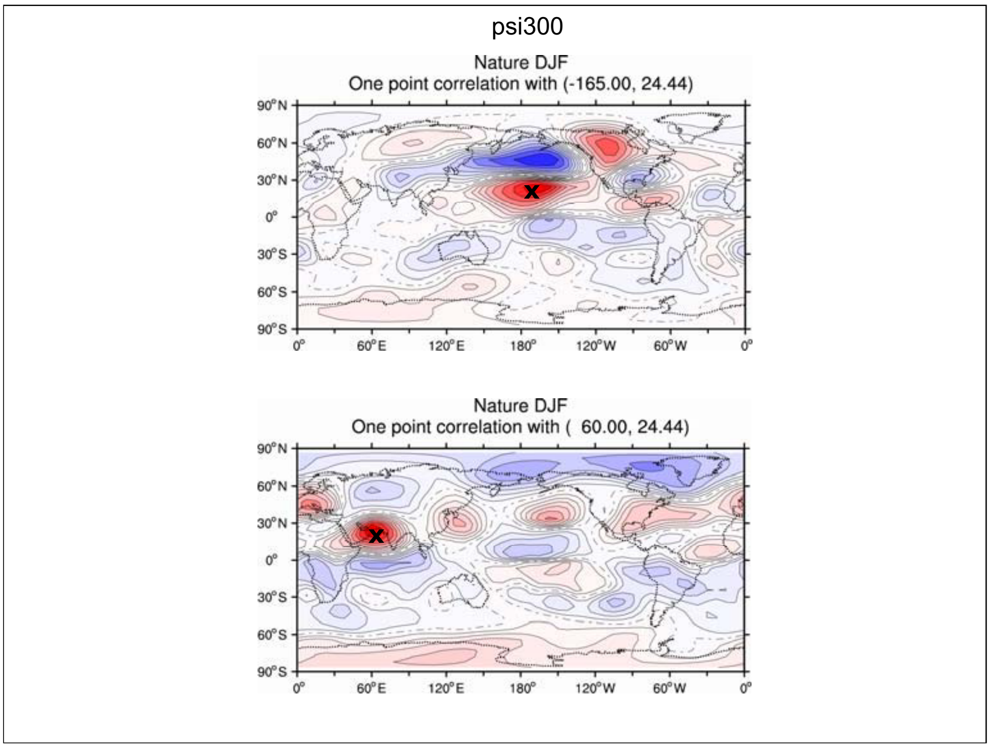
I. Linear and Nonlinear Signatures of Modes of Variability

Recognizing that much of variability on monthly and longer timescales is composed of a few recurring patterns is helpful for understanding variability on these timescales as well as its predictability characteristics. But there is more than one idea as to what are the fundamental dynamical processes that produce these patterns. In particular it is not clear how important nonlinear processes are in their formation and behavior.

What are natural modes of variability?

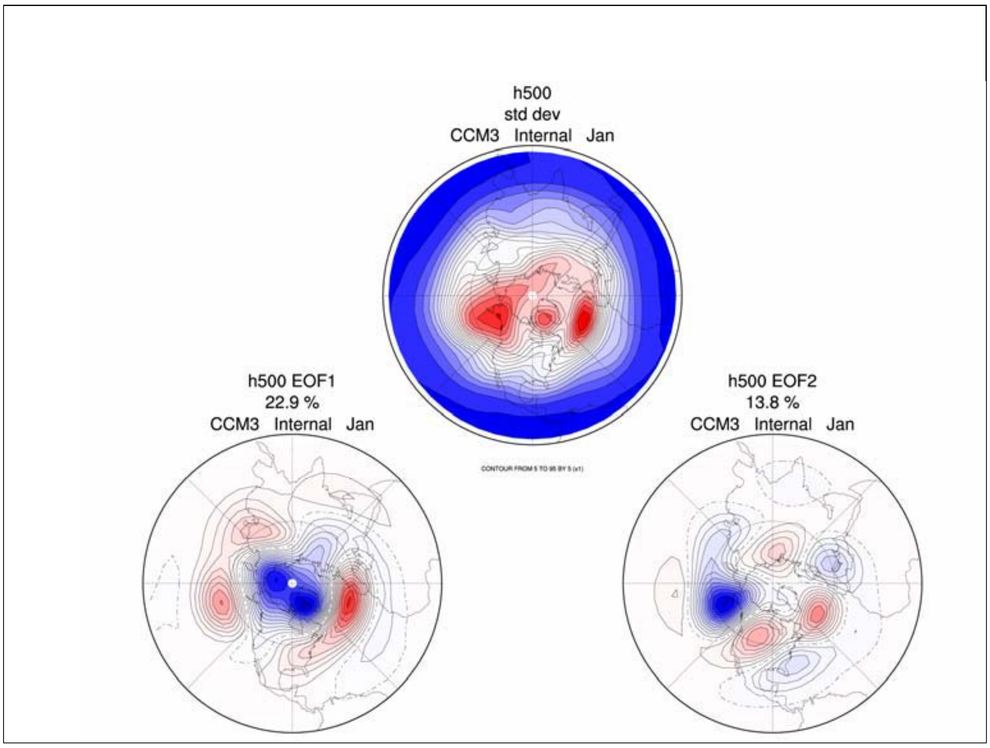


There is no consensus as to the exact meaning of mode of variability, but one of the consequences of their existence is that a large fraction of variability can be explained by just a few patterns.



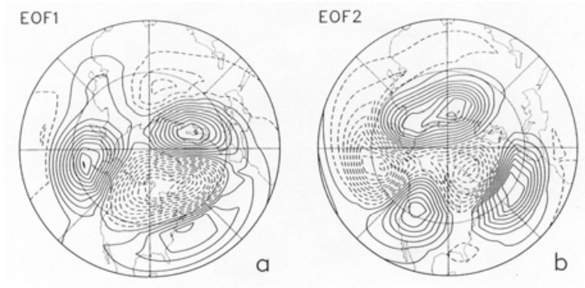
Another consequence is that there is strong covariability between widely spaced points on the globe.

But the pattern of co-varying points varies markedly from place to place on the globe.

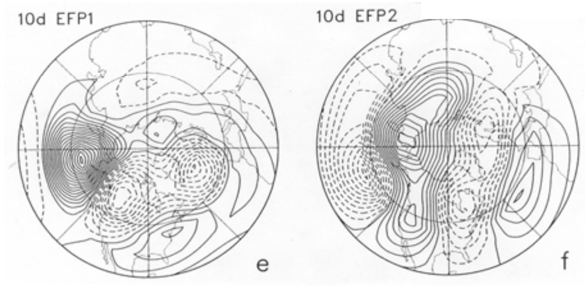


Many researchers associate some of the prominent modes of variability with special external forcing. For example, the pattern on the right, which is often called the Pacific North American pattern can be stimulated by tropical Pacific rainfall associated with El Nino and La Nina events. But natural modes of atmospheric variability are produced by intrinsic atmospheric processes and do not rely on special external forcing events for their existence. In fact, the patterns shown in this diagram result from a numerical integration of a general circulation model in which external conditions are kept fixed from one year to the next.

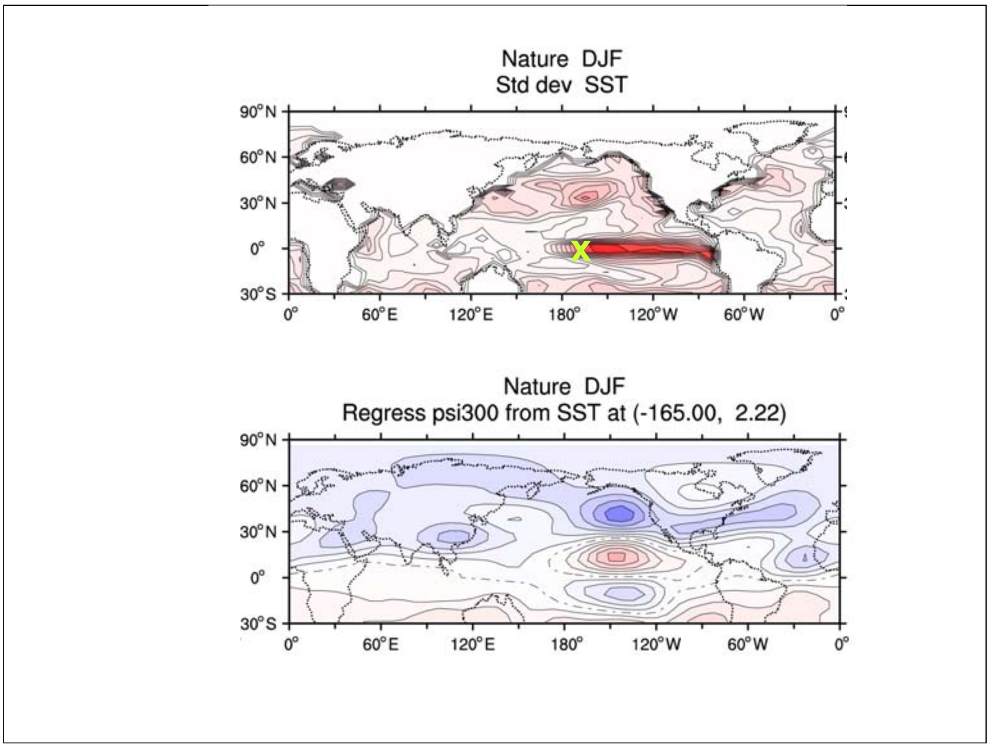
h500 EOFs



Most predictable patterns for 10d forecasts

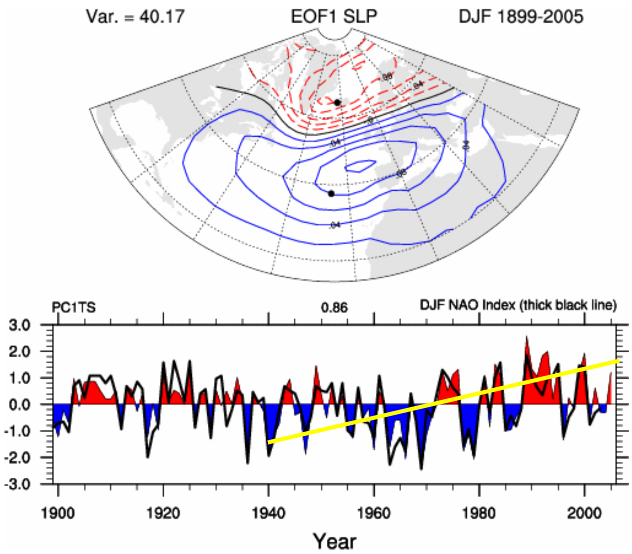


The existence of prominent modes of variability has many important consequence. As we shall see, they are patterns that are easily generated and which naturally last a long time. This makes them easier to forecast for long periods than most patterns are to forecast. Here we see that the two best forecast patterns at a 10 day range by an operational forecast model are very similar to the most prominent patterns in nature.



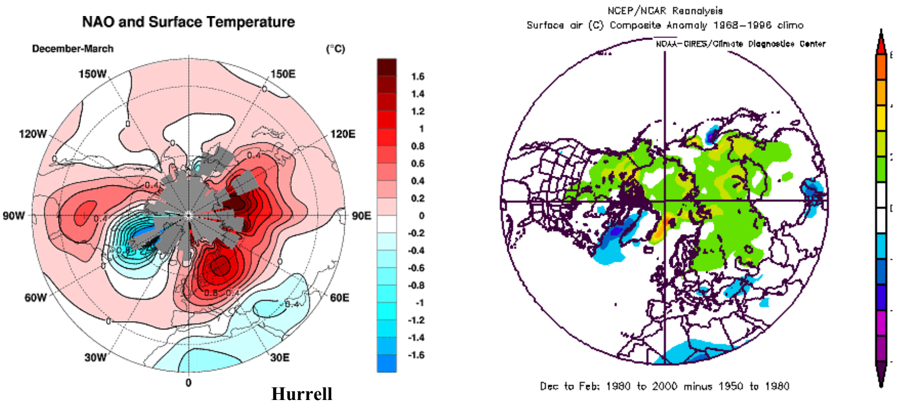
Though they do not rely on special forcing for their existence, modes of variability often play prominent roles in the atmospheric response to forcing events. Here we see that the average response of the atmosphere to El Nino events has a midlatitude structure that is a combination of two prominent modes, the PNA and a second mode that we will encounter later.

Are modes affecting observed climate change?



Another possible example of intrinsic modes being important for understanding why the structure of the atmospheric response to external conditions concerns global warming. One of the most famous modes of variability is the North Atlantic Oscillation.

Over the last 40 or so years there has been a distinct trend in the amplitude of the NAO.



This trend in the NAO appears to be related to a concurrent trend in surface temperatures in that the observed trends (right) have similar structure to the surface temperature distribution associated with strong NAO events (left).

Two Competing Definitions of "Mode"

$$\frac{\partial \varsigma}{\partial t} = -\vec{v}_\psi \cdot \nabla(\varsigma + f)$$

$$(\bullet) = \frac{1}{T}\int_0^T (\bullet)dt + (\bullet)' = \overline{()} + (\bullet)'$$

$$\frac{\partial \varsigma'}{\partial t} = -\bar{\vec{v}}_\psi \cdot \nabla(\bar{\varsigma} + f) - \bar{\vec{v}}_\psi \cdot \nabla\varsigma' - \vec{v}'_\psi \cdot \nabla(\bar{\varsigma} + f) - \vec{v}'_\psi \cdot \nabla\varsigma'$$

$$\bar{\vec{v}}_\psi \cdot \nabla(\bar{\varsigma} + f) = -\overline{\vec{v}'_\psi \cdot \nabla\varsigma'}$$

$$\frac{\partial \varsigma'}{\partial t} = -\bar{\vec{v}}_\psi \cdot \nabla\varsigma' - \vec{v}'_\psi \cdot \nabla(\bar{\varsigma} + f) - (\vec{v}'_\psi \cdot \nabla\varsigma' - \overline{\vec{v}'_\psi \cdot \nabla\varsigma'})$$

$$\frac{\partial \varsigma'}{\partial t} \cong -\bar{\vec{v}}_\psi \cdot \nabla\varsigma' - \vec{v}'_\psi \cdot \nabla(\bar{\varsigma} + f) \quad \text{if } (\bullet)' \text{ is small}$$

Given the prominence of intrinsic modes of variability and their importance to forecasts from the weekly to interannual to secular timescales, we would like to understand what produces them. But there is no consensus as to what the essential processes are. An interesting consequence of this lack of agreement is that there is not even agreement as to what the definition of "modes of variability" is. The disagreement about processes and definition can be traced to the question of whether modes are a result of linear processes or whether nonlinearities must be included to reproduce their characteristics.

Linear Initial Value Problem Using Normal Mode Basis

$$\frac{\partial \varsigma'}{\partial t} = -\bar{\vec{v}}_\psi \cdot \nabla \varsigma' - \vec{v}'_\psi \cdot \nabla(\bar{\varsigma} + f)$$

$$= -iL\varsigma'$$

Say $LE = \sigma E$

Then $\varsigma'(t) = Ee^{-\sigma t} = (E_R + iE_I)e^{\sigma_I t}(\cos\sigma_I t - i\sin\sigma_I t)$

$= e^{\sigma_I t}\{E_R \cos\sigma_I t + E_I \sin\sigma_I t\} + i\{...\}$ is a solution.

Thus, if $\varsigma'(t=0) = \sum_j a_j E^j$ is real,

then $\varsigma'(\mathrm{t}) = \sum_j a_j e^{\sigma_I^j t}\{E_R^j \cos\sigma_I t + E_I^j \sin\sigma_I t\}$

That linear processes alone have the ability to produce many of the attributes of modes of variability can be seen by analyzing the behavior of a linearized system in terms of a basis consisting of its normal modes. One finds that for the unforced initial value problem, after a time just a few, and eventually only one, oscillating pattern will dominate a solution. These dominant patterns corresponds to the normal modes of the problem whose natural (complex) eigenfrequencies either grow the fastest or decay the slowest.

Linear Response Problem Using Normal Mode Basis

$$\frac{\partial \varsigma'}{\partial t} = -\bar{\vec{v}}_\psi \cdot \nabla \varsigma' - \vec{v}'_\psi \cdot \nabla(\bar{\varsigma} + f) + R'$$

$$= -iL\varsigma' + R'$$

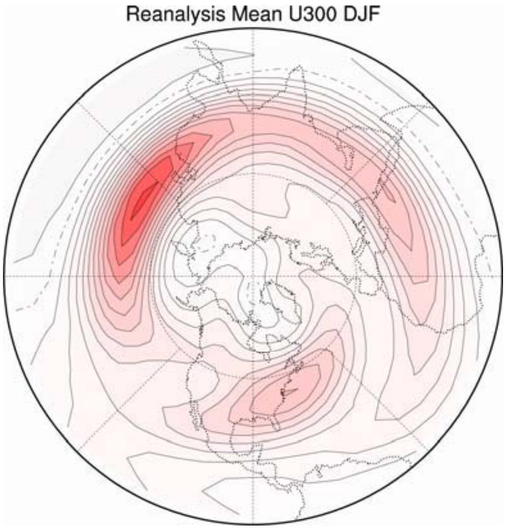
$$\Rightarrow \quad \varsigma' = -iL^{-1}R'$$

Suppose $\varsigma' = \sum_j a_j E^j$ and $R' = \sum_j r_j E^j$

Then $\sum_j a_j E^j = -L^{-1} \sum_j r_j E^j = -\sum_j i(\frac{r_j}{\sigma_j}) E^j$

So $a_j = \frac{r_j}{i\sigma_j}$

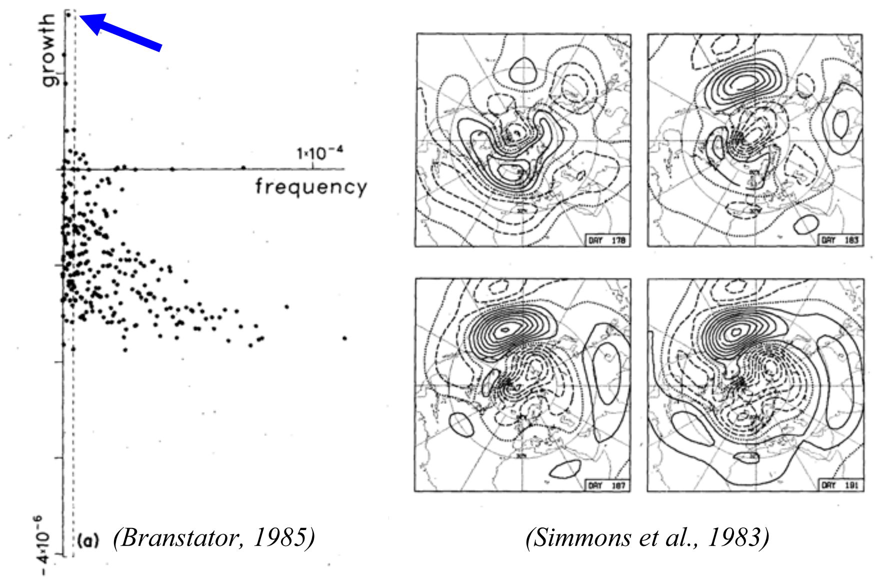
Similarly, using an eigenbasis demonstrates how some normal modes will dominate over others for forced linear systems. Here we see that for the steady problem, it is modes that have small (complex) eigenfrequencies that will be easiest to excite.



Linear Vorticity Equation

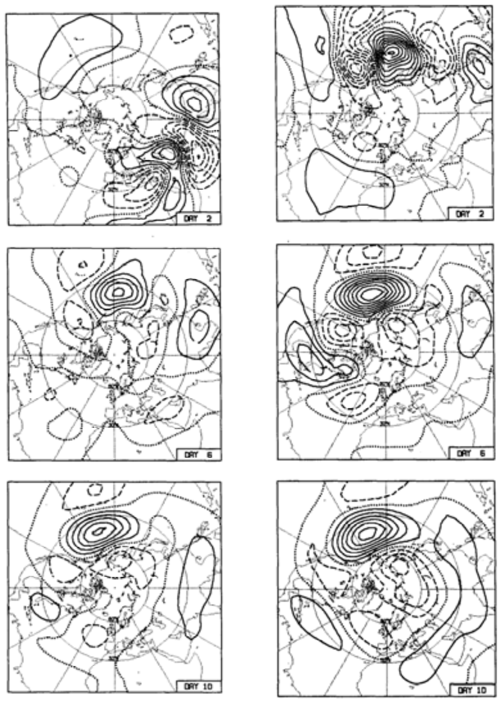
$$\frac{\partial}{\partial t}\zeta' = -\overline{u}\frac{\partial \zeta'}{\partial x} - \overline{v}\frac{\partial \zeta'}{\partial y} - u'\frac{\partial}{\partial x}\overline{\zeta} - v'\frac{\partial}{\partial y}(\overline{\zeta}+f) + \mathrm{Diss}$$

To see how these ideas might apply to the earth's atmosphere, we consider the leading terms of the one level vorticity equation linearized about the mean state from the upper troposphere.



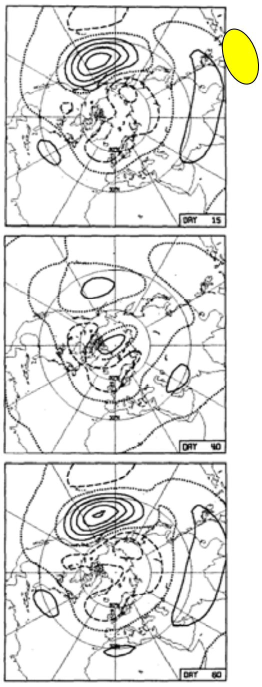
(Branstator, 1985) (Simmons et al., 1983)

The spectrum from an eigenanalysis of that model is very peaked, with a few modes having much faster growth rates than all others. (Also, with realistic levels of damping these same normal modes have the smallest amplitude eigenvalues. Thus they are the easiest to stimulate.) Interestingly, the structure of these leading normal modes is very reminiscent of some of the observed most prominent patterns of variability.



Simmons et al. (1983)

The dominance of this normal mode for initial value problems can be seen in these two solutions, which demonstrate that for very different initial perturbations, the solutions tend to take on the character of the leading mode within a couple of weeks of initiation.



Simmons et al. (1983)

Similarly, this solution shows the prominence of the leading normal mode for steadily forced solutions. Here is shown the response to forcing over the eastern Indian Ocean takes on the structure of the leading normal mode.

$$\frac{\partial \varsigma}{\partial t} = -\vec{v}_{\psi} \cdot \nabla(\varsigma + f)$$

$$\frac{\partial \varsigma'}{\partial t} = -\overline{\vec{v}}_{\psi} \cdot \nabla \varsigma' - \vec{v}'_{\psi} \cdot \nabla(\overline{\varsigma} + f) - (\vec{v}'_{\psi} \cdot \nabla \varsigma' - \overline{\vec{v}'_{\psi} \cdot \nabla \varsigma'})$$

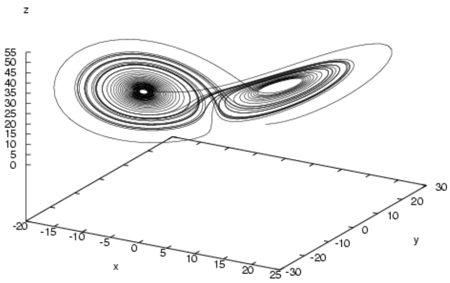
$$\cong -\overline{\vec{v}}_{\psi} \cdot \nabla \varsigma' - \vec{v}'_{\psi} \cdot \nabla \overline{\varsigma} + damping + noise$$

Signatures of linear behavior:

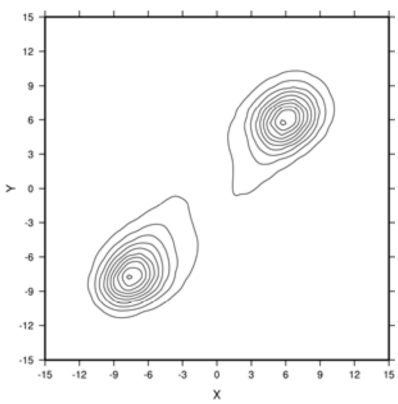
- **Gaussian PDFs**
- **Elliptical trajectories**

For comparison to behavior of the atmosphere it is worth noting a couple of attributes of a system where linear dynamics are prominent. In general one cannot ignore the nonlinearities because the are responsible for keeping the system from reaching unbounded amplitudes and because they serve to scatter energy from one mode into another. But these processes can be approximated by linear damping and random forcing. So if we want to assume that atmospheric dynamics are fundamentally linear we can take the linearized equations and add these two additional terms. In this case our (stochastic) linear system will have two important signatures. First, it will have Gaussian PDFs. Second average trajectories will be sinusoidal oscillations.

Why does a system have natural modes of variability? (Possibility II – Nonlinear System)



PDF

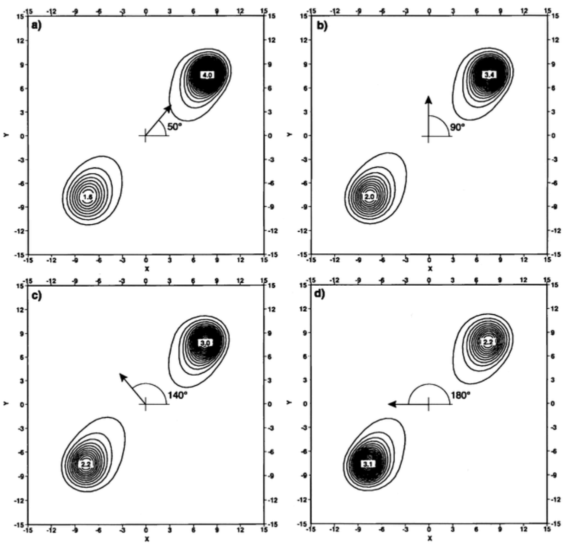


Lorenz63

$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

As similar as linear behavior seems to be to certain characteristics of prominent atmospheric modes, there is some reason to believe that linear theory is not capturing important processes that affect these modes. In particular nonlinear terms appear to be too large to ignore. Examples of simple nonlinear systems show how nonlinearity can completely change the characteristics of a dynamical system. One such system that is often used to demonstrate the possible effects of nonlinearity is Lorenz's 1963 model. For time averages the PDF of the system is completely different from a linear system, with two distinct maxima. The dominant pattern of variability consists of the difference between the centroids of these features as the system jumps from one lobe to the other. In this situation the term "modes of the system" refers to the two local maxima (that is "modes" in the statistical sense), not to normal modes of a linearization of the governing equations. Note in this case, as in the linear system, modes are an intrinsic property of the system and are not dependent on special forcing.

PDFs of Forced Lorenz' 3-Parameter Model



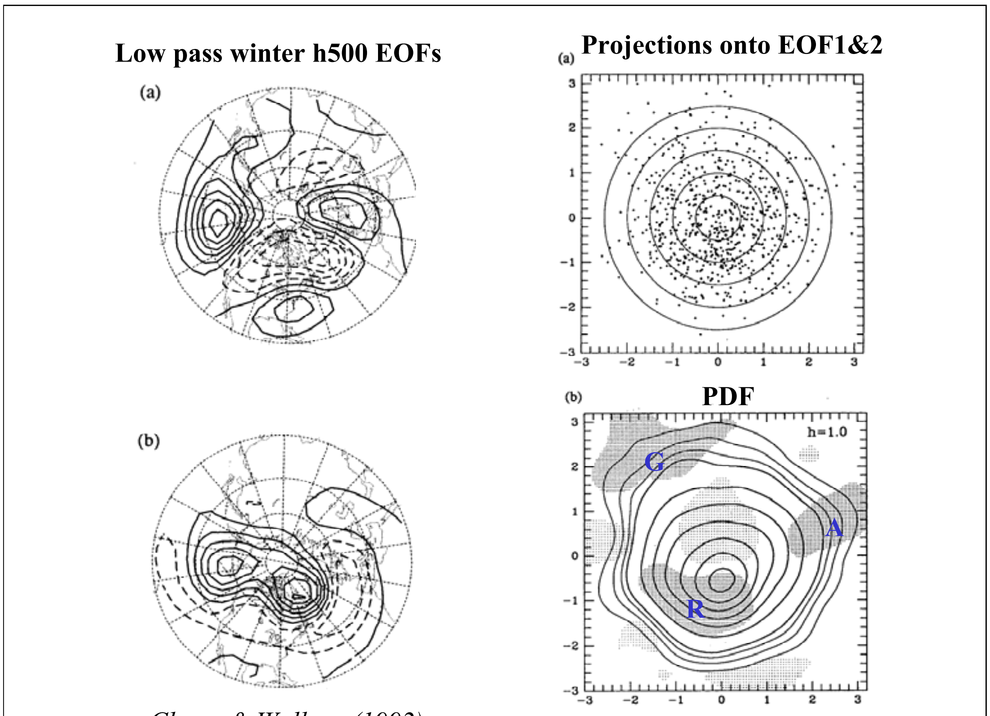
Palmer (1999)

Another property of this nonlinear system that matches observed behavior is that its response to forcing is similar in structure to the leading pattern of variability. This is because if one forces the system, its modes (i.e. maxima) do not change position. Rather one lobe simply becomes more populated than the other.

Signatures of nonlinear behavior:

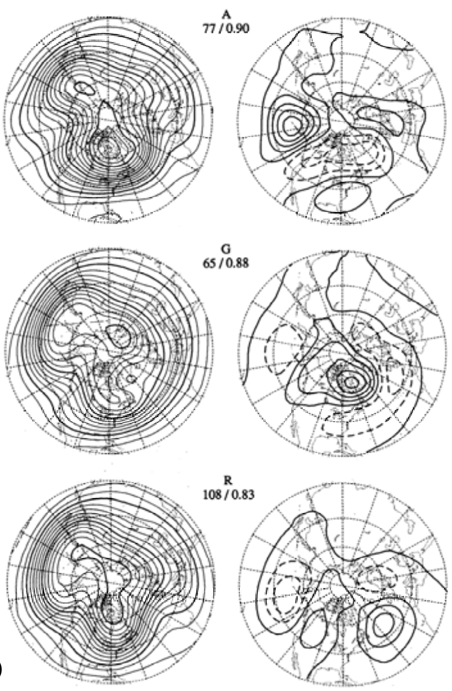
- **Nongaussian PDF**
- **Modes (i.e. PDF maxima) insensitive to forcing**
- **Nonelliptical trajectories**

From the Lorenz example we see features that can distinguish nonlinear behavior from linear behavior.



Cheng & Wallace (1992)

Many researchers have considered PDFs of leading patterns from nature to see if they have evidence of nonlinear modes. They find slight departures from Gaussianity.

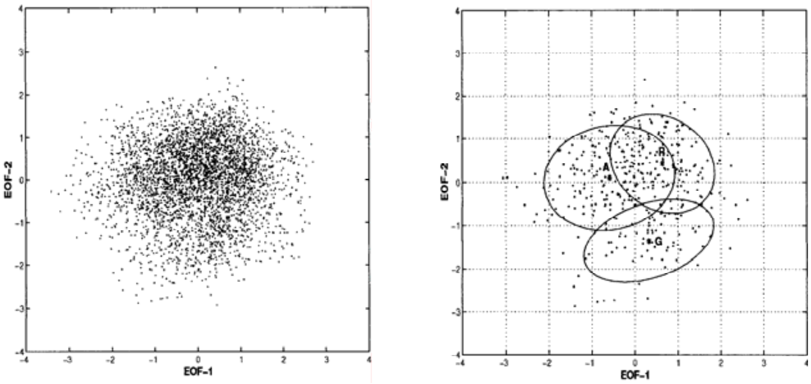


Cheng & Wallace (1992)

The modes from PDF analysis tend to match an alternative analysis in which one searches for “clusters” of similar states. The modes found with either of these methods are structuraly similar to the patterns found from EOF analysis.

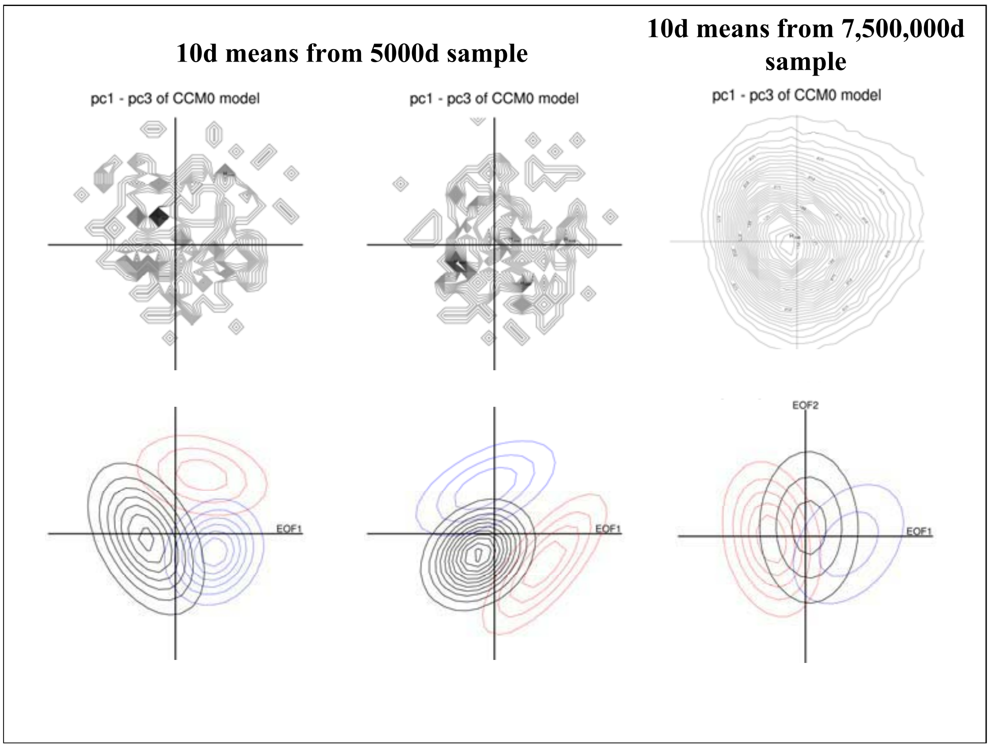
Projections of low pass winter h500 onto EOFs

Fit mixture of 3 Gaussians

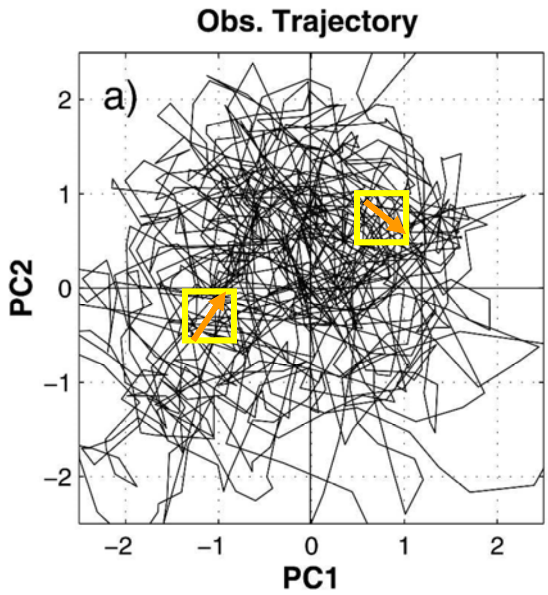


Smyth, Ide & Ghil (1999)

One technique sometimes used to characterize PDF nonGaussianity is to fit a mixture (i.e. combination) of several Gaussian distributions to the data.



Unfortunately our tests with AGCM data indicate that the dataset from nature is too short to identify structure using Gaussian mixtures. Here we see that two dataset (left and center), each with a length similar to the observed record, have very different mixture fits even though they are taken from the same model. In fact the true distribution (right), found using more than a 1000 times as many samples, is completely different from either estimate.

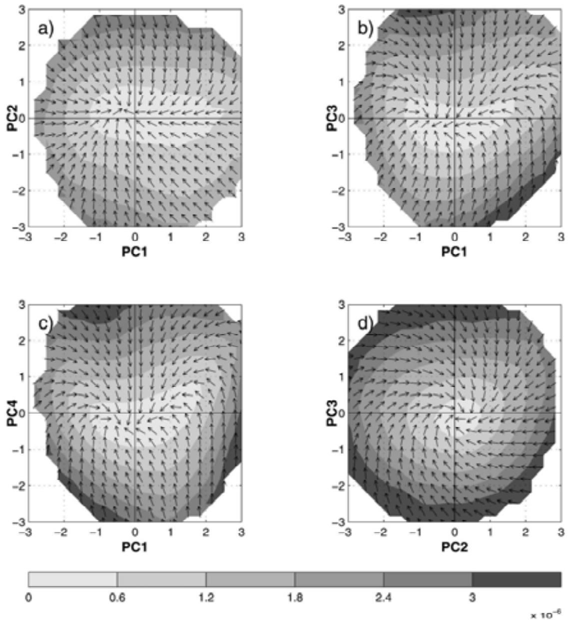


Branstator & Berner (2005)

If one looks at trajectories of the atmospheric state by looking at projections onto a low-dimensional state space, it is difficult to see any organization. Here we see part of the observed NH wintertime trajectory in a two-dimensional state space defined by projections onto the two leading EOFs of 500mb heights.

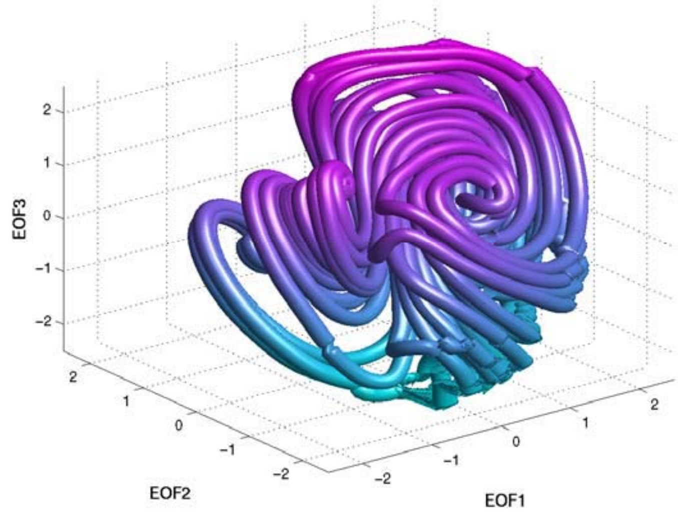
If, however, we focus on a small region of the phase space and find the average of all trajectories in that region, then we do see evidence of organization in the evolution of the system.

Mean 24hr increments



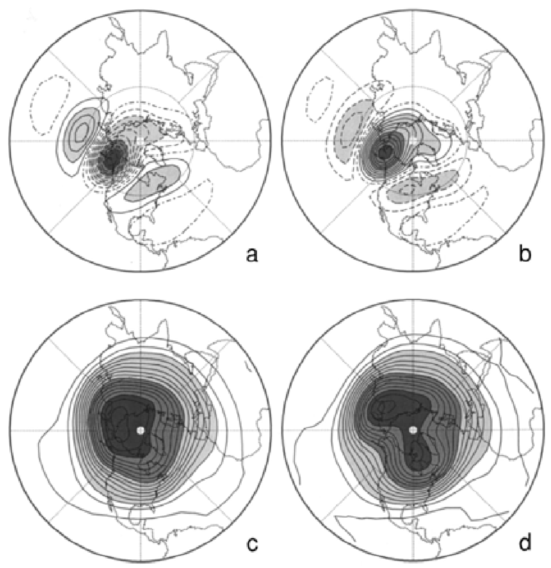
Branstator & Berner (2005)

It turns out that for nature there is not enough data to find mean trajectories with statistical confidence. But if one carries out a mean trajectory calculation for very long integrations of general circulation models, then one does find distinctive structure in the mean trajectories, as in this example. In some state space planes the trajectories (for example the EOF2-3 plane in the lower right hand corner of this figure) have the elliptical shaped trajectories of linear dynamics. But in others (for example the EOF1-3 and EOF1-4 planes) the trajectories have highly nonlinear characteristics.



Branstator & Berner (2005)

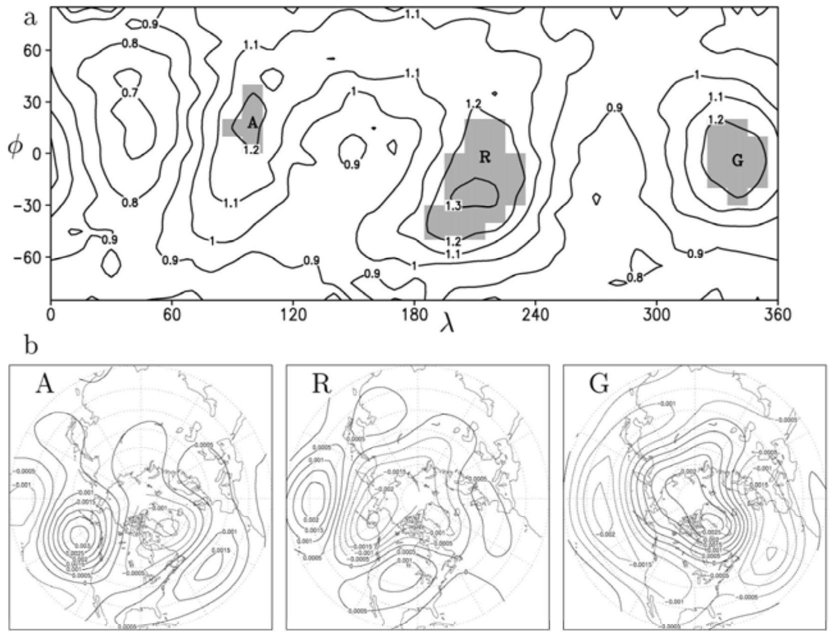
In three dimensions, the nonlinear component of mean trajectories is highly suggestive of a system that is under the influence of two distinct regimes.



Branstator & Berner (2005)

Interestingly, the centers of the two regimes in the mean phase space trajectories correspond to Pacific blocking and anti-blocking events. These are similar to the states that Charney and DeVore (1980) first proposed as regimes based on a highly reduced, nonlinear version of the barotropic quasi-geostrophic potential vorticity equation, which has two stable equilibria.

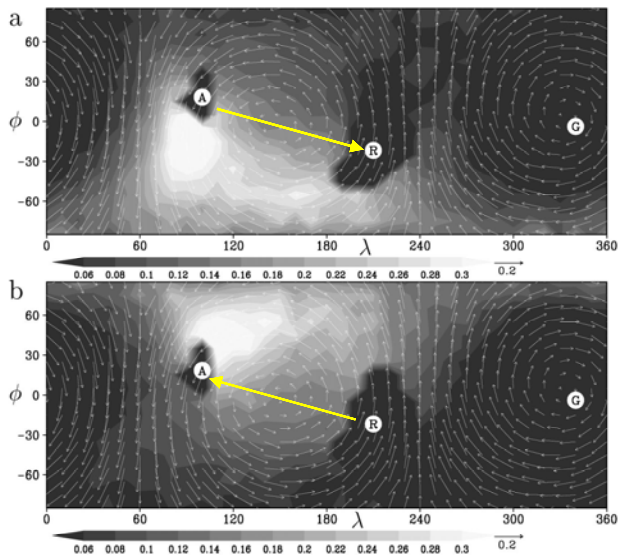
PDF of EOF1,2,3



Selten & Branstator (2004)

Another indication of the influence of nonlinearities can be seen in a three-level quasi-geostrophic model examined by Selten and Branstator (2004). This model has three local maxima in its distribution of states in state space, though this is most evident only if one factors out distinctions in pattern amplitude.

Preferred transition routes



Selten & Branstator (2004)

The routes taken by the system as it moves from one PDF maximum to another are very organized. Here we see that in moving from maximum A to maximum R the system takes a completely different route than if it is moving from maximum R to maximum A. However, though the local maxima in PDFs are indicative of the influence of nonlinear processes, the trajectories indicated by this figure are well-approximated by linear dynamics. From all of the above examples, we see that in many ways the distribution of states and the trajectories taken by atmospheric systems are consistent with linear behavior, but there are some characteristics of behavior that can only be explained by resorting to nonlinear effects.