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Leaves on Shimura Varieties

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ICTP, Trieste.

The goals of these lectures are

- Introduce the moduli spaces of PEL-type.
- Introduce leaves (F. Oort) and the theory of canonical coordinates on leaves (C.-L. Chai).
- Deduce the dimension formula for leaves.

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§ Moduli spaces of PEL-type

P = polarization, E = endomorphism, L = level structure

moduli spaces of abelian varieties with a polarization, endomorphisms and level structure.

<u>Def.</u> A PEL-datum is $\mathcal{D} = (B, *, V, \psi)$, where

- B is finite-dim'l semi-simple $/\mathbb{Q}$ with a positive involution $*$, i.e $\mathrm{tr}_{B/\mathbb{Q}}(bb^*) > 0 \ \forall b \neq 0 \in B$.
- V: finite faithful B-module,
- $\psi : V \times V \to \mathbb{Q}$ non-degenerate, alternating s.t. $\psi(bx, y) = \psi(x, b^* y)$, $\forall b \in B$, $x, y \in V$.

A PEL datum $\mathcal{D}$ gives rise to $\boxed{G}$, G_1, $\boxed{X}$:

- G; G_1 reductive alg gps $/\mathbb{Q}$

$$\underline{G}(\mathbb{Q}) = \{ h \in \mathrm{End}_B(V)^\times \mid h' \cdot h \in \mathbb{Q}^\times \}$$

$$G_1(\mathbb{Q}) = \{ h \in \mathrm{End}_B(V)^\times \mid h' \cdot h = 1 \}$$

$h \mapsto h'$ the adjoint w.r.t. ψ

- let $h_0 : \mathbb{C} \to \mathrm{End}_B(V_{\mathbb{R}})$ $\mathbb{R}$-linear alg hom.

s.t. $h(z)' = h(\bar{z})$ and the symmetric form

$(x, y) := \psi(x, h_0(i) y)$ is positive definite.

$\underline{X} \subset \mathrm{Hom}_{\mathbb{R}\text{-gp}}(\mathbb{C}^\times, G_{\mathbb{R}})$

= the $G(\mathbb{R})$-conjugacy class of h_0.

The pair (G, X) satisfies the axioms for defining Shimura varieties. For any $U \subset G(\mathbb{A}_f)$ open compact

$$Sh_U(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / U,$$ quasi-projective alg var $/\mathbb{C}$ (Baily-Borel)

$$Sh(G, X) = \{ Sh_U(G, X) \}_{U \subseteq G(\mathbb{A}_f)}$$

the Shimura variety associated to the pair (G, X).

- $Sh_U(G,X)$ has a moduli interpretation as a moduli space of abelian varieties with a polarization, a B-action and level structure.
- Using the moduli interpretation, each $Sh_U(G,X)$ is defined over a number field $E(G,X) \subseteq \mathbb{C}$, (the reflex field)
- The Hecke operators $Sh(G,X) \circlearrowleft G(\mathbb{A}_f)$ are defined over $E(G,X)$.

$h_0(i)$ complex structure on $V_{\mathbb{R}}$ commuting with B

$$V_{\mathbb{C}} = \underset{\substack{\circlearrowleft \\ B}}{V^{-1,0}} \oplus V^{0,-1}, \quad \text{eigensubspace with e.v. } i, -i.$$

$\underline{E(G,X)}$ = the field of definition of the complex algebra representation $V^{-1,0}$ of B

In fact $E(G,X) = \mathbb{Q}\left(\mathrm{Tr}(b \mid V^{-1,0})\right)_{b \in B}$

Recall Shimura datum:

A pair (G, X), where

- G is a connected reductive algebraic group $/\mathbb{Q}$
- X is a $G(\mathbb{R})$-conjugacy class of morphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times =: \mathbb{S} \longrightarrow G_{\mathbb{R}}$$

is called a Shimura datum if

(1) The Hodge structure on $\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$ corresponding to the representation

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$$

has Hodge types $(-1,1), (0,0), (1,-1)$

(2) $\mathrm{Int}\, h(i)$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$.

i.e $\{ g \in G^{ad}(\mathbb{R}) \mid h(i) \cdot g\, h(i)^{-1} = g \}$ is a maximal compact subgp.

(3) G^{ad} has no $\mathbb{Q}$-simple factor H s.t. $H(\mathbb{R})$ is compact.

<u>Remark</u> :

(1) $\Rightarrow$ $h(\mathbb{G}_{m,\mathbb{R}}) \subseteq Z(G_{\mathbb{R}})$

(2) $\Leftrightarrow$ $\exists$ <u>natural structure</u> of complex manifold on X and any conn. comp. of X is a Hermitian symmetric space.

Natural :

$\forall\ \rho : G_{\mathbb{R}} \to V_{\mathbb{R}}$ such that the Hodge structure $\mathbb{S} \xrightarrow{h} G \xrightarrow{\rho} \mathrm{Aut}(V_{\mathbb{R}})$ has pure weight

Then $X \longrightarrow$ Partial flag variety is holomorphic.

$h \longmapsto F^{\bullet}_{\rho(h)}$: the Hodge filtration

(1) $\overset{!}{\Longleftrightarrow}$ $\{\mathrm{Ad} \circ h\}_{h \in X}$ a family of HS on $\mathfrak{g}_{\mathbb{R}}$, is a variation of Hodge structures.

(Deligne : Travaux de Griffiths)

(3) $\Rightarrow$ the strong approximation holds for $\widetilde{G^{der}}$: the simply-connected cover of G^{der}.

$\boxed{\text{Reflex field } E(G,X)}$

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}}, \qquad h_{\mathbb{C}} : \underset{\substack{\| \\ (z_1,z_2) \ni \mathbb{C}^\times \times \mathbb{C}^\times}}{\mathbb{S}(\mathbb{C})} \longrightarrow G_{\mathbb{C}}$$

$$\mu_h(z) := h_{\mathbb{C}}(z,1) : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$$

$X' \subset \mathrm{Hom}_{\mathbb{C}\text{-gps}}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$ the $G(\mathbb{C})$-conjugacy chass of μ_h

$E(G,X) :=$ the field of definition of X'

i.e $H = \{ \sigma \in \mathrm{Aut}(\mathbb{C}) \mid X'^{\sigma} = X' \}$

$E(G,X) = \mathbb{C}^H$.

<u>Theorem</u> (Existence of the canonical model)

$\forall (G,X)$ and $U \subset G(\mathbb{A}_f)$

- $Sh_U(G,X)$: quasi-projective scheme/ $E(G,X)$ s.t

- When (G,X) comes from a PEL-datum $Sh_U(G,X)$ is extended by the modular interpretation.
- Hecke operations are defined over $E(G,X)$.
- If $(G_1,X_1) \xrightarrow{\varphi} (G_2,X_2)$ a morphism of Shimura data $\varphi(U_1) \hookrightarrow U_2$.

$\varphi_* : Sh_{U_1}(G_1,X_1) \longrightarrow Sh_{U_2}(G_2,X_2)$ defined / $E(G_1,X_1) \cdot E(G_2,X_2)$

Moduli interpretation

$\mathfrak{D} = (B, *, V, \psi)$ PEL datum $\rightsquigarrow$ G, G_1, X.

(I) $Sh_U(G,X)(\mathbb{C})$ = the set of isom classes of

$(A, \lambda, \iota, \bar{\eta})$, where

- A : g-dim'l abelian variety over $\mathbb{C}$ up to isogeny, $g = \frac{1}{2}\dim_{\mathbb{Q}} V$,
- $\lambda : A \to A^t$ polarization up to $\mathbb{Q}$-multiple.
- $\iota : B \to \mathrm{End}(A)$ s.t. $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$.

(A, λ, ι) polarized abelian B-variet up to isogeny

- $\bar{\eta}$: U-orbit of $B \otimes \mathbb{A}_f$-linear isomorphisms

$$\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} H_1(A, \mathbb{A}_f) \quad \text{s.t. } \exists \alpha : \mathbb{A}_f \to \mathbb{A}_f(1)$$

s.t. the diagram

$$\begin{tikzcd} V_{\mathbb{A}_f} \times V_{\mathbb{A}_f} \arrow[r, "(\eta,\eta)"] \arrow[d, "\psi"] & H_1(\mathbb{A}_f) \times H_1(\mathbb{A}_f) \arrow[d, "e_\lambda"] \\ \mathbb{A}_f \arrow[r, "\alpha"] & \mathbb{A}_f(1) \end{tikzcd}$$

commutes.

Such that

(a) $\exists \beta \in \bar{\eta} : V \xrightarrow{\sim} H_1(A, \mathbb{Q})$, compatible with the add. structures

(b) $\mathrm{tr}(b; V^{-1,0}) = \mathrm{tr}(b; \mathrm{Lie}(A))$. $\forall b \in B$

<u>Discussion</u> :

The level structure $\bar{\eta} \Rightarrow V \otimes \mathbb{Q}_\ell \simeq V_\ell(A)$ compatible

with the additional structures, $\forall \ell$

(b) $\Longleftrightarrow$ $V \otimes \mathbb{R} \simeq H_1(X, \mathbb{R})$

If the Hasse principle holds for $H^1(\mathbb{Q}, G)$

then (a) is satisfied.

II Choose $V_{\mathbb{Z}}$ ($\mathcal{O}_B$-lattice, $\mathcal{O}_B$-order of B.) s.t. $\psi(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subseteq \mathbb{Z}$, and U satabilizes $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$

$Sh_U(G, X)(\mathbb{C}) = \{(A, \lambda, \iota, \bar{\eta})\} / \simeq$, where

- A: g-dim'l ab. var. $/\mathbb{C}$,
- $\lambda : A \to A^t$
- $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ s.t. $\lambda\, \iota(b^*) = \iota(b)^t \lambda$

(A, λ, ι): polarized abelian $\mathcal{O}_B$-variety

- $\bar{\eta}$: U-orbit of $\mathcal{O}_B \otimes \hat{\mathbb{Z}}$-linear isomorphisms

$\eta : V \otimes \hat{\mathbb{Z}} \xrightarrow{\sim} H_1(A, \hat{\mathbb{Z}})$ s.t. $\exists\, \alpha : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}(1)$

such that the diagram

$$\begin{tikzcd} V_{\hat{\mathbb{Z}}} \times V_{\hat{\mathbb{Z}}} \arrow[r, "(\eta,\eta)"] \arrow[d, "\psi"] \arrow[dr, phantom, "\square"] & H_1(\hat{\mathbb{Z}}) \times H_1(\hat{\mathbb{Z}}) \arrow[d, "e_\lambda"] \\ \hat{\mathbb{Z}} \arrow[r, "\alpha"] & \hat{\mathbb{Z}}(1) \end{tikzcd}$$

commutes

such that

(a), (b) as before.

Extension $/E = E(G, X)$ using moduli interpretation

S: E-scheme, connected

$F_I(S) = \{ (A, \lambda, \iota, \bar{\eta}) \} / \simeq$, where

- (A, λ, ι) g-dim'l polarized abelian B-scheme over S up to isogeny,

- $\bar{\eta}$: U-orbit of $B \otimes \mathbb{A}_f$-linear ψ-similitudes

$$\eta : V \otimes \mathbb{A}_f \simeq H_1(A_{\bar{s}}, \mathbb{A}_f)$$

which is $\pi_1(S, \bar{s})$-invariant, $\bar{s}$: a geometric pt of S.

such that

(a) $\quad tr(b; V^{-1,0}) = tr(b; Lie(A)_s)$ in $\mathcal{O}_s$.

The functor F_I is representable by a quasi-projective E-scheme S_U, $\quad S_U = \coprod_{i=1}^{r} S_{U,i} \,/E$, $\quad S_{U,i}(\mathbb{C}) = Sh_U(G^{(i)}, X^{(i)})$.

(when U is small enough)

S: connected E-scheme.

$F_{II}(S) = \{ (A, \lambda, \iota, \bar{\eta}) \} / \simeq$, where

- (A, λ, ι) : g-dim'l polarized abelian $\mathcal{O}_B$-scheme $/S$,
- $\bar{\eta}$: U-orbit of $\mathcal{O}_B \otimes \hat{\mathbb{Z}}$-linear ψ-similitudes.

$$\eta : V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}} \simeq H_1(A_{\bar{s}}, \hat{\mathbb{Z}})$$

which is $\pi_1(S, \bar{s})$-invariant, $\bar{s}$: a geom. pt. of S.

Satisfying the trace condition.

The functor $F_I \simeq F_{II}$ is representable by a quasi-projective E-scheme S_U, a finite disjoint of $Sh_U(G^{(i)}, X^{(i)})$.

<u>Remark</u>.

(1) It is more flexible using I to define Hecke operators

$$(A, \lambda, \iota, \bar{\eta}) \mapsto (A, \lambda, \iota, \overline{\eta g}), \quad Sh_U \to Sh_{gUg^{-1}}$$
$$g \in G(\mathbb{A}_f)$$

For II, we use isogenies to define Hecke correspondences.

(2) It is II that is used to show the representability.

$F_{II} \to \mathcal{A}_{g,d,N}$ is relatively representable.

$(A, \lambda, \iota, \bar{\eta}) \mapsto (A, \lambda, \bar{\eta})$

Integral models at p

Fix p: rational prime, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ embedding

Want to extend the Shimura variety $Sh_{U/E}$ to over $\mathcal{O} := \mathcal{O}_{E,\mathfrak{p}}$.

$\tilde{\mathfrak{D}} = (B, *, V, \psi, \mathcal{O}_B, V_{\mathbb{Z}})$ p-integral PEL datum

- $\mathfrak{D} = (B, *, V, \psi)$ rational PEL datum.
- $\mathcal{O}_B$: order of B stable under $*$, maximal at p. (always exists)
- $V_{\mathbb{Z}}$: $\mathcal{O}_B$-lattice of V, $\psi(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subseteq \mathbb{Z}$, and

  $V_{\mathbb{Z}} \otimes \mathbb{Z}_p$: self-dual w.r.t. ψ (such $V_{\mathbb{Z}}$ exists if we modify ψ).

Choose $\zeta_N \in \overline{\mathbb{Q}}$: primitive Nth root of unity, $N \geq 3$. $(p, N) = 1$

$U_N^N := \ker(G_1(\hat{\mathbb{Z}}) \to G_1(\hat{\mathbb{Z}}/N\hat{\mathbb{Z}}))$. $g = \frac{1}{2}\dim_{\mathbb{Q}} V$.

X = the $G(\mathbb{R})$-conj class of

$$h_0 : \mathbb{S} \to G_{\mathbb{R}}$$

which satisfies the Riemann condition as before.

Kottwitz determinant condition :

Choose a basis $e_1, \cdots, e_n$ for $\mathcal{O}_B$ over $\mathbb{Z}$.

For $h \in X$, let $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ the decomposition

where $h(z)$ acts on $V^{-1,0}$ by z (on $V^{0,-1}$ by $\bar{z}$).

$B \circlearrowright V^{-1,0}$, $\quad F_X(X_1, \cdots, X_n) := \det\left(X_1 e_1 + \cdots + X_n e_n \,;\, V^{-1,0}\right) \in \mathbb{C}[X_1, \cdots]$.

- F_X : independent of the choice of h, depends on the choice of basis e
- $F_X \in \mathcal{O}_E[X_1, \cdots, X_n] \subseteq \mathcal{O}_{E,\mathfrak{g}}[X_1, \cdots, X_n]$

$S = \mathcal{O}_{E,\mathfrak{g}}$-sch

(A, λ, ι) : g-dim'l polarized abelian $\mathcal{O}_B$-scheme $/S$ satisfies the <u>Kottwitz determinant condition</u>

if $\quad \det\left(\underbrace{X_1 e_1 + \cdots + X_n e_n}_{\in\, \mathcal{O}_S[X_1, \cdots, X_n]} \,;\, \mathrm{Lie}(A)_S\right) = F_X \qquad (*)$

Note : • Though F_X depends on $\{e_i\}$, the condition $(*)$ is independent of $\{e_i\}$.

• The determinant condition is closed; it is also open if $G_{\mathbb{Q}_p}$ is unramified.

With $\tilde{\mathcal{D}}$, ξ_N as before.

$\mathcal{M}_{\tilde{\mathcal{D}},N}$: the moduli space over $\mathcal{O}_{E,\mathfrak{p}}[\xi_N]$ of

g-dim'l p-principally polarized abelian

$\mathcal{O}_B$-varieties with a $(V_{\hat{\mathbb{Z}}^{(p)}}, U_N)$-level

structure w.r.t. ξ_N.

S : connected $\mathcal{O}_{E,\mathfrak{p}}$-scheme,

$\mathcal{M}_{\tilde{\mathcal{D}},N}(S) = \{(A, \lambda, \iota, \bar{\eta})_S\}/\simeq$, where

- (A, λ, ι) : g-dim'l polarized abelian $\mathcal{O}_B$-scheme over S such that $(\deg \lambda, p) = 1$.
- $\bar{\eta}$: a $(V_{\hat{\mathbb{Z}}^{(p)}}, U_N)$-level structure w.r.t. ξ_N :

  a U_N-orbit of $\mathcal{O}_B \otimes \hat{\mathbb{Z}}^{(p)}$-linear isoms

  $$\eta : V_{\hat{\mathbb{Z}}^{(p)}} \xrightarrow{\sim} H_1(A_{\bar{s}}, \hat{\mathbb{Z}}^{(p)})$$

  with
  $$\begin{tikzcd} V_{\hat{\mathbb{Z}}^{(p)}} \times V_{\hat{\mathbb{Z}}^{(p)}} \arrow[r, "(\eta,\eta)"] \arrow[d] \arrow[dr, phantom, "\square"] & H_1(\hat{\mathbb{Z}}^{(p)}) \times H_1(\hat{\mathbb{Z}}^{(p)}) \arrow[d, "e_\lambda"] \\ \hat{\mathbb{Z}}^{(p)} \arrow[r, "\xi_N"] & \hat{\mathbb{Z}}^{(p)}(1) \end{tikzcd}$$

  which is $\pi_1(S, \bar{s})$-invariant.

Such that

(A, λ, ι) satisfies the Kottwitz determinant condition.

- $\mathcal{M}_{\tilde{D},N}$ is representable by a quasi-projective $\mathcal{O}_{E,\mathfrak{p}}$-scheme,

  and the generic fibre $\mathcal{M}_{\tilde{D},N} \otimes_{\mathcal{O}_{E,\mathfrak{p}}} E = \coprod_i Sh_{U_N U_p}$, a disjoint union of the can. model Shimura varieties of PEL-type

- The reason of replacing the trace condition by the determinant condition is the following:

  B: finite-dim'l semi-simple alg $/k$

  V, W finite dim'l vect. spaces$/k$ with k-linear action by B.

  Then $\det_V(X) = \det_W(X) \iff V \simeq W$ as $B \otimes k$-modules.

- Let $x_1, x_2 \in \mathcal{M}_{\tilde{D},N}(k)$, $\underline{A}_{x_1}, \underline{A}_{x_2}$ corresponding objects.

  Then $T_\ell(\underline{A}_{x_1}) \simeq T_\ell(\underline{A}_{x_2}) \quad \forall\, \ell \neq \operatorname{char} k$.

  If $\operatorname{char} k = 0$, we don't have (examples in type D)

  $$T_p(\underline{A}_x) \simeq V_{\mathbb{Z}} \otimes \mathbb{Z}_p \quad \forall\, x \in \mathcal{M}_{\tilde{D},N}.$$

- $G(\mathbb{Z}_p)$ is a special maximal subgp of $G(\mathbb{Q}_p)$.

- The integral model $\mathcal{M}_{\tilde{D},N}$ is NOT flat$/\mathcal{O}_{E,\mathfrak{p}}$ in general. One needs to modify the model (Pappas - Rapoport).

Natural coordinates on complex Shimura varieties

$\mathfrak{D}$, G, X, $\quad Sh_U(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/U$

$$= \coprod_i \Gamma_i \backslash X \quad \text{complex uniformization}$$

$x \in Sh_U(\mathbb{C})$, $\quad h_0 \in X$, $\quad h_0 \mapsto x$

$T_x Sh_U = T_{h_0}(X) = \mathfrak{g}/\mathfrak{k} = \mathfrak{p} \qquad X = G(\mathbb{R})/Stab(h_0)$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

$\Rightarrow \dim Sh_U = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{p}$

$\mathfrak{p} \simeq \mathbb{R}^N$: global coordinates give rise to, via the exponential function, coordinates on a nbd of h_0.

$\because$ X is homog, the local chart around $h_0 \rightsquigarrow$ a local char around any point h of X.

However, this system of coordinates is real analytic not holomorphic. We want a natural system of holomorphic coordinates.

$$h : \mathbb{S} \to G_{\mathbb{R}} \subset GL(V_{\mathbb{R}})$$

$$V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$$

$$z, \quad \bar{z} \qquad h(z)$$

$$z, \quad 1 \qquad \mu_h(z) := h_{\mathbb{C}}(z,1)$$

$$V^{0,-1} = F^0 V_{\mathbb{C}}, = \{ v \in V_{\mathbb{C}} \mid \mu_h \cdot v = v \}.$$

$$X \longrightarrow \mathrm{Grass}(V_{\mathbb{C}}, g) \quad \text{holomorphic inclusion.}$$

$$h \longmapsto F^0 V_h$$

$$h_0 \in X, \quad P_- = \{ g' \in G_{\mathbb{C}} \mid g' \cdot (F^0 V_{h_0}) = F^0 V_{h_0} \}$$

$$X = G(\mathbb{R}) / \underset{\substack{\| \\ K}}{\mathrm{Stab}(h_0)} \hookrightarrow \underset{\substack{\| \\ \hat{X}}}{G_{\mathbb{C}}/P_-} \subset \mathrm{Grass}(V_{\mathbb{C}}, g)$$

open

(Borel embedding) $\qquad \hat{X}$: compact dual

$$g = k + p \qquad\qquad K_{\mathbb{C}} = \text{Levi-factor of } P_-.$$

$$g_{\mathbb{C}} = \underbrace{k_{\mathbb{C}} + u_-}_{\mathrm{Lie}(P_-)} + u_+$$

The nilpotent Lie subalgebra u_+ gives rise to a local chart at h_0, and a local chart at every point in X.

$$\Rightarrow \quad \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} \mathcal{U}_+$$

Observation: $\langle \mu_{h_0}, \alpha \rangle = 1 \quad \forall \alpha \in R(\mathcal{U}_+)$

$$\dim_{\mathbb{C}} X = \sum_{\alpha \in R(\mathcal{U}_+)} \langle \mu_{h_0}, \alpha \rangle .$$

<u>Example</u>: GS_{2g},

$$T = \left\{ \left(\begin{array}{ccc|ccc} \lambda\lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda\lambda_g & & & \\ \hline & & & \lambda_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \lambda_g^{-1} \end{array} \right) \right\} \simeq \{ (\lambda, \lambda_1, \cdots, \lambda_g) \in \mathbb{C}^{\times\, g+1} \}$$

$$\mu_{h_0}(z) = \left\{ \left(\begin{array}{ccc|ccc} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array} \right) \right\} . \quad P_- = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in GSp_{2g} \right\} .$$

$$\mathcal{U}_+ = \{ B \in M_g(\mathbb{C}) \,;\, B^t = B \} = \mathbb{C}\langle e_{ij} \rangle_{i \le j} \quad e_{ij} = E_{ij} + E_{ji}$$

$$\alpha_{ij}(\lambda, \lambda_1, \cdots, \lambda_g) = \lambda \cdot \lambda_i \cdot \lambda_j$$

$$\Rightarrow \alpha_{ij} \circ \mu_{h_0}(t) = t .$$

We call μ_{h_0} the <u>slope homo</u> associated the Hodge structure h_0, and call the Hodge filtration $F^0 V_{h_0} \subset V_{\mathbb{C}}$ the <u>slope filtration</u> ass. to h_0

Newton polygon strata :

$\mathcal{A}_g = \mathcal{A}_{g,1,N}$: the moduli space over $\overline{\mathbb{F}}_p$ of g-dim'l

$\ni (A, \lambda, \eta)$ principally polarized abelian varieties.

$k = \bar{k}$, char $k = p$, $(A, \lambda, \eta) \in \mathcal{A}_{g,1,N}(k)$

M = the (covariant) Dieudonné module of A.

free $W(k)$-module of rank $2g$, + σ-linear F

σ^{-1}-linear V st $FV = VF = p$.

Dieudonné - Manin

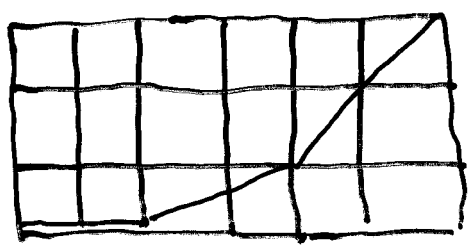
$$M \simeq \bigoplus_{i=1}^{s} \left(W[F] / F^{a_i} - p^{b_i} \right)^{m_i} \qquad \begin{array}{l} a_i \geq 1 \\ (a_i, b_i) = 1 \end{array}$$

Isog classes $\longleftrightarrow$ $\left\{ \underbrace{\frac{b_1}{a_1}, \cdots \frac{b_1}{a_1}}_{m_1 a_1}, \underbrace{\frac{b_2}{a_2} \cdots \frac{b_2}{a_2}}_{m_2 a_2}, \cdots \right\}$ the slope sequence.

$\longleftrightarrow$ Newton polygon, denoted by $NP(A)$

Example

$\{0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1\}$



In general

$0 \leq \lambda_1 < \cdots < \lambda_r \leq 1$ • $\lambda_i + \lambda_{r+1-i} = 1$ (symmetry)

multiplicities m_i • $m_i \cdot \lambda_i \in \mathbb{Z}$ (integrality)

ξ : symmetric Newton polygon of height $2g$, dim g.

$$W_\xi^\circ = \{ x \in \mathcal{A}_g \mid NP(x) = \xi \} \subseteq \mathcal{A}_g$$

reduced, locally closed subscheme of $\mathcal{A}_g$.

Partial order on NP's:

$$\xi_1 \prec \xi_2 \iff \xi_1 \text{ is lying over } \xi_2 \text{ or } \xi_1 = \xi_2$$

<u>Thm</u> (Oort): If $\xi_1 \prec \xi_2$, then $W_{\xi_1}^\circ \subseteq \overline{W_{\xi_2}^\circ} =: W_{\xi_2}$

The dimension of W_ξ° is described by purity:
(de Jong & Oort).

$g=3$

	ξ	$\dim W_\xi^\circ$	$\dim \mathcal{C}(x)$, $\mathcal{C}(x) \subseteq W_\xi^\circ$
ordinary	$0\,0\,0\,1\,1\,1$	6	6
	$\succ$		
	$0\,0\,\frac{1}{2}\,\frac{1}{2}\,1\,1$	5	5
	$\succ$		
	$0\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,1$	4	3
	$\succ$		
	$\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\,\frac{2}{3}\,\frac{2}{3}\,\frac{2}{3}$	3	2
	$\succ$		
super-singular	$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}$	2	0

Leaves

Fix $x = (A_x, \lambda_x, \eta_x) \in \mathcal{A}_g(k)$, $k = \bar{k}$, char $k = p$

$$\mathcal{C}(x) := \left\{ y \in \mathcal{A}_g \otimes k \;\middle|\; \begin{array}{l} \exists\, \Omega / k(y),\ \bar{\Omega} = \Omega \text{ s.t.} \\ (A_y, \lambda_y)[p^\infty] \otimes_{k(y)} \Omega \simeq (A_x, \lambda_x)[p^\infty] \otimes_k \Omega. \end{array} \right\}$$

called the leaf passing through x.

Prop. (Zink)

$\mathcal{C}(x) \subseteq \mathcal{A}_g \otimes k$ is a construcible subset.

(= a finite disjoint union of locally closed subsets)

Using the key lemma:

$\forall\, h \geq 1$, $\exists\, N(h) \in \mathbb{N}$, s.t

$\forall\, n \geq N(h)$, $\forall\, X_1, X_2$ p-divisible groups of heigh h/k

with $X_1[p^n] \simeq X_2[p^n]$,

one has $X_1 \simeq X_2$

$\xi = NP(x)$, the Newton polygon of x

W^0_ξ : the NP stratum

<u>Theorem</u> (Oort)

① $\mathcal{C}(x)$ is a reduced locally closed subscheme of $\mathcal{A}_g \otimes k$. Moreover, $\mathcal{C}(x) \subset W^0_\xi$ is closed.

② $\mathcal{C}(x)$ is smooth, quasi-affine, equi-dimensional, and $\dim \mathcal{C}(x)$ only depends on ξ.

(i.e. $x, y \in W^0_\xi$, then $\dim \mathcal{C}(x) = \dim \mathcal{C}(y)$)

①: Oort shows the second statement based on Zink's proposition and the slope filtration (Oort-Zink), and then concludes the first statement.

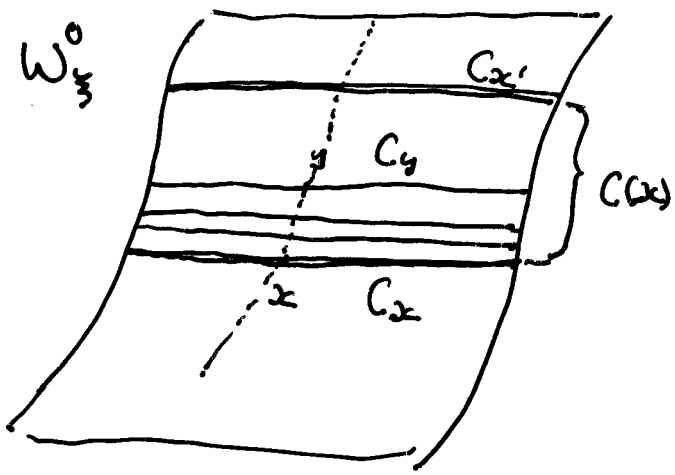
②. $\forall x_1, x_2 \in \mathcal{C}(x)$, we have $\mathcal{C}(x)^\wedge_{x_1} \simeq \mathcal{C}(x)^\wedge_{x_2}$ (Serre-Tate)

. quasi-affineness follows quasi-affineness of EO strata.

. the last statement follows from a result on isogeny correspondences.

Why is $\mathcal{C}(x)$ called a "leaf" ?

Foliation structure on W_ξ^0 (Oort) :



C_x : irreducible component of $\mathcal{C}(x)$ containg x.

$$W_\xi^0 = \coprod_{\substack{x \text{ moves} \\ \text{in an "alg." family.}}} \mathcal{C}_x$$: form a foliation structure

and $\{\mathcal{C}_x\}$ are leaves in this foliation.

Later : Chai-Oort : $\mathcal{C}(x)$ is irreducible is x is not supersingular.

<u>Theorem</u> (Oort) (almost product structure)

$\exists\ T \times J \xrightarrow{\Psi} W_\xi^0$ finite surjective morphism

s.t $\Psi(T \times \{j\}) = \mathcal{C}_z$ for some z.

Leaves on $\mathcal{M}_{\tilde{D}}$

$\tilde{\mathcal{D}} = (B, *, V, \Psi, \mathcal{O}_B, V_{\mathbb{Z}})$ p-integral PEL datum

$g = \frac{1}{2}\dim_{\mathbb{Q}} V$, $N \geq 3$, $(p,N)=1$, $\zeta_N \subseteq \overline{\mathbb{Q}}$. $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$

$\mathcal{M}_{\tilde{D}}$: the moduli space over $\overline{\mathbb{F}}_p$ of g-dim'l

$\ni$

$(A, \lambda, \iota, \bar{\eta})$ p-principally polarized abelian $\mathcal{O}_B$-varieties

with a level-N structure w.r.t. ζ_N.

Fix

$x = (A_x, \lambda_x, \iota_x, \bar{\eta}_x) \in \mathcal{M}_{\tilde{D}}(k)$, $k = \bar{k}$.

$$\mathcal{C}_{\tilde{D}}(x) := \left\{ y \in \mathcal{M}_{\tilde{D}} \otimes k \;\middle|\; \begin{array}{l} \exists\, \Omega/k(y),\ \bar{\Omega} = \Omega \text{ s.t.} \\ \underline{A}_y[p^\infty] \otimes_{k(y)} \Omega \simeq \underline{A}_x[p^\infty] \otimes_k \Omega \end{array} \right\}$$

called the <u>leaf</u> of $\mathcal{M}_{\tilde{D}}$ passing through x.

Using Oort's theorem and the fact that

the forgetful morphism $\mathcal{M}_{\tilde{D}} \longrightarrow \mathcal{A}_{g,d}$

is finite and unramified (locally injective),

we deduce

<u>Theorem</u> $\xi = NP(x)$ the Newton vector, $W^0_{\tilde{D},\xi}$ the (Kottwitz) NP stratum.

① $\mathcal{C}_{\tilde{D}}(x)$ is a reduced locally closed subscheme of $\mathcal{M}_{\tilde{D}}$. Moreover, $\mathcal{C}_{\tilde{D}}(x) \subset W^0_{\tilde{D},\xi}$ is closed.

② $\mathcal{C}_{\tilde{D}}(x)$ is smooth, quasi-affine, equi-dim'l and $\dim \mathcal{C}_{\tilde{D}}(x)$ depends only on ξ.

<u>Remark</u>: (i) This is proved by E. Mantovan in the smooth case. She also proves the almost product structure for the NP strata.

(ii) The "formal isogeny" strata and the NP strata are different for general $\tilde{D}$.

Ξ = the set of isogeny classes of $\underline{M}$ arising from points in $\mathcal{M}_{\tilde{D}}$.

$\gamma = [x]$, $\mathcal{V}^0_\gamma := \{ y \in \mathcal{M}_{\tilde{D}} \mid [y] = \gamma \}$

Then $\mathcal{V}^0_{\gamma,\tilde{D}} \subset W^0_{\xi,\tilde{D}}$ open and closed.

$\Rightarrow$ $\mathcal{C}_{\tilde{D}}(x) \subseteq W^0_{\tilde{D},\xi}$ closed $\Leftrightarrow$ $\mathcal{C}_{\tilde{D}}(x) \subseteq \mathcal{V}^0_{\tilde{D},\gamma}$ closed.

(iii) We start with $\tilde{\mathfrak{D}}$, define the moduli space $\mathcal{M}_{\tilde{\mathfrak{D}}}$, and then define the subvariety $\mathcal{C}_{\tilde{\mathfrak{D}}}(x)$. However we can define $\mathcal{C}_{\tilde{\mathfrak{D}}}(x)$ directly.

Fix $x_0 = (A_0, \lambda_0, \iota_0)$ g-dim'l polarized abelian $\mathcal{O}_B$-variety $/k$.

$\rightsquigarrow M_0 = M(A_0)$, $\quad T_0^{(p)} = \prod_{\ell \neq p} T_\ell(\underline{A}_0)$

$N \geq 3$, $(p, N) = 1$.

$\underline{A} = (A, \lambda, \iota)$: g-dim'l polarized abelian $\mathcal{O}_B$-variety $/k$

An <u>level-N on $\underline{A}$ w.r.t. $T_0^{(p)}$</u> is a U_N-orbit of isomorphism $\eta : T_0^{(p)} \simeq T^{(p)}(\underline{A})$ compatible with $\mathcal{O}_B$-action and

$$\begin{tikzcd} T_0^{(p)} \times T_0^{(p)} \arrow[r, "(\eta,\eta)"] \arrow[d, "e_{\lambda_0}"] \arrow[dr, phantom, "\square"] & T^{(p)}(\underline{A}) \times T^{(p)}(\underline{A}) \arrow[d, "e_\lambda"] \\ \hat{\mathbb{Z}}^{(p)}(1) \arrow[r, equal] & \hat{\mathbb{Z}}^{(p)}(1) \end{tikzcd}$$

where $U_N = \ker\left(\mathrm{Aut}(T_0^{(p)}) \longrightarrow \mathrm{Aut}(T_0^{(p)}/N T_0^{(p)})\right)$.

$x_0 = (A_0, \lambda_0, \iota_0, \overline{\mathrm{id}})$

Then we have objects $\underline{A} = (A, \lambda, \iota, \bar{\eta})_{/S}$, S : k-scheme.

S : k-scheme.

$F_{x_0}(S)$: the set of isom classes of $(A, \lambda, \iota, \bar{\eta})/_S$

s.t. $(A_s, \lambda_s, \iota_s)[p^\infty] \simeq (A_0, \lambda_0, \iota_0)[p^\infty]$ $\forall s \in S(k)$.

Then $\exists$ reduced scheme $\mathcal{C}(x_0)$ of finite type,

unique up to a unique isom, and an object

$\underline{\mathcal{X}} \to \mathcal{C}(x_0)$ such that

$\forall$ <u>reduced</u> k-scheme S, and any object $\underline{A}_s \in F_{x_0}(s)$

$\exists!\ \alpha : S \to \mathcal{C}(x_0)$ s.t $\underline{A}_s \xrightarrow{\sim} \alpha^* \underline{\mathcal{X}}$.

Note : We don't impose the determinant condition

in the definition of the functor F_{x_0}.

Reason : If $\underline{M}_y \simeq \underline{M}_0$

then $\det(\underline{X}; \underset{\substack{\| \\ M_y/vM_y}}{\mathrm{Lie}(A_y)}) = \det(\underline{X}; \underset{\substack{\| \\ M_0/vM_0}}{\mathrm{Lie}(A_0)})$

Dimension formula for leaves:

Siegel case : $\mathcal{M}_{\tilde{B}} = \mathcal{A}_g$.

* The first formula is given by Oort. It is expressed in terms of combinatorics on the NP.
* We discuss Chai's dimension formula

Suppose

$$\xi = NP(x) : \quad \mu_1 > \mu_2 > \cdots > \mu_m \ , \quad \mu_i + \mu_{m+1-i} = 1$$

with multiplicity: $h_1 \quad h_2 \qquad h_m$

$$\left(\sum h_i = 2g, \quad \sum \mu_i h_i = g \right)$$

$$\boxed{\begin{aligned} \dim \mathcal{C}(x) = & \frac{1}{2} \sum_{\substack{i<j \\ \mu_i + \mu_j \neq 1}} (\mu_i - \mu_j) h_i h_j \\ & + \frac{1}{2} \sum_{i \leq \frac{m}{2}} (2\mu_i - 1) h_i (h_i + 1) \end{aligned}}$$

$$(2\mu_i - 1) = (\mu_i - \mu_j)$$

with $\mu_i + \mu_j = 1$

Examples ① One slope μ_1

$$\dim \mathcal{C}(x) = \boxed{0}$$

② Two slopes : $\mu_1 > \mu_2$ multip. h

$$\dim \mathcal{C}(x) = \frac{1}{2}(2\mu_1 - 1)\, h(h+1)$$

000111 $\quad \mu_1 = 1 > \mu_2 = 0$, $h = 3$

$$\dim = \frac{1}{2}\cdot(2-1)\cdot 3\cdot 4 = \boxed{6}$$

$\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}$, $\quad \mu_1 = \frac{2}{3}$, $\mu_2 = \frac{1}{3}$, $h = 3$

$$\dim = \frac{1}{2}\cdot\left(\frac{4}{3} - 1\right)\cdot 3\cdot 4 = \boxed{2}$$

③ Three slopes : $\mu_1 > \mu_2 > \mu_3$, $h_1 \quad h_2 \quad h_1$

$(1,2) \qquad (2,3) \qquad (1,3)$

$$\frac{1}{2}(\mu_1 - \mu_2)\, h_1 h_2 + \frac{1}{2}(\mu_2 - \mu_3)\, h_1 \cdot h_2 + \frac{1}{2}(2\mu_1 - 1)\, h_1(h_1+1).$$

$00\frac{1}{2}\frac{1}{2}11$ $\quad 1 > \frac{1}{2} > 0$, $2, 2, 2$

$$\frac{1}{2}\cdot\frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot\frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 1\cdot 2\cdot 3 = \boxed{5}$$

$0\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}1$ $\quad 1 > \frac{1}{2} > 0$, $1, 4, 1$

$$\frac{1}{2}\cdot\frac{1}{2}\cdot 1\cdot 4 + \frac{1}{2}\cdot\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 1\cdot 1\cdot 2 = \boxed{3}$$

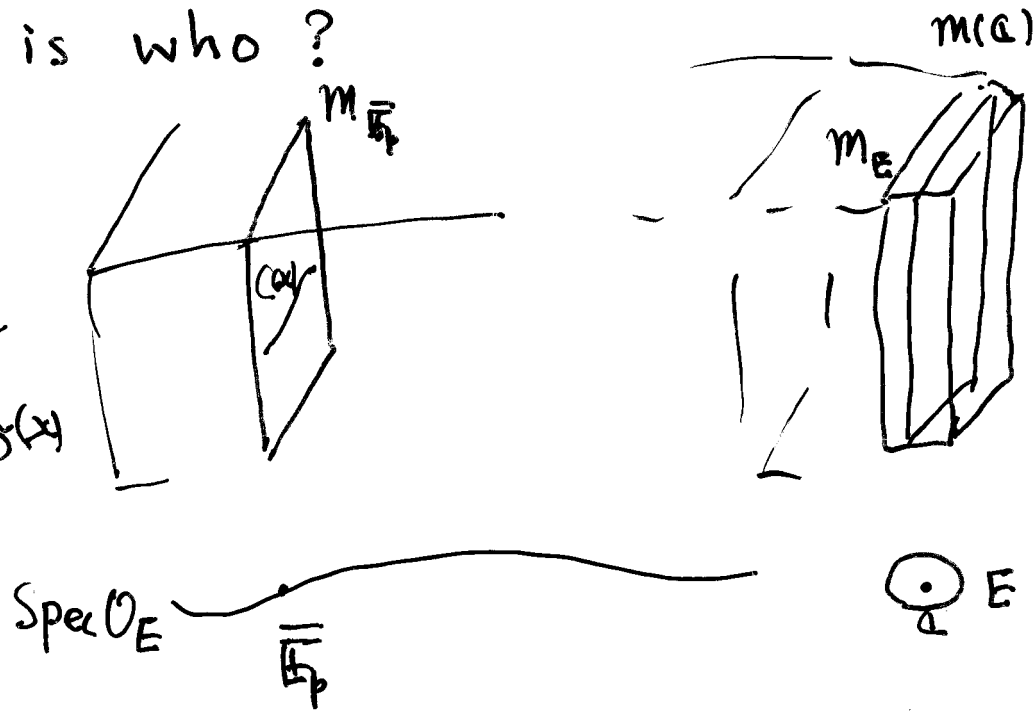
Shimura varieties and leaves

Who is who ?

(I)

$\mathcal{M} = \mathcal{M}_{\tilde{D}}$

$\mathcal{C}(x) = \mathcal{C}_{\tilde{D}}(x)$



Chai : $\mathcal{C}(x)$ is a "Shimura variety in char p".

- It has lots of symmetry, Hecke correspondences from $G(\mathbb{A}_f^{(p)})$.
- It's "homogeneous" locally. $\mathcal{C}(x)^{\wedge}_{x_1} \simeq \mathcal{C}(x)^{\wedge}_{x_2}$ look like ↓ the same $x_1, x_2 \in \mathcal{C}(x)$
- It has nice group-theoretic properties.

($\mathcal{C}(x)^{\wedge}_{x_1}$ is built up from formal p-divisible groups.
Chai's theory of canonical coordinates
= the generalization of Serre-Tate coordinates)

Behavior of $\mathcal{C}(x)$ = Behavior of Shimura varieties in char 0.

(II) Shimura varieties are "leaves".

Using the same definition of leaves, but not requiring the field k to be of char p....

$x_0 = (A_0, \lambda_0, L_0)$: g-dim'l polarized abelian $\mathcal{O}_B$-variety $/k$, $k = \bar{k}$ (for simplicity)

$\mathcal{C}(x_0)$: the moduli space over k of g-dim'l polarized abelian $\mathcal{O}_B$-varieties $\underline{A}$ with a level-N structure.

satisfies : (a) $\underline{A}[p^\infty] \simeq \underline{A}_0[p^\infty]$ if char $k = p$

(b) $\det(\underline{X}; \mathrm{Lie}(A)) = \det(\underline{X}; \mathrm{Lie}(A_0))$.

(i) char $k = p > 0$. the determinant condition is satisfied automorphically. This follows from (a).

We get the leaf $\mathcal{C}(x_0)$ defined as before

(ii) $k = \mathbb{C}$, $x_0 \rightsquigarrow \underset{V_{\mathbb{Z}}}{\underset{=}{H_1(A_0, \mathbb{Z})}}$, $V = H_1(A_0, \mathbb{Q})$. $\langle , \rangle = \psi$

$\tilde{D} = (B, *, V, \psi, \mathcal{O}_B, V_{\mathbb{Z}})$

given at the beginning

supplied by $\underline{A}_0$

Then $\mathcal{C}(x_0) = \mathcal{M}_{\tilde{D}}(\mathbb{C})$

What is happening?

Why "leaves" = "Shimura Varieties"

Serre - Tate: (char p)

$$\begin{pmatrix}\text{deforming an abelian} \\ \text{variety } /k,\ \mathrm{ch}\, k = p\end{pmatrix} \simeq \begin{pmatrix}\text{deforming its} \\ p\text{-divisible group} \\ \text{(preserving a fixed pol.)}\end{pmatrix}$$

$G \leftrightarrow M, \tau$

In other words:

$$\begin{pmatrix}\text{moving algebraic} \\ \text{structures}\end{pmatrix} \simeq \begin{pmatrix}\text{moving the} \\ \text{Frobenius structures}\end{pmatrix}$$

GAGA (char 0)

$$\begin{pmatrix}\text{moving algebraic} \\ \text{structures through an} \\ \text{abelian variety } / \mathbb{C}\end{pmatrix} \simeq \begin{pmatrix}\text{moving complex} \\ \text{structures through} \\ \text{an abelian variety} \\ \text{(preserving a fixed pol.)}\end{pmatrix}$$

$$\simeq \begin{pmatrix}\text{moving Hodge structures} \\ \text{through an abel variety} \\ \text{(preserving a fixed pol.)}\end{pmatrix}$$

Note: The set of <u>isom. classes</u> of Hodge structures is discrete.

Ex: $\{\text{lines in } \mathbb{C}^n\} \simeq \mathbb{P}^{n-1}$, but <u>only one</u> isom. class.

Unlike in the char zero,

The set of isom. classes of Frobenius structures is NOT discrete.

<u>Manin (1963)</u> The set of isom. classes of Dieudonné modules $/k=\bar{k}$ with height h is a construcible set.

So: $\begin{pmatrix}\text{moving Frob.}\\ \text{structures}\end{pmatrix} \approx \overset{A}{\begin{pmatrix}\text{moving Frob.}\\ \text{structures within}\\ \text{an isom. class}\end{pmatrix}} \times \overset{B}{\begin{pmatrix}\text{moving Frob.}\\ \text{structures so}\\ \text{that ONLY change}\\ \text{isom classes}\end{pmatrix}}$

Only means: the map $B \to \begin{pmatrix}\text{isom. classes}\\ \text{of Frob. structures}\end{pmatrix}$ is <u>injective</u>

Oda-Oort (1977):

$$\begin{pmatrix}\text{moving Frob. str.}\\ \text{along an isogeny}\\ \text{family}\end{pmatrix} \hookrightarrow \begin{pmatrix}\text{isom. classes}\\ \text{of Frob. structures}\end{pmatrix}$$

Maybe an maximal isogeny family exhausts B?

Rapoport - Zink study (1994)

$$\left\{ (G, \varphi) \;\middle|\; \begin{array}{c} \varphi : G_0 \to G \\ \text{quasi-isogeny} \end{array} \right\} \quad \text{(Rapoport - Zink space)}$$

They call it a <u>period domain for p-divisible groups</u>

Zink, later Oort :

$\begin{pmatrix} \text{Frob structures} \\ \text{within an} \\ \text{isom class} \end{pmatrix}$ is <u>locally closed</u> !

(now called a leaf)

So

$$\begin{pmatrix} \text{moving Frob.} \\ \text{structures along} \\ \text{an isogeny} \\ \text{class} \end{pmatrix} \approx \begin{pmatrix} \text{moving Frob.} \\ \text{structures along} \\ \text{a leaf} \end{pmatrix} \times \begin{pmatrix} \text{moving Frob.} \\ \text{structures along} \\ \text{a maximal isogeny} \\ \text{family} \end{pmatrix}$$

Oort (almost product structure) :

We have $\underset{\uparrow}{=}$!

This is a bit too good to have.
In general one should expect a
fibration structure.

Back to leaves

Our "leaf" in char p (= the original leaf $\mathcal{C}(x)$) parametrizes (at least locally)

Frob. structures within an isom. class on a fixed lattice.

Now in char 0:

Since the determinant condition determines isom. classes of Hodge structures,

Our "leaf" over $\mathbb{C}$ (= the original moduli space $\mathcal{M}_{\mathcal{D}}(\mathbb{C}$ parametrizes

Hodge structures within an isom. class on a fixed lattice.

Note: we already have a dimension formula for "leaves" $/\mathbb{C}$. The same formula should compute the dimensions of "leaves" $/\overline{\mathbb{F}}_p$.

Dimension formula for $\mathcal{C}(x)$

$x_0 = \underline{A}_0 = (A_0, \lambda_0, \iota_0)/k$, $k = \bar{k}$, char $k = p > 0$.

$W(k)$ = the ring of Witt vectors $/k$.

$L = \mathrm{Frac}\, W(k)$.

$\underline{M}_0 = (M_0, \overset{\psi_0}{\overset{\shortparallel}{\langle\ \rangle_0}}, \iota_0)$: covariant Dieudonné module of $\underline{A}_0$

$N_0 = M_0 \otimes_W L$

$N_0 = N_{\mu_1} \oplus N_{\mu_2} \oplus \cdots \oplus N_{\mu_m}$: isotypic component.

$\mu_1 > \mu_2 \qquad > \mu_m$

t acts by $t^{\mu_1} \quad t^{\mu_2} \qquad t^{\mu_m}$

$\nu_{x_0} : \mathbb{D} \longrightarrow T \subset G(L) = \mathrm{Aut}_B(N_0, L^\times \cdot \psi_0) \subseteq GL(N_0)$

$X^*(\mathbb{D}) = \mathbb{Q}$, $\nu_{x_0} \in X_*(T)_{\mathbb{Q}}$: <u>slope homomorphism</u>

(The slope homo. is constructed by Kottwitz using the Tannakian theory. for general G

For PEL-type cases, we don't need the Tannakian theory.

$$N_0 = N_{\mu_1} \oplus N_{\mu_2} \oplus \cdots \oplus N_{\mu_m}$$

$$\forall r \in \mathbb{Q}, \quad F^r = \bigoplus_{\mu_i \geq r} N_{\mu_i} :$$

$$F^\bullet : F^{\mu_m} \supsetneq F^{\mu_{m-1}} \supsetneq \cdots \supsetneq F^{\mu_1} = N_{\mu_1} \supsetneq 0$$

<u>slope filtration</u>.

$P_- \subset G_{\mathbb{Z}}$ the stabilizer of the slope filtration $F^\bullet$.

$$\mathfrak{g}_{\mathbb{Z}} = \mathrm{Lie}(G_{\mathbb{Z}}) = \underbrace{\mathfrak{m} \oplus \mathfrak{u}_-}_{\mathrm{Lie}(P_-)} \oplus \mathfrak{u}_+ \qquad P_- = MU_-$$

Theorem (Chai.–)

$$\boxed{\dim C(x_0) = \sum_{\alpha \in R(\mathfrak{u}_+)} \langle \nu_{x_0}, \alpha \rangle}$$

<u>Example</u>: $G_{\mathbb{T}} = G_{\mathbb{T}} Sp_{2g}$, three slopes.

$$\mu_1 > \mu_2 > \mu_3$$

$$h_1 \qquad \underset{2h}{\overset{\shortparallel}{h_2}} \qquad h_1$$

$$J_g = \left(\begin{array}{c|c} & \tilde{I}_g \\ \hline -\tilde{I}_g & \end{array}\right), \quad \tilde{I}_g = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

$$\begin{array}{c} \\ h_1 \\ h_2 \\ h_1 \end{array} \begin{array}{c} \begin{array}{ccc} h_1 & h_2 & h_1 \end{array} \\ \left(\begin{array}{c|c|c} \mu_1 & & \\ \hline & \mu_2 & \\ \hline & & \mu_3 \end{array}\right) \end{array}, \quad P_- = \left\{ \begin{pmatrix} A_{11} & 0 & 0 \\ * & A_{22} & 0 \\ * & * & A_{33} \end{pmatrix} \right\}, \quad J_g = \begin{pmatrix} & & \tilde{I}_{h_1} \\ & J_h & \\ -\tilde{I}_{h_1} & & \end{pmatrix}$$

$$\mathcal{U}_+ = \left\{ \begin{pmatrix} 0 & A & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} \tilde{I}_{h_1} B^t = B \tilde{I}_{h_1} \\ J_h A^t = C \cdot \tilde{I}_{h_1} \end{array} \right\} = \underbrace{\left\{ \begin{pmatrix} & A & \\ & & C \\ & & \end{pmatrix} \right\}}_{\mathcal{U}_{+,1}} + \underbrace{\left\{ \begin{pmatrix} & B & \\ & & \\ & & \end{pmatrix} \right\}}_{\mathcal{U}_{+,2}}$$

$$\sum_{\alpha \in R(\mathcal{U}_+)} \langle \nu_{x_0}, \alpha \rangle = \sum_{\alpha \in R(\mathcal{U}_{+,1})} \langle \nu_{x_0}, \alpha \rangle + \sum_{\alpha \in R(\mathcal{U}_{+,2})} \langle \nu_{x_0}, \alpha \rangle$$

$$= (\mu_1 - \mu_2) h_1 h_2 + (\mu_1 - \mu_3) \frac{h_1(h_1+1)}{2}$$

$$= \frac{1}{2}(\mu_1 - \mu_2) h_1 h_2 + \frac{1}{2}(\mu_2 - \mu_3) h_1 h_2$$

$$+ \frac{1}{2}(2\mu_1 - 1) h_1 (h_1 + 1)$$