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Soliton solutions of the 3D Gross-Pitaevskii equation by the potential control method.

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This work has been carried out in collaboration with

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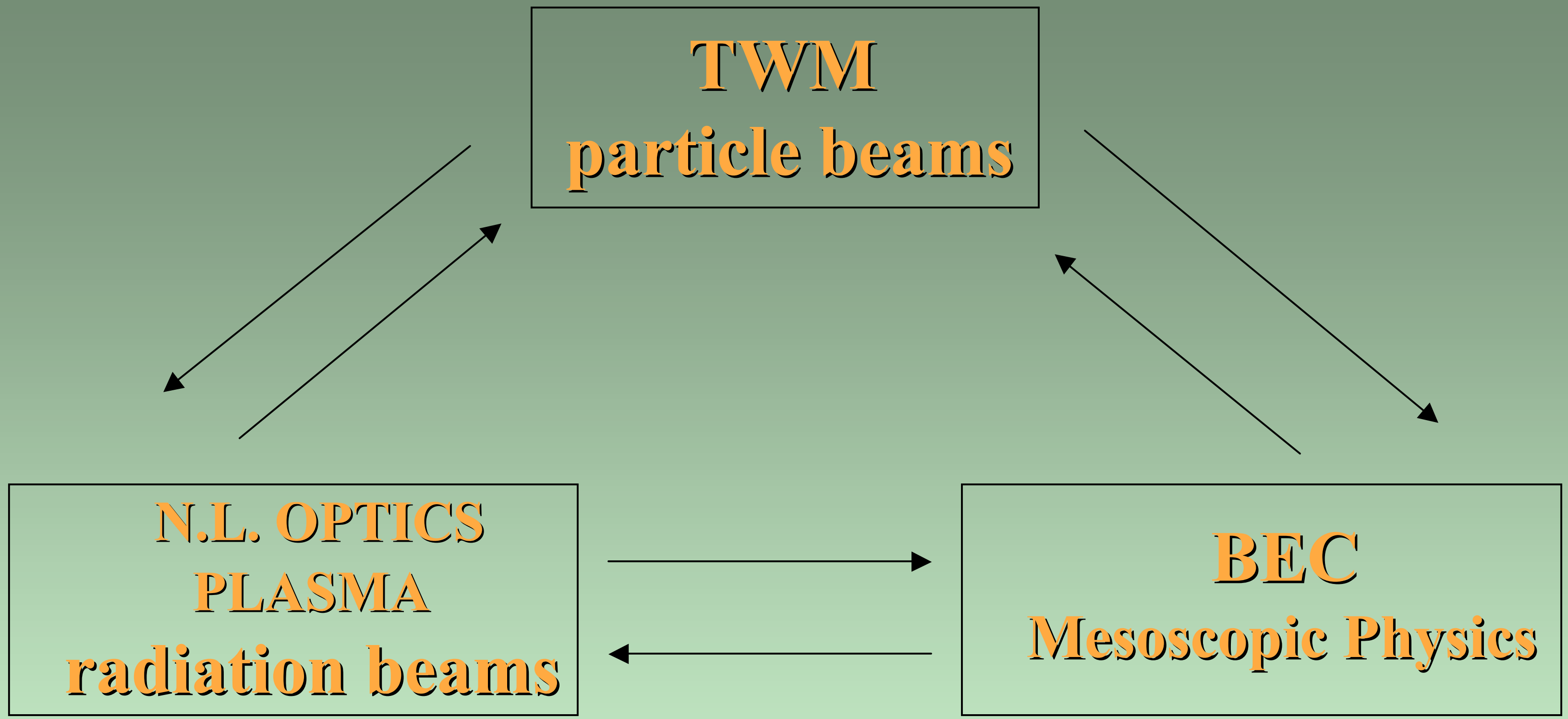

- Different systems described by the same formalism
- Transfer of know how from one discipline to another one
- Alternative “keys of reading” for each discipline
- New insights and possibility of new predictions

The dynamics of BECs in a spatially nonuniform confining potential well is governed by the GPE

E. P. Gross, Nuovo Cimento 20, 454 (1961)

L. P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961)

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_a}\nabla^2\Psi + U\left[\mathrm{r}, t, \left|\Psi(\mathrm{r}, t)\right|^2\right]\Psi$$

$$U(\mathbf{r},\mathrm{t}, |\Psi\,(\mathbf{r},\,\mathrm{t}))|^2) = \mathrm{U}_{ext}\,(\mathbf{r},\mathrm{t}) + \mathrm{g}_0\mathrm{N}\,|\Psi\,(\mathbf{r},\,\mathrm{t}))|^2$$

$\Psi\,(\mathbf{r}, t)$ = macroscopic wave function of the condensate

m_a = atomic mass,

$U_{ext}\,(\mathbf{r},t)$ = external confining potential for BECs

$g_0 = 4\pi\,\nabla^2\, a/\, m_a$ = coupling constant (related to the short range scattering (*s-wave*))

a = length representing the interactions between atomic particles

N = number of atoms

$$V \equiv \frac{U}{m_a c^2} \qquad s \equiv ct \qquad \lambda_c \equiv \frac{h}{m_a c}$$

$$V_{ext} \equiv \frac{U_{ext}}{m_a c^2} \qquad q \equiv \frac{g_0 N}{m_a c^2}$$

$$i\lambda_c \frac{\partial \Psi}{\partial s} = -\frac{\lambda_c^2}{2}\nabla^2\Psi + V\left[\mathbf{r}, s, \left|\Psi(\mathbf{r}, s)\right|^2\right]\Psi$$

$$V[\mathbf{r},\mathrm{s}, |\Psi(\mathbf{r}, \mathrm{s}))|^2] = V_{ext}(\mathbf{r},\mathrm{s}) + \mathrm{q}\,|\Psi(\mathbf{r}, \mathrm{s}))|^2$$

Assumptions for V_{ext} (r,s)

We assume that the $V_{ext}(\mathbf{r},s)$ is the sum of quadratic terms:

$$V_{ext}(\mathbf{r},s) = V_{\perp}(\mathbf{r}_{\perp},s) + V_{z}(z,s)$$

$$\mathbf{r} = (x,y,z) = (\mathbf{r}_{\perp}, z)$$

$$\begin{cases} V_{\perp} \equiv \dfrac{1}{2}K_x(s)x^2 + \dfrac{1}{2}K_y(s)y^2 \\ V_z \equiv \dfrac{1}{2}K_z(s)z^2 \end{cases}$$

THE "STANDARD" GPE

$$i\lambda_c \frac{\partial \Psi}{\partial s} = -\frac{\lambda_c^2}{2}\nabla^2\Psi + \frac{1}{2}\left[K_x x^2 + K_y y^2 + K_z z^2\right]\Psi + q|\Psi|^2\Psi$$

- **Difficulties to find soliton solutions in one or more dimensions**
- **Several kind of solitons have been found in some approximations**

In order to get exact soliton structures a sort of "control of the system" seems to be necessary

“CONTROLLED” 3D GPE

A correct analysis of the system should include the *control potential term* in the GPE

$$\underbrace{i\lambda_c \frac{\partial \Psi}{\partial s} = -\frac{\lambda_c^2}{2}\nabla^2\Psi + \frac{1}{2}\left[K_x x^2 + K_y y^2 + K_z z^2\right]\Psi + q|\Psi|^2\Psi}_{\text{standard GPE}} + \underbrace{V_{contr}(x,y,z,s)}_{\text{control potential}}\Psi$$

$$V_{contr} = V_{contr}[|\Psi(\mathbf{r}_\perp, z, s)|^2, s]$$

The control potential has to be determined dynamically by the system itself

JETP Letters, Vol. 80, No. 8, 2004, pp. 535–539. From Pis'ma v Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 80, No. 8, 2004, pp. 609–613.

Controlling Potential Traps for Filtering Solitons in Bose–Einstein Condensates¶

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We present a controlling potential method for solving the three-dimensional Gross–Pitaevskii equation (GPE), which governs the nonlinear dynamics of the Bose–Einstein condensates (BECs) in an inhomogeneous potential trap. Our method allows one to construct ground and excited matter wave states whose longitudinal profiles can have bright solitons. This method provides the confining potential that filters and controls localized BECs. Moreover, it is predicted that, while the BEC longitudinal soliton profile is controlled and kept unchanged, the transverse profile may exhibit oscillatory breathers (the unmatched case) or move as a rigid body in the form of either coherent states (performing the Lissajous figures) or a Schrödinger cat state (matched case). © *2004 MAIK "Nauka/Interperiodica".*

PACS numbers: 03.65.Ge; 03.75.Lm; 05.30.Jp; 05.45.Yv

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THE EUROPEAN
PHYSICAL JOURNAL B

1D stability analysis of filtering and controlling the solitons in Bose-Einstein condensates

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Abstract. We present one-dimensional (1D) stability analysis of a recently proposed method to filter and control localized states of the Bose–Einstein condensate (BEC), based on novel trapping techniques that allow one to conceive methods to select a particular BEC shape by controlling and manipulating the external potential well in the three-dimensional (3D) Gross–Pitaevskii equation (GPE). Within the framework of this method, under suitable conditions, the GPE can be exactly decomposed into a pair of coupled equations: a transverse two-dimensional (2D) linear Schrödinger equation and a one-dimensional (1D) longitudinal nonlinear Schrödinger equation (NLSE) with, in a general case, a time-dependent nonlinear coupling coefficient. We review the general idea how to filter and control localized solutions of the GPE. Then, the 1D longitudinal NLSE is numerically solved with suitable non-ideal controlling potentials that differ from the ideal one so as to introduce relatively small errors in the designed spatial profile. It is shown that a BEC with an asymmetric initial position in the confining potential exhibits breather-like oscillations in the longitudinal direction but, nevertheless, the BEC state remains confined within the potential well for a long time. In particular, while the condensate remains essentially stable, preserving its longitudinal soliton-like shape, only a small part is lost into "radiation".

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations – 05.45.Yv Solitons – 05.30.Jp Boson systems – 03.65.Ge Solutions of wave equations: bound states

We look for a normalized localized solution of the controlled 3D GPE in the form:

$$\Psi(\mathbf{r},s) = \Psi_\perp(\mathbf{r}_\perp,s)\,\Psi_z\,(z,s)$$

under the *controlling condition*

$$\boxed{\int V_{contr} \left|\Psi_\perp\right|^2 d^2 r_\perp = 0}$$

$$\int \left|\Psi(\mathbf{r}_\perp, z, s)\right|^2 d^3 r = 1$$

$$\int \left|\Psi_\perp(\mathbf{r}_\perp, s)\right|^2 d^2 r_\perp = 1 \quad \Longrightarrow \quad \int \left|\Psi_z(z, s)\right|^2 d^2 z = 1$$

$$\Psi_z\left[i\lambda_c\frac{\partial\Psi_\perp}{\partial s}+\frac{\lambda_c^2}{2}\nabla_\perp{}^2\Psi_\perp-\frac{1}{2}\left[K_x x^2+K_y y^2\right]\Psi_\perp\right]=$$

$$=-\Psi_\perp\left[i\lambda_c\frac{\partial\Psi_z}{\partial s}+\frac{\lambda_c^2}{2}\frac{\partial^2\Psi_z}{\partial z^2}-\frac{1}{2}K_z z^2\Psi_z-q\left|\Psi_\perp\right|^2\left|\Psi_z\right|^2\Psi_z-V_{contr}\Psi_z\right]$$

We assume that:

$$i\lambda_c\frac{\partial\Psi_\perp}{\partial s}+\frac{\lambda_c^2}{2}\nabla_\perp{}^2\Psi_\perp-\frac{1}{2}\left[K_x x^2+K_y y^2\right]\Psi_\perp=0$$

$$K_x>0,\quad K_y>0$$

$$\Rightarrow \quad \Psi_\perp=\Psi_\perp(\mathbf{r}_\perp,s)$$

Hermite-Gauss modes

The *controlled longitudinal GPE:*

$$i\lambda_c \frac{\partial \Psi_z}{\partial s} + \frac{\lambda_c^2}{2}\frac{\partial^2 \Psi_z}{\partial z^2} - \frac{1}{2}K_z z^2 \Psi_z - q|\Psi_\perp|^2 |\Psi_z|^2 \Psi_z - V_{contr}\Psi_z = 0$$

After multiplying by $\Psi_\perp \Psi_\perp{}^*$, integrating over all the transverse space and using the controlling condition,we get the *controlled 1D GPE*:

$$\boxed{i\lambda_c \frac{\partial \Psi_z}{\partial s} + \frac{\lambda_c^2}{2}\frac{\partial^2 \Psi_z}{\partial z^2} - \frac{1}{2}K_z z^2 \Psi_z - q_{1D}(s)|\Psi_z|^2 \Psi_z = 0}$$

$$q_{1D}(s) \equiv q\int |\Psi_\perp(r_\perp, s)|^4 \, d^2 r_\perp$$

q_{1D} depends on the shape of the transverse density profile

Note that:

•A soliton solution of the controlled 1D GPE is a one-dimensional solution of the controlled longitudinal GPE corresponding to the following control potential:

$$V_{contr} = [q_{1D}(s) - q|\Psi_\perp|^2]|\Psi_z|^2$$

Such a solution is fully controlled by the transverse dynamics, namely $\Psi_\perp$

•Once the transverse and the longidinal solutions are found, namely $\Psi_\perp(r_\perp,s)$ and $\Psi_z(z,s)$, the explicit dependence in *space* and *time* of the control potential, $V_{contr} = V_{contr}(r,s)$, that allows to keep and control the 3D solution

•For cilindrical simmetry, namely $K_x = K_y = K_f$:

$$\boxed{q_{1D}(s) \propto \frac{1}{\sigma_\perp^2(s)}}$$

$\sigma_\perp(s)$ = r.m.s. of the transverse Hermite-Gauss/Laguerre-Gauss modes

$$\frac{d^2\sigma_\perp}{ds^2} + K_\perp\sigma_\perp - \frac{\lambda_c^2}{\sigma_\perp^3} = 0$$

Pinney-Ermakov equation

SOLITON SOLUTION OF THE CONTROLLED 1D GPE

$$i\lambda_c \frac{\partial \Psi_z}{\partial s} + \frac{\lambda_c^2}{2}\frac{\partial^2 \Psi_z}{\partial z^2} - \frac{1}{2}K_z z^2 \Psi_z - q_{1D}(s)\left|\Psi_z\right|^2 \Psi_z = 0$$

Madelung's fluid representation

$$\Psi_z(z,s) \longrightarrow \quad \rho = \left|\Psi_z\right|^2, \;\; \mathrm{v} = \frac{\lambda_c}{i}\frac{\partial}{\partial z} Arg[\Psi_z]$$

$$\Psi_z(z,s) = \sqrt{\rho(z,s)}\, exp\left[\frac{i}{\lambda_c}\vartheta(z,s)\right]$$

$$\rho(z,s) = \left|\Psi_z(z,s)\right|^2, \;\; \mathrm{v}(z,s) = \frac{\partial}{\partial z}\vartheta(z,s)$$

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THE EUROPEAN PHYSICAL JOURNAL B

Solitary waves in the Madelung's fluid: Connection between the nonlinear Schrödinger equation and the Korteweg-de Vries equation

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Abstract. An investigation to deepen the connection between the family of nonlinear Schrödinger equations and the one of Korteweg-de Vries equations is carried out within the context of the Madelung's fluid picture. In particular, under suitable hypothesis for the current velocity, it is proven that the cubic nonlinear Schrödinger equation, whose solution is a complex wave function, can be put in correspondence with the standard Korteweg-de Vries equation, is such a way that the soliton solutions of the latter are the squared modulus of the envelope soliton solution of the former. Under suitable physical hypothesis for the current velocity, this correspondence allows us to find envelope soliton solutions of the cubic nonlinear Schrödinger equation, starting from the soliton solutions of the associated Korteweg-de Vries equation. In particular, in the case of constant current velocities, the solitary waves have the amplitude independent of the envelope velocity (which coincides with the constant current velocity). They are *bright* or *dark* envelope solitons and have a phase linearly depending both on space and on time coordinates. In the case of an arbitrarily large stationary-profile perturbation of the current velocity, envelope solitons are *grey* or *dark* and they relate the velocity u_0 with the amplitude; in fact, they exist for a limited range of velocities and have a phase nonlinearly depending on the combined variable $x - u_0 s$ (s being a time-like variable). This novel method in solving the nonlinear Schrödinger equation starting from the Korteweg-de Vries equation give new insights and represents an alternative *key of reading* of the dark/grey envelope solitons based on the fluid language. Moreover, a comparison between the solutions found in the present paper and the ones already known in literature is also presented.

PACS. 52.35.Mw Nonlinear phenomena: waves, wave propagation, and other interactions (including parametric effects, mode coupling, ponderomotive effects, etc.) – 05.45.Yv Solitons – 42.65.-k Nonlinear optics – 67.57.Jj Collective modes

Physica Scripta. Vol. 65, 502–508, 2002

Envelope Solitons versus Solitons

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Abstract

A theory involving a correspondence between envelope solitonlike solutions of the generalized nonlinear Schrödinger equation (GNLSE) and solitonlike solutions of the generalized Korteweg–de Vries equation (GKVdE) is developed within the context of the Madelung's fluid description (fluid counterpart description of the GNLSE). This correspondence, which, under suitable constraints, can be made invertible, seems to be very helpful for finding one family of solutions (whether envelope solitonlike solutions of the GNLSE or solitonlike solutions of the GKdVE) starting from the knowledge of the other family of solution (whether solitonlike solutions of the GKdVE or envelope solitonlike solutions of the GNLSE). The theory is successfully applied to wide classes of both modified nonlinear Schrödinger equation (MNLSE) and modified Korteweg–de Vries equation (MKVdE), for which bright and gray/dark solitonlike solutions are found. In particular, bright and gray/dark solitary waves are determined for the MNLSE with a quartic nonlinear potential in the modulus of the wavefunction (i.e. $q_1|\Psi|^2 + q_2|\Psi|^4$) as well as for the associated MKdVE. Furthermore, the well known bright and gray/dark envelope solitons of the cubic NLSE and the corresponding solitons of the associated standard KdVE are easily recovered from the present theory. Remarkably, this approach opens up the possibility to transfer all the know how concerning the instability criteria for solitonlike solutions of the MKdVE to the instability theory of envelope solitonlike solutions of the MNLSE.

where a and v are real constants[2], and $G[u]$ is a real functional of u. In both equations, x is the 1-D configurational space coordinate and s is the timelike coordinate. In particular, in this letter, special attention will be devoted to a correspondence between the special case of (1) when $U[|\Psi|^2] = q_0|\Psi|^{2\beta}$, q_0 and β being real and positive real numbers, respectively, namely

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0|\Psi|^{2\beta}\Psi = 0, \tag{3}$$

and the special case of (2) when $G[u] = p_0 u^\gamma$, p_0 and γ being real and positive real numbers, respectively, namely

$$a\frac{\partial u}{\partial s} - p_0 u^\gamma\frac{\partial u}{\partial x} + \frac{v^2}{4}\frac{\partial^3 u}{\partial x^3} = 0. \tag{4}$$

According to other authors [2–5], let us define (3) as *modified nonlinear Schrödinger equation* (MNLSE) and (4) as *modified Korteweg–de Vries equation* (MKdVE). Our goals are: (*i*) to construct a correspondence between (1) and (2) in such a way that each positive stationary-profile solution of (2),

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Solitons in the Madelung's Fluid*

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Abstract

We review the results of recent investigations dealing with a connection between the envelope soliton-like solutions of a wide family of nonlinear Schrödinger equations (NLSEs) and the soliton-like solutions of a wide family of Korteweg-de Vries equations (KdVEs). The investigation is carried out within the context of the Madelung's fluid picture, which plays the role of the fluid counterpart of the NLSE. In two different fluid motion regimes (uniform current velocity and stationary-profile current velocity variation, respectively), bright and gray/dark soliton-like solutions of both modified NLSE and modified KdVE are found. Remarkably, the present approach represents an alternative *key of reading* for the envelope soliton theory of the NLSE. In particular, the well known envelope soliton solutions and soliton solutions of the cubic NLSE and the standard KdVE, respectively, are recovered from the general approach and in terms of the fluid language presented in this paper.

of the cubic NLSE, starting from the knowledge of the corresponding bright and gray/dark solitons of the standard KdVE [6–9]. Furthermore, a more general approach which involves envelope solitons of the following generalized nonlinear Schrödinger equation (GNLSE)

$$i\alpha \frac{\partial \Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} + U\left[|\Psi|^2\right]\Psi, \tag{1}$$

($U[|\Psi|^2]$ and α being an arbitrary real functional of the complex wavefunction Ψ and an arbitrary real dispersion/diffraction coefficient, respectively) and the solitons of the following generalized Korteweg-de Vries equation (GKdVE)

$$\begin{cases} -\dfrac{\partial \mathrm{v}}{\partial s}\rho + \mathrm{v}\dfrac{\partial \rho}{\partial s} + 2\left[c_0(s) - \displaystyle\int \dfrac{\partial \mathrm{v}}{\partial s}dz\right]\dfrac{\partial \rho}{\partial z} - \left(\dfrac{\partial U_z}{\partial z}\rho + 2U_z\dfrac{\partial \rho}{\partial z}\right) + \dfrac{\lambda_c^2}{4}\dfrac{\partial^3 \rho}{\partial z^3} = 0 \\ \dfrac{\partial \rho}{\partial s} + \dfrac{\partial}{\partial z}\left(\rho \mathrm{v}\right) = 0 \end{cases}$$

$$U_z = U_z(z,s) \equiv \frac{1}{2}K_z z^2 + q_{1D}(s)\left|\Psi_z(z,s)\right|^2$$

$$c_0(s) = \textit{arbitrary function of } s$$

•**Assumption of quadratic phase:**

$$\vartheta(z,s) = \frac{1}{2} g(s) z^2 + \vartheta_0(s)$$

$$g(s) = \textit{function to be determined}$$

•**Provided that** $g(s)$ **is a solution of the following** ***Riccati's equation*****:**

$$\frac{dg}{ds} + g^2 + K_z = 0$$

the pair of fluid equations can be reduced to the following one (KdV equation + continuity equation):

$$2c_0(s)\frac{\partial \rho}{\partial z} - 3q_{1D}(s)\rho\frac{\partial \rho}{\partial z} + \frac{\lambda_c^2}{4}\frac{\partial^3 \rho}{\partial z^3} = 0$$

$$\frac{\partial \rho}{\partial s} + g(s) z \frac{\partial \rho}{\partial z} + g(s)\rho = 0$$

After some manipulations and transformations, the following soliton-like solution is obtained:

$$\rho(z,s) = a\sqrt{q_{1D}(s)}\,\mathrm{sech}^2\left(\frac{\sqrt{q_{1D}(s)}}{\delta}z\right)$$

$$\Psi_z(z,s) = \sqrt{a\sqrt{q_{1D}(s)}}\,\mathrm{sech}\left(\frac{\sqrt{q_{1D}(s)}}{\delta}z\right)\exp\left[\frac{i}{\lambda_c}\left(\frac{1}{2}g(s)z^2 + \vartheta_0\right)\right]$$

$$K_z > 0$$

with the following additional controlling conditions

$$\frac{1}{q_{1D}}\frac{dq_{1D}}{ds} = -2g(s), \qquad \frac{1}{c_0}\frac{dc_0}{ds} = -2g(s)$$

Properties of Riccati's equation:

$$\frac{dg}{ds} + g^2 + K_z = 0 \xrightarrow{g = \frac{1}{w}\frac{dw}{ds}} \frac{d^2 w}{ds^2} + K_z w = 0$$

$$q_{1D}(s) \equiv \frac{\delta_z}{w^2(s)}$$

on the other hand

$$\frac{d^2\sigma_\perp}{ds^2} + K_\perp \sigma_\perp - \frac{\lambda_c^2}{\sigma_\perp^3} = 0$$

$$q_{1D}(s) \equiv q\int |\Psi_\perp(r_\perp, s)|^4 \, d^2 r_\perp$$

$$q_{1D}(s) \equiv \frac{\delta_\perp}{\sigma_\perp^2(s)}$$

Condition of consistency:

$$\boxed{\frac{w^2(s)}{\delta_z} = \frac{\sigma_\perp{}^2(s)}{\delta_\perp}}$$

$$w^2(s) = \left(w_{01}\sin\sqrt{K_z}\,s + w_{02}\cos\sqrt{K_z}\,s\right)^2$$

$$= \frac{w_{01}{}^2 + w_{02}{}^2}{2} + \frac{w_{02}{}^2 - w_{01}{}^2}{2}\cos 2\sqrt{K_z}\,s + w_{01}w_{02}\sin 2\sqrt{K_z}\,s$$

$$\sigma_\perp{}^2(s) = \left(\frac{\gamma_0{}^2}{2K_\perp} + \frac{\lambda_c{}^2}{2K_\perp\sigma_{\perp 0}{}^2} + \frac{\sigma_{\perp 0}{}^2}{2}\right) +$$

$$\left(\frac{\sigma_{\perp 0}{}^2}{2} - \frac{\gamma_0{}^2}{2K_\perp} - \frac{\lambda_c{}^2}{2K_\perp\sigma_{\perp 0}{}^2}\right)\cos 2\sqrt{K_\perp}\,s + \frac{\gamma_0\sigma_{\perp 0}}{\sqrt{K_\perp}}\sin 2\sqrt{K_\perp}\,s$$

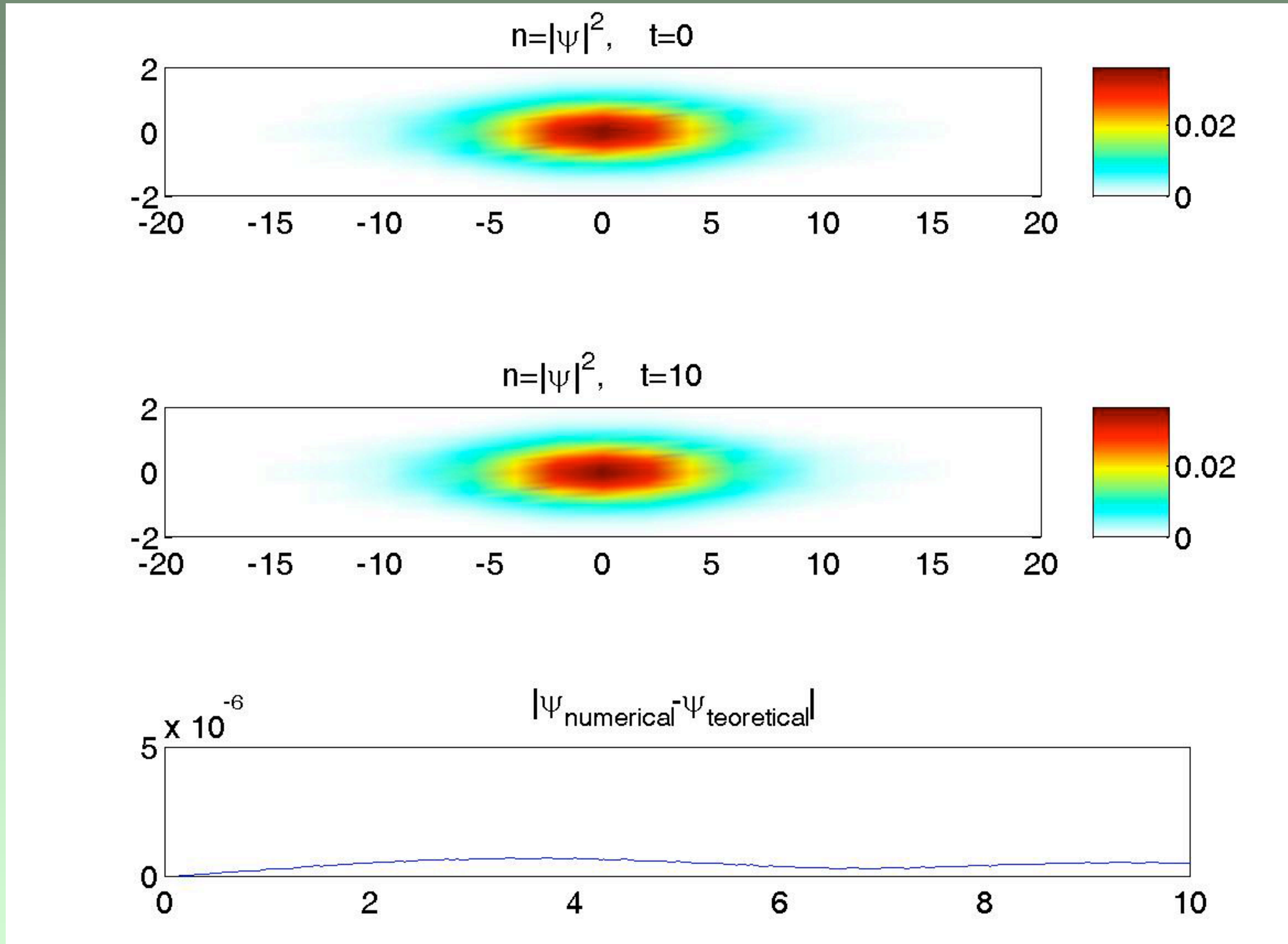
$$\sigma_{\perp 0} = \sigma_\perp(s=0), \qquad \gamma_0 = \left(\frac{d\sigma_\perp}{ds}\right)_{s=0}$$

$$K_z = K_\perp$$

$$\frac{w_{01}^{\ 2} + w_{02}^{\ 2}}{\delta_z} = \frac{1}{\delta_\perp}\left(\frac{\gamma_0^{\ 2}}{K_\perp} + \frac{\lambda_c^{\ 2}}{K_\perp \sigma_{\perp 0}^{\ 2}} + \sigma_{\perp 0}^{\ 2}\right)$$

$$\frac{w_{02}^{\ 2} - w_{01}^{\ 2}}{\delta_z} = \frac{1}{\delta_\perp}\left(\sigma_{\perp 0}^{\ 2} - \frac{\gamma_0^{\ 2}}{K_\perp} - \frac{\lambda_c^{\ 2}}{K_\perp \sigma_{\perp 0}^{\ 2}}\right)$$

$$\frac{w_{01} w_{02}}{\delta_z} = \frac{1}{\delta_\perp} \frac{\gamma_0 \sigma_{\perp 0}}{\sqrt{K_\perp}}$$



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0.02
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-2
-20 -15 -10 -5 0 5 10 15 20
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Thank you
for your attention!