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Clebsch parameterization - theory and applications

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parameterization of vectors

- “state” = “vector” (in a Hilbert space)
- “observation” = “parameterization”

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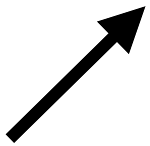

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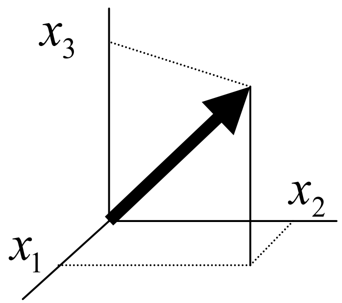


observation
(analysis)

some basic notions

- Parameterization of a vector depends on the choice of “frame”, while scalars are frame-independent.

(→ covariance arguments)

- “functional” : vector → scalar

(→ variational principles)

- Linear functional = $\langle f |$

(→ observable arguments; Riesz’ theorem)

some different parameterizations (forms) of (3D) vector fields

- Cartesian: $\boldsymbol{u} = u_x \boldsymbol{e}_x + u_y \boldsymbol{e}_y + u_z \boldsymbol{e}_z$
- Covariant form (1-form): $\boldsymbol{u} = \sum u_j \nabla \varphi^j$
- Contravariant form: $\boldsymbol{u} = \sum \nabla u_j \times \nabla \varphi^j$
- Mixed form (Grad form): $\boldsymbol{u} = \omega \nabla \varphi + \nabla \psi \times \nabla \varphi$
- Clebsch form: $\boldsymbol{u} = \nabla \varphi + \psi \nabla \xi$

frame $\rightarrow$ potential

- curl-free (irrotational) field: $\boldsymbol{E} = -\nabla\phi$
- div-free (solenoidal) field: $\boldsymbol{B} = \nabla \times \boldsymbol{A}$
- Poincare-de Rham-Hodge-Kodaira-Wyle theorem: $u = \nabla\varphi \oplus \nabla \times \boldsymbol{A} \oplus \nabla\Theta$

Θ=multi-valued potential = angle

application I (streamlines)

- Grad form of a symmetric vector field
  - $\to$ integrable streamlines
  - $\to$ integrals of hyperbolic PDE (Casimir)

$$\boldsymbol{B} = \nabla\psi \times \nabla\varphi + \left(B_\varphi r\right)\nabla\varphi$$

$$\frac{dr}{\boldsymbol{B}\cdot\nabla r} = \frac{dz}{\boldsymbol{B}\cdot\nabla z} = \frac{d\varphi}{\boldsymbol{B}\cdot r\nabla\varphi} = d\boldsymbol{\tau}$$

$$\begin{cases} \dfrac{dr}{d\boldsymbol{\tau}} = \dfrac{\partial\psi}{\partial z} \\[2ex] \dfrac{dz}{d\boldsymbol{\tau}} = -\dfrac{\partial\psi}{\partial r} \end{cases}$$

Nonlinear elliptic-hyperbolic PED theory
-- including Cauchy problem to generate Casimirs

ideal MHD

$$\begin{cases} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla p, \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

ideal fluid (Euler eq.)

$$\begin{cases} (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = -\nabla (p + \boldsymbol{v}^2 / 2), \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases}$$

$$\boldsymbol{u} = \nabla \psi \times \nabla \varphi + \omega \nabla \varphi$$

↓

$$\{\omega, \psi\} = 0 \quad \longrightarrow \quad \begin{aligned} &\omega = \omega(\psi), P = P(\psi) \\ &-L\psi = P'(\psi) + \omega(\psi)\omega'(\psi) \end{aligned}$$

- double contravariant form (Boozer form) in a toridal (ρ, θ, ϕ) frame

  → Hemiltonian form of general (non-integrable) streamlines

$$\boldsymbol{B} = \nabla\Psi \times \nabla\phi + \nabla\chi \times \nabla\theta$$

$$\begin{cases} \dfrac{d\chi}{d\phi} = \dfrac{\partial\Psi}{\partial\theta} \\[2ex] \dfrac{d\theta}{d\phi} = -\dfrac{\partial\Psi}{\partial\chi} \end{cases}$$

application II (canonical form)

- Hamiltonian fluid mechanics:

irrotational $\rightarrow$ zero-helicity $\rightarrow$ general

???

$$H = \int \left[\frac{1}{2} u^2 + \varepsilon(\rho) \right] \rho \, dx$$

$$\boldsymbol{u} = \nabla \varphi \quad \longrightarrow \quad \begin{cases} \partial_t \rho = \partial_\varphi H = -\nabla \cdot (\rho \boldsymbol{u}) \\ \partial_t \varphi = -\partial_\rho H = -\left(u^2 / 2 + h \right) \end{cases}$$

$$\longrightarrow \quad \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\nabla h$$

$$\boldsymbol{u} = \nabla\varphi + \alpha\nabla\beta$$

$\longrightarrow$

$$\begin{cases} \partial_t \rho = \partial_\varphi H = -\nabla\cdot(\rho\boldsymbol{u}) \\ \partial_t \varphi = -\partial_\rho H = -\left(u^2/2 + h\right) \end{cases}$$

$$(\alpha' = \alpha\rho)$$

$$\begin{cases} \partial_t \alpha' = \partial_\beta H = -\nabla\cdot(\alpha'\boldsymbol{u}) \\ \partial_t \beta = -\partial_{\alpha'} H = -\boldsymbol{u}\cdot\nabla\beta \end{cases}$$

$\longrightarrow$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla h$$

local $\rightarrow$ global

$$\{\text{ exact forms } \quad \omega = d\varphi \} \subset \{\text{ closed forms } \quad d\omega = 0 \}$$

$$\downarrow$$

the gap = topology of the domain (cohomology)

helicity (topological degree)

- Gauss' liking number:

$$l(C_1, C_2) = \oint_{C_1}\oint_{C_2} d\boldsymbol{x}_1 \times \boldsymbol{K}(\boldsymbol{x}_1, \boldsymbol{x}_2) \cdot d\boldsymbol{x}_2$$

$$\boldsymbol{K}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \frac{1}{4\pi} \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^3} \quad \text{(Biot – Savart's kernel)}$$

- helicity:

$$C = \int curl^{-1} \boldsymbol{w} \cdot \boldsymbol{w} \, dx \quad \longleftarrow$$

Casimir invariant of ideal vortex dynamics

incompleteness of the Clebsch form and its amendment

- The Clebsch form $\boldsymbol{u} = \nabla\varphi + \alpha\nabla\beta$ is incomplete

  (1) to describe streamline chaos,

  (2) to represent a field with a helicity.

- A complete form: $\boldsymbol{u} = \nabla\varphi + \sum \alpha_j \nabla\beta^j$

Theorem (complete Clebsch form)

Let $\boldsymbol{u}$ *be a continuous 1-form in* Ω:

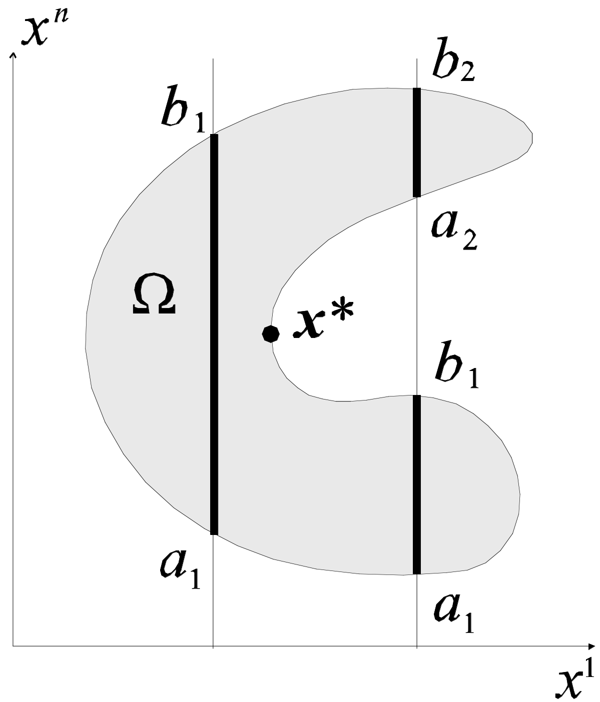
(*) $$\boldsymbol{u} = \sum_{j=1}^{n} u_j dx^j$$

With chosing a complete 1-form $d\varphi$ *, we can transform* (*) *into*

(**) $$\boldsymbol{u} = \sum_{j=1}^{n-1} u_j' dx^j + d\varphi$$

where

$$u_j' = u_j - \partial_{x^j}\varphi$$



$$\varphi(x^1,\ldots,x^n) = \int_{a_m}^{x^n} u_n(x^1,\ldots,x^{n-1},y)\,dy$$

$$+\,C(x^1,\ldots,x^{n-1};m)$$

remarks

- Counting the number of fields does not suffice.

- The set $\{\nabla\times(\nabla\varphi+\alpha\nabla\beta)=\nabla\alpha\times\nabla\beta\}$ is NOT a linear space.

- The set $\{\nabla\alpha_1\times\nabla\beta^1+\nabla\alpha_2\times\nabla\beta^2\}$ is equal to the linear space of solenoidal fields, proving the completeness of the Boozer form [Z. Yoshida, Phys. Plasmas **1** (1994), 208-209].