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Modular Curves

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Modular Curves, Trieste, July 2007. / 4 times 1 hour

References:

1. Brian Conrad, Modular forms .... (2007) 480 pages.
2. Diamond and Im, Modular forms... (1995)
3. E., Couveignes, .. on the Computation (arxiv) 2006. (Ch9

1. The Gal. repr. ass. to mod forms: main results.

Thm. 1 (Eichler-Shimura for $k=2$, Deligne 1969 for $k \geq 2$). Let $N \geq 1$ and $k \geq 2$ be integers, $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a char., and $f = \sum_{n\geq 1} a_n(f) \cdot q^n$ a normalised ($a_1(f) = 1$) newform of type (N, k, ε). Then $K_f := \mathbb{Q}(a_1(f), a_2(f), \ldots)$ is finite over $\mathbb{Q}$, and $\forall$ prime l $\exists$ a unique continuous repr. $\rho_{f,l} : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l \otimes K_f)$ that is unramified at all $p \nmid Nl$ and such that $\forall p \nmid Nl$: $\det(\rho_{f,l}\, \mathrm{Frob}_p) = \varepsilon(p) \cdot p^{k-1}$, $\mathrm{tr}(\rho_{f,l}\, \mathrm{Frob}_p) = a_p(f)$

0. These $\rho_{f,l}$ are not smooth.

Remarks. 1. $\mathbb{Q}_l \otimes K_f = \prod_{\lambda | l} K_{f,\lambda}$.

2. Kuga & Sato & Shimura had already treated higher weight cases ~~for~~ certain Shimura curves (no cusps...) (relation with zeta function...)

3. For uniqueness in thm: the $\rho_{f,l}$ are irreducible (Ribet, Deligne?)

4. Deligne-Serre proved the thm. for $k=1$. Then the $\rho_{f,l}$ have finite image, indep. of l. These cannot be constructed in the same way as the others.

The thm. above gives us information on $(\rho_{f,l})_p := \rho_{f,l}|_{G_{\mathbb{Q}_p}}$ for $p \nmid Nl$: $(\rho_{f,l})_p$ is unramified, and:

$$\det(1 - t \cdot \rho_{f,l}(\mathrm{Frob}_p)) = 1 - a_p(f) \cdot t + \varepsilon(p) \cdot p^{k-1} t^2.$$

To describe $(\rho_{f,l})_p$ for $p | N$ $(p \neq l)$ one needs repr. theory.

Recall $f \rightsquigarrow \varphi$ on $\mathrm{GL}_2(\mathbb{A}) \rightsquigarrow \pi_f$ cuspidal irred. autom. repr. in $\mathcal{A}_0(\mathrm{GL}_{2,\mathbb{Q}}, \varepsilon)$

$$\pi_f \overset{\text{Nyssen}}{=} \bigotimes_v{}' \pi_{f,v}$$

Thm 2 (Langlands, Deligne, Carayol) (vaguely formulated).

$\forall l, \forall p \neq l$: $(\rho_{f,l})_p^{F\text{-s.s.}}$ and $\pi_{f,p}$ correspond to each other via a
$\forall \lambda,$

suitably normalised local Langlands correspondence.

Rem. 1: $\begin{cases} \text{up to } W \mapsto W^\vee \text{ and } \rho \mapsto \rho \otimes \chi_l^m \ (m \in \mathbb{Z}) \text{ at Galois side.} \\ \text{up to } \pi \mapsto \pi^\vee \text{ and } \pi \mapsto \pi \otimes (|\cdot| \circ \det)^{m(k)} \end{cases}$

$\chi_l : G_{\mathbb{Q}} \twoheadrightarrow \underset{\mathrm{Aut}(\overline{\mathbb{Q}}^\times[l^\infty])}{\overset{\mathrm{Aut}(\overline{\mathbb{Q}}^\times[l^\infty])}{}} \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times$, $\mathbb{Z}_l^\times = \mathrm{Aut}(\overline{\mathbb{Q}}^\times[l^\infty])$

2. π_f is complex, but can naturally be defined over K_f, hence also $\pi_{f,p}$.

3. F-s.s. = Frob. semi-simplification. Functor, if $\exists\ \alpha : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_\ell^\times$ s.t. $(\rho_{f,\lambda})_p \otimes \alpha$ is unram., then it makes $((\rho_{f,\lambda})_p \otimes \alpha)(\mathrm{Frob}_p)$ semi-simple. Conjecturally, this is never necessary. Details: Tate "Number theoretic background" (1979).

uses Fontaine functor.... + forget filtration

4. For $p = \ell$: $(WD\,(\rho_{f,\lambda})_p)^{F\text{-s.s}} \longleftrightarrow \pi_{f,p}$. (Saito, 1997)
p-adic LL : $(\rho_{f,\lambda})_p \longleftrightarrow\ ?$

5. All this is crucial for the recent work of Wiles, Taylor, Khare, Kisin.....

Goal of my series of 4 lectures: sketch a proof of Thm.1, sketch Deligne's proof that $(\rho_{f,\lambda})_p$ is determined by $\pi_{f,p}$ if $\pi_{f,p}$ is cuspidal.

The $\rho_{f,\lambda}$ are constructed in the cohom. of modular curves (Jacobian if $k=2$). So now we will turn to modular curves.

2. Modular curves / $\mathbb{C}$.

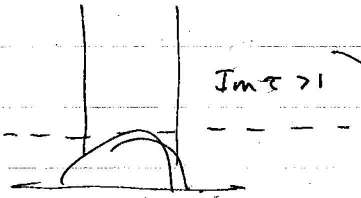
For $\Gamma \subset Sl_2(\mathbb{Z})$: $Y_\Gamma(\mathbb{C}) := \Gamma\backslash\mathbb{H}$, 1-dim. complex manifold.

Example:
$$\begin{tikzcd} \mathbb{H} \arrow[r, "j"] \arrow[d, two heads] & \mathbb{C} \\ Y_{Sl_2(\mathbb{Z})}(\mathbb{C}) \arrow[ur] & \end{tikzcd}$$
, for $\Gamma \subset Sl_2(\mathbb{Z})$ finite index:

$Y_\Gamma(\mathbb{C}) \to Y_{Sl_2(\mathbb{Z})}(\mathbb{C})$ finite possibly ramified cover.

Compactification.
(by "normalisation")

$$\begin{tikzcd} Y_\Gamma(\mathbb{C}) \arrow[r] \arrow[d, hook] & Y_{Sl_2(\mathbb{Z})}(\mathbb{C}) \arrow[r, phantom, "="] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] \\ X_\Gamma(\mathbb{C}) \arrow[r] & X_{Sl_2(\mathbb{Z})}(\mathbb{C}) \arrow[r, phantom, "="] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$



$q = (\tau \mapsto e^{2\pi i\tau})$

$D^* \subset D = D^* \cup \{0\}$.
$D^* = \{z \in \mathbb{C} \mid 0 < |z| < e^{-2\pi}\}$

inv. image of D^* is a finite cover of D^*, hence a finite union of D_i^* ......

We have: $X_\Gamma(\mathbb{C}) = Y_\Gamma(\mathbb{C}) \cup \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$.

GAGA: $X_\Gamma(\mathbb{C})$ is also a projective complex alg. curve.
$Y_\Gamma(\mathbb{C})$ is an affine complex alg. curve.

Lecture 2

3. Elliptic curves / $\mathbb{C}$.

As analytic spaces: V/Λ, V 1-dim. $\mathbb{C}$-v.sp., $\Lambda \subset V$ a lattice. $\mathrm{Hom}(V/\Lambda, V'/\Lambda') = \{f : V \to V' \ \mathbb{C}\text{-linear} \mid f\Lambda \subset \Lambda'\}$.

Intrinsically: $\Lambda = H_1(V/\Lambda, \mathbb{Z})$, $V = \mathbb{R} \otimes \Lambda$ + $\mathbb{C}$-vect. sp. str.
$= T_{V/\Lambda}(0)$.

$$\mathbb{H} \xleftrightarrow{\sim} \{(V, \Lambda, \varphi)\}/\cong$$

$\begin{cases} V: \text{ 1-dim } \mathbb{C}\text{-v.sp.} \\ \Lambda \subset V \text{ lattice} \\ \varphi: \mathbb{Z}^2 \xrightarrow{\sim} \Lambda, \text{ s.t. } \mathrm{Im}(\varphi\binom{1}{0}/\varphi\binom{0}{1}) > 0 \end{cases}$

$$\tau \longmapsto (\mathbb{C}, \mathbb{Z}\tau + \mathbb{Z}, \tbinom{n}{m} \mapsto n\tau + m)$$

$$\varphi\tbinom{1}{0}/\varphi\tbinom{0}{1} \longleftarrow\!\shortmid (V, \Lambda, \varphi)$$

$\{(V, \Lambda, \varphi)\}/\cong \ \xleftrightarrow{\wr} \ \{(E, \varphi)\}$ E ell. curve / $\mathbb{C}$
$\varphi: \mathbb{Z}^2 \xrightarrow{\sim} H_1(E, \mathbb{Z})$

$Sl_2(\mathbb{Z})$-action: $(\tau \mapsto \gamma\cdot\tau) \longleftrightarrow (E, \varphi) \mapsto (E, \varphi\circ\gamma^t)$.

Hence $Y_\Gamma(\mathbb{C}) \longleftrightarrow \{(E, \varphi)\}/\cong$, $\varphi \in \Gamma\backslash\mathrm{Isom}^+(\mathbb{Z}^2, H_1(E, \mathbb{Z}))$

<u>Examples</u>: let $N \geq 1$. $SL_2(\mathbb{Z}) \xrightarrow{f} SL_2(\mathbb{Z}/N\mathbb{Z}) \circlearrowright (\mathbb{Z}/N\mathbb{Z})^2$.

$\Gamma(N) := \ker(f)$, $\Gamma_1(N) := f^{-1}(\mathrm{Stab}_{\binom{1}{0}}) \neq \Gamma_0(N) = f^{-1}\mathrm{Stab}_{\overline{\binom{1}{0}}}$. $\in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

$Y_{\Gamma(N)}(\mathbb{C}) = \{(E, \varphi : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$ symplectic isom.$)\}/\cong$

$Y_{\Gamma_1(N)}(\mathbb{C}) = \{(E, P) \mid P \in E$ order $N\}/\cong$

$Y_{\Gamma_0(N)}(\mathbb{C}) = \{(E, G) \mid G \subset E$ cyclic subgrp. order $N\}/\cong$.

Actually, over $\mathbb{H}$ we have a $SL_2(\mathbb{Z})$-equivariant family of elliptic curves:

$$\begin{tikzcd}
\mathbb{Z}^2 \times \mathbb{H} \arrow[r, hook] \arrow[dr] & \mathbb{C} \times \mathbb{H} \arrow[r, two heads] \arrow[d] & \mathbb{E} \arrow[dl] \\
& \mathbb{H} &
\end{tikzcd}$$

$(\binom{n}{m}, \tau) \longmapsto (n\tau + m, \tau)$

action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$: $(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}))$

$$(\binom{n}{m}, \tau) \longmapsto (\gamma^{-1,t}\binom{n}{m}, \gamma\cdot\tau) \qquad (z, \tau) \longmapsto (\frac{z}{c\tau+d}, \gamma\cdot\tau)$$

If $\Gamma \subset SL_2(\mathbb{Z})$ acts freely on $\mathbb{H}$, we get an ell. curve $\mathbb{E} \to Y_\Gamma(\mathbb{C})$, by taking the quotient. True for: $\Gamma_1(N) : N \geq 4$, $\Gamma(N) : N \geq 3$, $\Gamma_0(N)$: never.

4. Modular forms.

$$\begin{tikzcd}
\mathbb{E} \arrow[d] \\
\mathbb{H} \arrow[u, bend right, "0"']
\end{tikzcd}$$

gives $\underline{\omega} := 0^* \Omega^1_{\mathbb{E}/\mathbb{H}}$, invertible $\mathcal{O}_{\mathbb{H}}$-module.

We have $\underline{\omega} = \mathcal{O}_{\mathbb{H}} \cdot dz$ (z the coord. on $\mathbb{C}$); $(\overset{\binom{a\,b}{c\,d}}{\gamma}\cdot)^* dz = \frac{1}{c\tau+d} \cdot dz$,

so $(\gamma\cdot)^*(f \cdot (dz)^{\otimes k}) = (f \circ \gamma) \cdot (c\tau+d)^{-k} \cdot (dz)^{\otimes k}$,

$f \cdot (dz)^{\otimes k}$ is Γ-invariant $\Longleftrightarrow$ $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k \cdot f(\tau)$ $\forall$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

$SL_2(\mathbb{Z})_\infty = \{\pm 1 \cdot \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$

If Γ acts freely on $\mathbb{H}$: get $\underline{\omega}$ on $Y_\Gamma(\mathbb{C})$.

If moreover Γ acts regularly at the cusps: get $\underline{\omega}$ on $X_\Gamma(\mathbb{C})$

at ∞: we have $\underline{\omega}$ on D^*, $\underline{\omega} = \mathcal{O}_{D^*} \cdot dz$ $\left((\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot)^* dz = dz \right)$,

declare $\underline{\omega} = \mathcal{O}_D \cdot dz$ on D.

Then; (for such Γ): $S_k(\Gamma) = H^0(X_\Gamma(\mathbb{C}), \underline{\omega}^{\otimes k}(-\text{cusps}))$.

Kodaira-Spencer isomorphism $\underline{\omega}^{\otimes 2} = \mathcal{O}_{\mathbb{H}} \cdot (dz)^{\otimes 2}$, $\Omega^1_{\mathbb{H}} = \mathcal{O}_{\mathbb{H}} \cdot d\tau$

$$\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathbb{H}} \quad SL_2(\mathbb{Z})\text{-equiv.}, \quad \underset{\overset{\shortparallel}{(2\pi i)^2 \cdot (dz)^{\otimes 2}}}{\left(\frac{dt}{t}\right)^{\otimes 2}} \longleftrightarrow \frac{dq}{q} = 2\pi i \cdot d\tau$$

$$t = e^{2\pi i z} \qquad \frac{dt}{t} = 2\pi i\, dz$$

Finally: $S_k(\Gamma) = H^0(X_\Gamma(\mathbb{C}), \Omega^1 \otimes \underline{\omega}^{\otimes(k-2)})$ $\left\{\begin{array}{l}\Gamma \text{ acting freely, reg. at cusps,} \\ k \geq 2.\end{array}\right.$

(on $X_\Gamma(\mathbb{C})$: $\underline{\omega}^{\otimes 2}(-\text{cusps}) \xleftrightarrow{\sim} \Omega^1_{X_\Gamma(\mathbb{C})}$; hence $\deg_{X_\Gamma(\mathbb{C})}(\underline{\omega}) > \frac{1}{2}\deg(\Omega^1_{X_\Gamma(\mathbb{C})})$
so R-R. gives dim. of $S_k(\Gamma)$ for $k \geq 2$)) (and $k \leq 0$ ..).

$k = 2$: $S_2(\Gamma) = H^0(X_\Gamma(\mathbb{C}), \Omega^1)$, $\forall\, \Gamma \subset SL_2(\mathbb{Z})$ finite index.

Eichler-Shimura isomorphism for $k=2$.

$$\begin{tikzcd} \mathbb{C} \otimes H^1(X_\Gamma(\mathbb{C}), \mathbb{Z}) \arrow[r, phantom, "="] \arrow[d, equal, "\div \otimes \mathrm{id}"'] & S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \arrow[dl, bend left=60] \\ \mathbb{C} \otimes_{\mathbb{R}} H^1\left(C^\infty_{\mathbb{R}}(X_\Gamma(\mathbb{C})) \xrightarrow{d} \begin{array}{c}\text{real } C^\infty \\ \text{1-forms}\end{array} \xrightarrow{d} \begin{array}{c}\text{real } C^\infty \\ \text{2-forms}\end{array} \right) & \end{tikzcd}$$

$$\gamma \longmapsto \left(\omega \mapsto \int_\gamma \omega\right)$$

Equivalently: $J_\Gamma(\mathbb{C}) := H^0(X_\Gamma(\mathbb{C}), \Omega^1)^\vee / H_1(X_\Gamma(\mathbb{C}), \mathbb{Z})$

jacobian var. of $X_\Gamma(\mathbb{C})$. $= S_2(\Gamma)^\vee / H_1(X_\Gamma(\mathbb{C}), \mathbb{Z})$

So: $S_2(\Gamma)^\vee$ = co-tangent space at 0 of $J_\Gamma(\mathbb{C})$.

$J_\Gamma(\mathbb{C}) = \mathrm{Pic}^0(X_\Gamma(\mathbb{C})) = \mathrm{Div}^0(X_\Gamma(\mathbb{C})) / \text{principal divisors}$

$$\left(\omega \mapsto \sum_i \int_\infty^{P_i} \omega\right) \longleftarrow\!\!\shortmid \; P_1 + \cdots + P_d - d \cdot \infty$$

Hecke operators. as endom. of $J_\Gamma(\mathbb{C})$.

For $N \geq 1$. $X_1(N) := X_{\Gamma_1(N)}(\mathbb{C})$ as alg. curve $/\mathbb{C}$.

Recall: $Y_1(N) = \{(E,P) \mid E$ ell. curve $/\mathbb{C}$, $P \in E$ order $N\}/\cong$.

We have $(\mathbb{Z}/N\mathbb{Z})^\times \circlearrowright X_1(N)$: $a \bullet \overline{(E,P)} = \overline{(E, a\cdot P)}$, $\langle a \rangle$, diamond operator: (related to the "R_p-operators" in Nair's lecture, $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$).); $\boxed{S_2(N) = \bigoplus_\varepsilon S_2(N,\varepsilon)}$

And we have, $\forall\, n \geq 1$, the correspondence T_n: ("$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$"... if $n = p$ prime)

$$\overline{(E,P)} \longmapsto \sum_{\substack{G \subset E \text{ order } n \\ \text{s.t. } \langle P\rangle \cap G = 0}} \overline{(E/G, \overline{P})}. \quad (\underset{T_p}{\uparrow}\text{"degree" } p+1 \text{ if } p \nmid N,\ p \text{ if } p \mid N).$$

$$X_1(N;n) \supset Y_1(N;n) = \{(E,P,G) \mid P \in E \text{ order } N,\ G \subset E \text{ order } n,\ \langle P \rangle \cap G = \{0\}\}/\cong.$$

$$\begin{tikzcd} & X_1(N;n) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X_1(N) & & X_1(N) \end{tikzcd}$$

Induces $T_n \in \operatorname{End}(J_1(N))$, $\overline{P_1 + \cdots + P_d - d.\infty} \mapsto \overline{T_n P_1 + \cdots + T_n P_d - d.T_n\infty}$

③ $\mathbb{T}_N :=$ the subring of $\operatorname{End}(J_1(N))$ generated by all T_n and $\langle a \rangle$.

(also generated by all T_p and $\langle a \rangle$); note: $\operatorname{End}(J_1(N))$ free f.g. $\mathbb{Z}$-module, $(\subset \operatorname{End}_{\mathbb{Z}} H_1(X_1(N), \mathbb{Z}).)$

Formulas on q-expansions etc. show:

- (i) the pairing $(\mathbb{T}_N)_\mathbb{C} \times S_2(N) \longrightarrow \mathbb{C}$, $(t, \omega) \mapsto a_1(t^*\omega)$ is perfect
- (ii) $S_2(N)^\vee$ is a free $(\mathbb{T}_N)_\mathbb{C}$-module of rank 1
- (iii) $S_2(N)$ is a free $(\mathbb{T}_N)_\mathbb{C}$-module of rank 1. (use $(\omega, \eta) \mapsto \frac{i}{2}\int_{X_1(N)} \omega\cdot\overline{\eta}$)
- (iv) $H_1(X_1(N), \mathbb{Q})$ is a free $(\mathbb{T}_N)_\mathbb{Q}$-module of rank 2.

For ℓ prime, $N \geq 1$, define $W_{N,\ell} := \mathbb{Q} \otimes \left(\varprojlim_n J_1(N)[\ell^n]\right) = H_1(X_1(N), \mathbb{Q}_\ell)$,

this is a free $(\mathbb{T}_N)_{\mathbb{Q}_\ell}$-module of rank 2.

ref: • book by Katz-Mazur

• article Deligne-Rapoport

• Conrad's book, • Diamond-Im.

5. Arithmetic moduli of elliptic curves.

Def. Let S be a scheme. An elliptic curve over S is:

$$\begin{tikzcd} E \arrow[d, "f"'] \\ S \arrow[u, bend right, "o"'] \end{tikzcd} \quad \circlearrowleft \mathrm{id}_S$$

f proper, smooth rel. dim. 1, geom. fibers connected, genus 1.

Equivalently: locally on S, given by a W-equation in $\mathbb{P}^2_S$:

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

s.t. discrim(..) is a unit; $o = (0:1:0)$.

$\mathrm{Pic}_{E/S} : Sch/S \to Set$, $\begin{smallmatrix} T \\ \downarrow \\ S \end{smallmatrix} \longmapsto \mathrm{Pic}(E_T)/f_T^*(\mathrm{Pic}\,T)$
(contravariant)

$$\begin{tikzcd} E_T \arrow[r] \arrow[d, "f_T"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

$\mathrm{Pic}_{E/S} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_{E/S}$

$E \xrightarrow[/]{} \mathrm{Pic}^1_{E/S}$: $E(T) \longrightarrow \mathrm{Pic}^1_{E/S}(T)$

isomorphism. $\quad P \longmapsto [\mathcal{O}_{E_T}(P)] = [I_P^\vee] \quad (I_P \rightarrowtail \mathcal{O}_{E_T} \twoheadrightarrow P_*\mathcal{O}_T)$

$$\begin{tikzcd} E_T \arrow[r] \arrow[d] & E \arrow[d] \\ T \arrow[r] \arrow[u, bend left, "o"] \arrow[u, bend left=70, "P"] \arrow[ur, "P"] & S \arrow[u, bend right, "o"'] \end{tikzcd}$$

hence: (i) $\mathrm{Pic}^1_{E/S}$ is representable

(ii) $E \xrightarrow{\sim} \mathrm{Pic}^1_{E/S} \xrightarrow{\sim} \mathrm{Pic}^0_{E/S}$
$[\mathcal{L}] \longmapsto [I_o \otimes \mathcal{L}]$ gives group str. on E.

Morphisms:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr] & & E_2 \arrow[dl] \\ & S \arrow[ul, bend left, "o_1"] \arrow[ur, bend right, "o_2"'] & \end{tikzcd}$$

are autom. group-morphisms.

Def. The stack $[Ell]$ is the category with objects: $\begin{tikzcd} E \arrow[d] \\ S \arrow[u, bend right, "o"'] \end{tikzcd}$,

and morphisms:

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E_2 \arrow[d] \\ S_1 \arrow[r] \arrow[u, bend left] & S_2 \arrow[u, bend right] \end{tikzcd}$$

We have $[Ell] \to (Sch)$: $(E \overset{o}{\rightleftarrows} S) \mapsto S$. Kind of "sheaf of categories".

Fact: $[Ell]$ has no final object. (trivial, bec. of $\pm 1 \in \mathrm{Aut}_S(E)$ for all E/S

For $N \geq 1$, $\forall E/S$: $[N] : E \to E$ is finite loc. free rank n^2, and étale if N is invertible on S.

For $N \geq 1$: $[\Gamma_1(N)]_{\mathbb{Z}[1/N]}$ is the cat. with objects $\left(\begin{smallmatrix} E \\ f\downarrow \uparrow 0 \\ S \end{smallmatrix}, P\right)$, where

$P \in E(S)$ is of order N in all fibres, morphisms:

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E_2 \arrow[d] \\ S_1 \arrow[u, bend left, "P_1"] \arrow[r] & S_2 \arrow[u, bend left, "P_2"] \end{tikzcd}$$

<u>Thm</u> (Igusa) For $N \geq 4$, $[\Gamma_1(N)]_{\mathbb{Z}[1/N]}$ has a final object:
$$\left(\begin{tikzcd} \mathbb{E} \arrow[d] \\ Y_1(N)_{\mathbb{Z}[1/N]} \arrow[u, bend right, "0"'] \end{tikzcd}, \mathbb{P}\right).$$
where $Y_1(N)_{\mathbb{Z}[1/N]}$ is a smooth affine curve $/\mathbb{Z}[1/N]$ with geom. irred. fibers, which can be uniquely compactified into a smooth proper curve:

$$\begin{tikzcd} Y_1(N)_{\mathbb{Z}[1/N]} \arrow[r, hook] \arrow[dr] & X_1(N)_{\mathbb{Z}[1/N]} \arrow[d] & \text{cusps} \arrow[l, hook'] \arrow[dl, "\text{finite étale.}"] \\ & \operatorname{Spec} \mathbb{Z}[1/N] & \end{tikzcd}$$

Let $J_1(N)_{\mathbb{Z}[1/N]} := \mathrm{Pic}^0_{X_1(N)_{\mathbb{Z}(1/N)}}$, abelian scheme $/\mathbb{Z}(1/N)$. $\mathbb{T}_N \hookrightarrow \underline{\underline{\mathrm{End}(J_1(N)_{\mathbb{Z}}}}$

Then $\forall \ell$ prime : $W_{N,\ell} = \mathbb{Q} \otimes \left(\varprojlim_n J_1(N)(\overline{\mathbb{Q}})[\ell^n] \right)$,

free $(\mathbb{T}_N)_{\mathbb{Q}_\ell}$-module of rank 2, + $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

Choice of basis gives:

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[rr, bend right, "\rho_{f,\ell}"'] & \mathrm{GL}_2((\mathbb{T}_N)_{\mathbb{Q}_\ell}) \arrow[r, two heads] & \mathrm{GL}_2(K_{f,\ell}) \end{tikzcd}$$

$(\mathbb{T}_N \to K_f,\ T_n \mapsto a_n(f))$

$\rho_{f,\lambda}$ unramified at $p \nmid N\ell$: $p \neq \ell$ and $J_1(N)_{\mathbb{F}_p}$ is an <u>ab. var.</u>

To prove : for $p \nmid N\ell$, $\mathrm{tr}(\rho_{f,\ell}\ \mathrm{Frob}_p) = a_p(f)$.

This follows from the Eichler-Shimura relation:

$$T_p = F + \langle p \rangle \cdot V \quad \text{in} \quad \mathrm{End}_{\mathbb{F}_p}(J_1(N)_{\mathbb{F}_p}).$$

T_p: $\mathrm{End}_{\mathbb{Z}[1/N]}(J_1(N)_{\mathbb{Z}[1/N]}) \hookrightarrow \mathrm{End}_{\overline{\mathbb{F}}_p}(J_1(N)_{\overline{\mathbb{F}}_p})$ (good reduction).

$J_1(N)_{\overline{\mathbb{F}}_p} = \mathrm{Pic}^{\circ}_{X_1(N)_{\overline{\mathbb{F}}_p}}$, T_p given by correspondence of <u>p-isogenies</u>.

$$\begin{tikzcd}
 & X_1(N;p)_{\overline{\mathbb{F}}_p} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
X_1(N)_{\overline{\mathbb{F}}_p} & & X_1(N)_{\overline{\mathbb{F}}_p}
\end{tikzcd}$$

s & t both finite loc. free of rank $p+1$.

But we <u>see</u>:

$$\left(\begin{tikzcd}
 & \mathbb{P} \arrow[rrrr, bend left] & & & & \mathbb{P}^{(p)} & \\
 & & \mathbb{E} \arrow[rr, "F_{\mathbb{E}/Y_1(N)_{\mathbb{F}_p}}"] \arrow[dl] \arrow[dr] & & \mathbb{E}^{(p)} \arrow[dl] \arrow[dr] & & \\
\mathbb{P} & \mathbb{E} \arrow[dr] & & Y_1(N)_{\mathbb{F}_p} \arrow[dl, "\mathrm{id}"', "\sim"] \arrow[dr, "F"] & & \mathbb{E} \arrow[dl] & \mathbb{P} \\
 & & Y_1(N)_{\mathbb{F}_p} & & Y_1(N)_{\mathbb{F}_p} & &
\end{tikzcd}\right) = F$$

$$\left(\begin{tikzcd}
 & \mathbb{P}^{(p)} & & & & p\cdot\mathbb{P} \arrow[ddr, leftrightarrow, bend left] & \\
 & & \mathbb{E}^{(p)} \arrow[rr, "V_{\mathbb{E}/Y_1(N)_{\mathbb{F}_p}}"] \arrow[dl] \arrow[dr] & & \mathbb{E} \arrow[dl] \arrow[dr] & & \\
\mathbb{P} & \mathbb{E} \arrow[dr] & & Y_1(N)_{\mathbb{F}_p} \arrow[dl, "F"'] \arrow[dr, "\langle p\rangle", "\sim"'] & & \mathbb{E} \arrow[dl] & \mathbb{P} \\
 & & Y_1(N)_{\mathbb{F}_p} & & Y_1(N)_{\mathbb{F}_p} & &
\end{tikzcd}\right) = \langle p\rangle . V$$

Bayer-Neukirch, 1981.

6. Construction of $\rho_{f,\lambda}$ for $k \geq 2$.

Ref: $\left\{\begin{array}{l}\text{Conrad's book?}\\ \text{T. Saito's article?!}\end{array}\right.$

Rem. One can use congruences with weight 2 forms of varying level $\ell^n \cdot N$ to construct $\rho_{f,\lambda}$, but that does not give the Ramanujan conjecture, for example. So we want the construction in the cohomology.

Assume $N \geq 5$: $\Gamma_1(N)$ acts freely on $\mathbb{H}$, regularly at the cusps.

$\mathbb{E}$ gives $(R^1 p_*)\mathbb{Z}_{\mathbb{E}}$ on $Y_1(N)(\mathbb{C})$, loc. const. sheaf of free $\mathbb{Z}$-modules of rank 2.

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] & \\ Y_1(N)(\mathbb{C}) \arrow[r, hook, "j"] & X_1(N)(\mathbb{C}) \end{tikzcd}$$

$\mathcal{F}_k := j_* \operatorname{Sym}^{k-2}((R^1 p_*)\mathbb{Z}_{\mathbb{E}})$ on $X_1(N)(\mathbb{C})$

Eichler-Shimura isomorphism: $\mathbb{C} \otimes H^1(X_1(N)(\mathbb{C}), \mathcal{F}_k) \xrightarrow{\sim} S_k(N) \oplus \overline{S_k(N)}$

This is a Hodge decomposition: $\underset{(k-1,0)}{S_k(N)} \quad \underset{(0,k-1)}{\overline{S_k(N)}}$

Rem. One can also embed $S_k(N)$ in $H^{k-1}(\mathbb{E}^{k-2,*}, \mathbb{C})$, where

$\mathbb{E}^{k-2}$ = $k-2$ fold fiber pr. of $\mathbb{E} \to Y_1(N)(\mathbb{C})$,

$\mathbb{E}^{k-2,*}$ a suitable non-sing. compactification, over $X_1(N)(\mathbb{C})$.

$$f \longmapsto f \cdot d\tau \cdot dz_1 \cdots dz_{k-2}.$$

Hecke operators T_n, $\langle a \rangle$ on $H^1(X_1(N)(\mathbb{C}), \mathcal{F}_k)$ f.g. $\mathbb{Z}$-module.

$\rightsquigarrow \mathbb{T}_N$, free f.g. $\mathbb{Z}$-module; $\mathbb{Q} \otimes H^1(X_1(N)(\mathbb{C}), \mathcal{F}_k)$ is free rk 2 $/ \mathbb{T}_{N,\mathbb{Q}}$

Let ℓ be prime. $\mathcal{F}_{k,\ell} := j_* \operatorname{Sym}^{k-2}(R^1 p_*)\mathbb{Q}_{\ell,\mathbb{E},et}$ where now $p: \mathbb{E}_{\mathbb{Q}} \to Y_1(N)_{\mathbb{Q}}$, ℓ-adic sheaf on $X_1(N)_{\mathbb{Q},et}$, it extends well over $X_1(N)_{\mathbb{Z}[1/N\ell]}$: "lisse" away from "cusps", tamely ramified along "cusps".

Put $W_{N,k,\ell} := H^1(X_1(N)_{\overline{\mathbb{Q}},et}, \mathcal{F}_{k,\ell})^\vee$; free rank 2 over $(\mathbb{T}_N)_{\mathbb{Q}_\ell}$.

Have:
$$\begin{tikzcd} \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[rr, bend right, "\rho_{f,\ell}"'] & GL_2((\mathbb{T}_N)_{\mathbb{Q}_\ell}) \arrow[r, two heads] & GL_2(\mathbb{Q}_\ell \otimes K_f) \end{tikzcd}$$

$\left(\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ acts as: } (id \times \operatorname{Spec}(\sigma^{-1}))^{*,\vee}\right)$

Unramifiedness and $\mathrm{tr}(\rho_{f,\lambda}\,\mathrm{Frob}_p) = a_p(f)$ are proved, modulo technicalities, as before.

7. What about $\rho_{f,\lambda,p}$ $\left(= \rho_{f,\lambda}|_{G_{\mathbb{Q}_p}}\right)$ for $p \mid N$, $p \neq \ell$?

Rem. The case $k \geq 2$ is not really harder than the case $k = 2$.
So we only discuss $k = 2$, now.

References. • Carayol's thesis (1986, Annales sc. E.N.S.). • Nyssen.
- Breuil-Conrad-Diamond-Taylor, 200? ; Appendix A of my B. lecture
- letter Deligne → P-S.

We must get the repr. theory of $GL_2(\mathbb{A}_f)$ in the picture: use the formalism of Shimura varieties.

The Shimura datum: $(GL_2, \mathbb{H}^{\pm})$, $\mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \overset{GL_2(\mathbb{R})}{\circlearrowleft}$

$\mathbb{H}^{\pm}$ = $GL_2(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{C}^\times, GL_2(\mathbb{R}))$ of $a+bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $GL_2(\mathbb{R})$ acts by composition with inner automorphisms.

($\mathbb{H}^{\pm}$ = set of $\mathbb{R}$-H.S. on $\mathbb{R}^2$ of type $(-1,0)+(0,-1)$.)

For $K \subset GL_2(\mathbb{A}_f)$ open compact subgrp: $Y_K(\mathbb{C}) := GL_2(\mathbb{Q}) \backslash \left(\mathbb{H}^{\pm} \times GL_2(\mathbb{A}_f)/K \right)$,

As $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f)/K$ is finite, $Y_K(\mathbb{C}) = \coprod_{\text{finite}} \Gamma_i \backslash \mathbb{H}^+$.

$X_K(\mathbb{C}) := Y_K(\mathbb{C}) \cup$ cusps; compact Riemann surface (not nec. connected)

$X_{K,\mathbb{C}}$:= the alg. curve / $\mathbb{C}$ associated to $X_K(\mathbb{C})$.

Moduli of elliptic curves: canonical model $X_{K,\mathbb{Q}}$ over $\mathbb{Q}$.

Example: $K_N := \ker\left(GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/N\mathbb{Z})\right)$, $N \geq 3$.

Then $Y_{K_N,\mathbb{Q}}\left(\begin{smallmatrix} S \\ \downarrow \\ \mathrm{Spec}(\mathbb{Q}) \end{smallmatrix}\right) = \left\{ (E/S, \varphi) \mid \varphi : (\mathbb{Z}/N\mathbb{Z})^2_S \xrightarrow{\sim} E[N] \right\} / \cong$

The $X_{K,\mathbb{Q}}$ form a projective system: $K' \subset K \rightsquigarrow X_{K',\mathbb{Q}} \overset{\text{finite}}{\longrightarrow} X_{K,\mathbb{Q}}$.

$X_{\mathbb{Q}} := \varprojlim_K X_{K,\mathbb{Q}}$; this is a scheme, profinite / j-line
(finite transition maps $\rightsquigarrow$ direct limit of structure sheaves...).

$\circlearrowleft$

$Gl_2(\mathbb{A}_f)$, "smooth" action: the stabiliser of every $\varphi \in \mathcal{O}_{X_{\mathbb{Q}}}(U)$ is open.

$\forall\, K \subset Gl_2(\mathbb{A}_f)$ open, compact: $X_{K,\mathbb{Q}} = X_{\mathbb{Q}}/K$.

All Hecke correspondences can be described from $X_{\mathbb{Q}} \circlearrowleft Gl_2(\mathbb{A}_f)$.

For ℓ prime, $H_\ell := \varinjlim_K \underbrace{H^1(X_{K,\overline{\mathbb{Q}},\text{et}}, \overline{\mathbb{Q}}_\ell)}$

$\circlearrowleft$ $\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\mathbb{C}$ $\quad G_{\mathbb{Q}} \times Gl_2(\mathbb{A}_f)$ $\qquad \left(\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n J_K(\overline{\mathbb{Q}})[\ell^n]\right)^\vee$.

$\cup$

Choose $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_\ell$.

$\cup$

$\mathbb{Q} \subset K_f = \mathbb{Q}(a_1(f), \dots)$

$\varinjlim_K \Omega^1(X_{K,\mathbb{Q}}(\mathbb{C}))$ $\quad$ pr: $X \to X_1(N)$

$\|$ $\qquad V_{\pi_f} = \mathbb{C}.\{g \cdot \mathrm{pr}^* f \mid g \in Gl_2(\mathbb{A}_f)$

From q-expansion formulas: $\Omega^1(X_{\mathbb{Q}}(\mathbb{C})) = \bigoplus_{f \text{ newforms weight } 2} \pi_f$.

Hodge decomposition: $H^1(X_{K,\mathbb{Q}}(\mathbb{C}), \mathbb{C}) = \Omega^1(X_{K,\mathbb{Q}}(\mathbb{C})) \oplus \overline{\Omega^1(X_{K,\mathbb{Q}}(\mathbb{C}))}$

<u>Conclusion:</u> $H_\ell = \bigoplus_{f \text{ newf. wt. } 2 / \overline{\mathbb{Q}}_\ell} \rho_f^\vee \otimes \pi_f$, ρ_f as characterized by <u>Thm</u>!

<u>Thm</u> (Deligne (1973)) $\forall\, p \neq \ell$: $\pi_{f,p}$ determines $\rho_{f,p}$.

Ingredients:
- a good model over $\mathbb{Z}$ of $X_{\mathbb{Q}}$ (Drinfeld level structures)
- vanishing cycle theory
- Serre-Tate theory at the supersingular points
- Jacquet-Langlands correspondence

Models over $\mathbb{Z}$. (Katz-Mazur)

For $N \geq 1$, E/S ell. curve: a Drinfeld level N str. on E/S is a $\varphi : (\mathbb{Z}/N\mathbb{Z})^2_S \to E$ s.t. $E[N] = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^2} \varphi(x)$ as divisors on $E \to S$. (relative eff. Cartier)

Basic result: for N divisible by at least 2 distinct prime powers ≥ 3, the stack $[\Gamma(N)]$ has a final object $(\mathbb{E} \to Y_{K_N}, \varphi)$, with Y_{K_N} an affine curve $/\mathbb{Z}$,

$$\begin{tikzcd}
 & Y_{K_N} \arrow[r, hook] \arrow[dr] & X_{K_N} \arrow[dll, "\text{geom. connected fibers}"'] \arrow[d, "\text{proper}"] & \text{cusps} \arrow[l, hook'] \\
\mathrm{Spec}\,\mathbb{Z}[\zeta_N] \arrow[rr] & & \mathrm{Spec}\,\mathbb{Z} &
\end{tikzcd}$$

, X_{K_N} regular.

(write $N = p^n \cdot N'$, $p \nmid N'$;

$$\begin{tikzcd}
X_{K_N, \bar{\mathbb{F}}_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{K_N} \arrow[d] \\
\mathrm{Spec}\,\bar{\mathbb{F}}_p \arrow[r] & \mathrm{Spec}\,\mathbb{Z}[\zeta_N]
\end{tikzcd}$$

irred. components $\leftrightarrow \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$, are smooth, intersect at all supersingular points:

$p=2$ $n=1$:

not a stable model.

Vanishing cycle theory.

$K_{n,N} := \overset{\substack{\text{at } p \\ \downarrow}}{K_{p^n}} \times \overset{\substack{\text{away from } p \\ \downarrow}}{K'_N}$, $p \nmid N$, $N \geq 5$.

$GL_2(\mathbb{Q}_p) \circlearrowright X_N := \varprojlim_n X_{K_{n,N}}$.

$H_{\ell,p} := H^1(X_{N,\bar{\mathbb{Q}}_p,\text{et}}, \bar{\mathbb{Q}}_\ell) \circlearrowleft G_{\mathbb{Q}_p} \times GL_2(\mathbb{Q}_p) \times \mathbb{T}'_N \leftarrow$ Hecke operators away from p.

$H^s_{\ell,p} := H^1(X_{N,\bar{\mathbb{F}}_p,\text{et}}, \bar{\mathbb{Q}}_\ell) \circlearrowleft$ same, $G_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_p^\times$.

$$H^s_{\ell,p} \rightarrowtail H_{\ell,p} \longrightarrow \overbrace{\bigoplus_{x \in \mathrm{Sing}(X_{N,\bar{\mathbb{F}}_p})} \Psi_x}^{\Psi} \twoheadrightarrow \text{small cokernel } (H^2\text{'s})$$

↑ explicitly known, $GL_2(\mathbb{Q}_p)$ acts through non-cuspidal (induced) rep's. (Langlands, 1972?)

Serre-Tate theory. Ψ_x depends only on the $\hat{\mathcal{O}}_{X_{K_n,N},x}$, i.e., on the deformation theory of E_x, hence, only on the def. theory of $E_x[p^\infty]$. Hence:

$$\hat{\mathcal{O}}_{X_{K_n,N},x} \circlearrowleft \mathrm{Aut}(E_x[p^\infty]) = (\mathbb{Z}_p \otimes B)^\times$$

$B := \mathrm{End}(E_x)$; max. order in $B_{\mathbb{Q}}$, the quat. alg. "ramified" at p and ∞.

$$\Psi \circlearrowleft G_{\mathbb{Q}_p} \times Gl_2(\mathbb{Q}_p) \times \Pi_N'$$

$$\bigoplus_i \rho_i \otimes \pi_i \otimes \pi_i'$$

$$\circlearrowleft G_{\mathbb{Q}_p} \times Gl_2(\mathbb{Q}_p) \times B_{\mathbb{Q}_p}^\times$$

Deligne's "fundamental local repr."
(sum of copies of, finitely many)

Π_N' acts as matrices with coeff. in $\overline{\mathbb{Q}}_\ell[B_{\mathbb{Q}_p}^\times]$, via $B_{\mathbb{Q}}^\times$.

Consequence: π_i' determines π_i and ρ_i

Jacquet-Langlands relates autom. repr. of $\underline{B}^\times$ to those of $Gl_{2,\mathbb{Q}}$.

It gives: π_i determines π_i' and hence also ρ_i.

Hence: $\pi_{f,p}$ determines $\rho_{f,p}$.