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Symmetries Conservation Laws

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Symmetries - Conservation Laws.

$$S = \int \mathcal{L}(\varphi, \partial_\mu \varphi)\, d^4x$$

Four - dimensional space - time.

$$V_\mu = \{V_0, \underset{\sim}{V}\} \qquad V_\mu V^\mu = -\lambda^2$$

An example of a Lagrangian

Scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2$$

$$= \frac{1}{2}\left((\partial_0 \varphi)^2 - (\underset{\sim}{\nabla}\varphi)^2 - m^2\varphi^2\right)$$

$\Rightarrow$ Derive the equation of motion K·G equation

$$\cancel{\frac{\partial^2 \varphi}{\partial t^2} - \cdots + m^2\varphi^2 = 0}$$

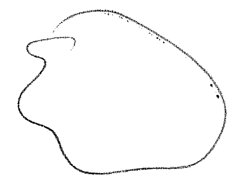
$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2\varphi^2 = 0$$

$$\Rightarrow \qquad \omega^2 = m^2 + k^2$$

Discuss $\rightarrow$ Lagrangian $\rightarrow$ why and how.

Formalism ;

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$



∂R

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \delta\varphi(x)$$

all variations are zero at the surface.

Also assume

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

some interaction with an external system.

$\delta\varphi(x)$ is only the functional variation in φ

$$\varphi'(x') = \varphi(x) + \Delta\varphi(x)$$

$$\Delta\varphi = \varphi'(x') - \varphi(x') + \varphi(x') - \varphi(x)$$
$$= \delta\varphi + (\partial_\mu \varphi)\,\delta x^\mu$$

$$\delta S = \int \mathcal{L}(\varphi', \partial_\mu \varphi', x'^\mu)\, d^4x' - \int \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)\, d^4x$$

$$d^4x' = J(x'/x)\, d^4x$$

$$\Rightarrow \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu \delta x^\mu$$

$$J(x'/x) = 1 + \partial_\mu(\delta x^\mu)$$

$$\delta S = \int \left(\delta \mathcal{L} + \mathcal{L}\, \partial_\mu x^\mu \right) d^4x$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underset{\partial_\mu(\delta\varphi)}{\delta(\partial_\mu \varphi)} + \frac{\partial \mathcal{L}}{\partial x^\mu}\, \delta x^\mu$$

$$\Downarrow$$

$$\delta S = \delta S^1 + \delta S^2$$

$$\delta S^1 = \int_R \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi \, d^4x$$

$$\delta S^2 = \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\, \delta\varphi + \mathcal{L}\, \delta x^\mu \right] d\sigma_\mu$$

From assumption : At ∂R , $\delta\varphi = 0 = \delta x^\mu$

$\therefore \ \delta S^2 = 0$

For stationary action, then

$\delta S^1 = 0 \quad \Rightarrow \quad \dfrac{\partial \mathcal{L}}{\partial \varphi} = \dfrac{\partial}{\partial x^\mu}\left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right)$

But Is that All ?

An equally powerful use of the variation principle is the exploration of conservation laws. R arbitrary – No requirement on $\delta\varphi, \delta x^\mu$ to be zero outside R.

$$\Delta\varphi = \delta\varphi + (\partial_\nu\varphi) x^\nu$$

$$\delta S^2 = \int_{\partial R} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \left[\delta\varphi + \partial_\nu\varphi\, \delta x^\nu\right] \Delta\varphi \; d\sigma_\mu \right.$$

$$- \int_{\partial R} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \partial_\nu\varphi - \eta^\mu{}_\nu \mathcal{L} \right) \delta x^\nu \, d\sigma_\mu$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$$

$$\boxed{\delta S^2 = \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} \Delta\varphi - T^\mu{}_\nu \delta x^\nu \right] d\sigma_\mu}$$

S invariant under a group of transformation

$$\Delta x^\mu = X^\mu{}_\nu \delta\omega^\nu$$

$$\Delta\varphi = \varphi_\mu \delta\omega^\mu$$

$\delta\omega^\nu$ are infinitesimal parameters

$$\delta S^2 = \int \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \tilde{\Phi}_\nu - \eta^\mu{}_\alpha X^\alpha{}_\nu \right] \underset{\substack{\Downarrow \\ \text{arbitrary}}}{dw^\nu} \, d\sigma_\mu$$

$$\delta S^2 = 0 \quad \Rightarrow$$

$$\int_{\partial R} J^\mu{}_\nu \, d\sigma_\mu = 0$$

Gauss's theorem

$$\boxed{\partial_\mu J^\mu{}_\nu = 0}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \tilde{\Phi}_\nu - T^\mu{}_\alpha X^\alpha{}_\nu$$

Consequences

$$\partial_t J^0{}_\nu + \nabla \cdot \underset{\sim}{J_\nu} = 0$$

$$\frac{d}{dt} \int J_\nu^0 \, d^3x = - \int d^3 \, \nabla \cdot \underset{\sim}{J_\nu} = \text{Surface term} \Longrightarrow \text{far way}$$

Conservation Law

$$Q_\nu = \int J_\nu^0 \, d^3x$$

Noether Theorem

A Simple Example

Translation in space - Time

$$\Delta x^{\mu} = \epsilon^{\mu} \qquad \Delta \varphi = 0$$

$$X^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} \qquad \varphi_{\mu} = 0$$

The Conserved current now, is

$$J^{\mu}_{\nu} = - T^{\mu}{}_{\nu}$$

Conservation law $\Rightarrow$

$$\frac{d}{dt} \int T^{0}{}_{\nu} \, d^3x = 0$$

$\int T^{0}{}_{\nu} \, d^3x = P_{\nu}$ is the four moment-

$$P_{\nu} = \{ E, \underset{\sim}{P} \}$$

Invariance ~~due~~ on Space - Time translations $\Longrightarrow$ Conservation of Energy and Momentum.

The Sine - Gordon Equation

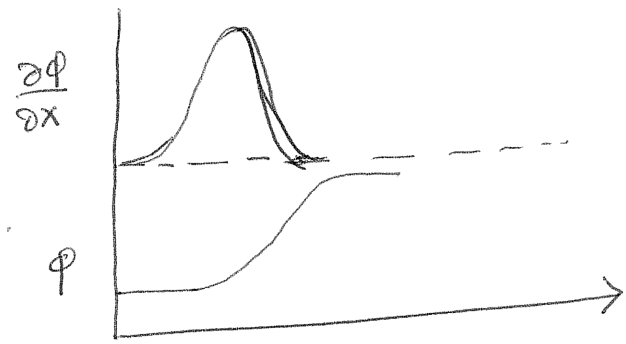
$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{b^2} \sin b\varphi = 0 \qquad (1)$$

scalar - field , a nonlinear wave function.

$$\varphi(x,t) = f(x - vt) = f(\xi)$$

$$f(\xi) = \frac{4}{b} \tan^{-1}\left\{ e^{\pm \frac{\gamma}{\sqrt{b}} \xi} \right]$$

$$\gamma^{-1} = \sqrt{1 - v^2}$$



Solitary wave -
propagates without dissipation.

No superposition

Eq. (1) clearly has an infinite number of constant solution

$$\varphi = \frac{2\pi n}{b} \qquad \text{for integral } n$$

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 - V(\varphi)$$

$$V(\varphi) = \frac{1}{b^2}\left[1 - \cos b\varphi\right]$$

For this choice $V_{\varphi = \frac{2\pi n}{b}} = 0$,

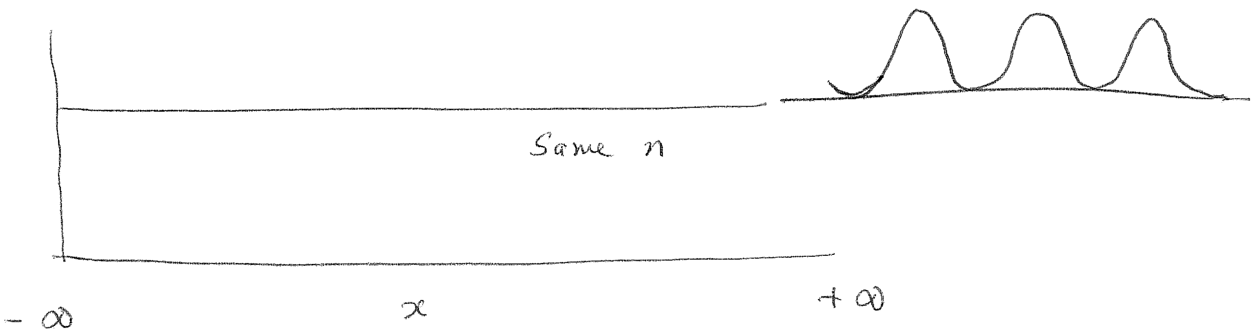
$$H = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + V(\varphi)$$

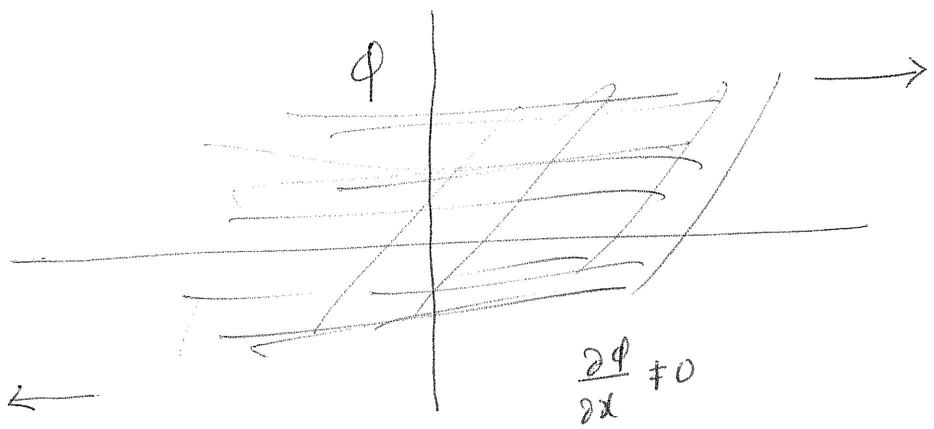
Analysis

$$V(\varphi) = \frac{\varphi^2}{2} - \frac{b^2}{4!}\varphi^4$$

unit mass

$b^2 = \lambda$





n_2

∞

$\frac{\partial \varphi}{\partial x} \neq 0$

φ as $x \to \infty$

n_1

φ as $x \to -\infty$

$\partial/\partial t = 0$

$\mathcal{H}$ is positive definite $\longrightarrow$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial V}{\partial \varphi} \qquad \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 = V(\varphi)$$

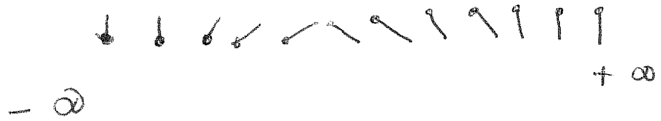
$$\left\{\begin{array}{ll} n_1 = 0 & \varphi \Rightarrow 2\pi/b \\ n_2 = 1 & \varphi \Rightarrow 0 \end{array}\right.$$

$$E = \int \mathcal{H}\, dx$$
$$= \int \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + V(\varphi)\right] dx$$
$$= \int 2V(\varphi)\, dx$$
$$= \int_0^{2\pi/b} [2V(\varphi)]^{1/2}\, d\varphi \qquad \equiv 8/b^2$$

$$E = \frac{8m^3}{\lambda}$$

Energy +ve definite

Energy goes as λ^{-1}



space is one dimensional →

The origin of the solution and why such solutions are stable ?

⇒ Energy arguement

⇒ Mathematically arguement

boundary conditions → In one-d space $(n, n' \neq n)$ solution cannot be continuously deformed to n, n. $(0,0$ = ground state$)$

Kink, is a ~~pure~~ topological object.

⇒ The stability further implies – yes – a conservation law — something which must remain 'constant' and that simply prevents $(1,0) \to (0,0)$.

Topological Charge.

The Conserved charge in this case is an integer N.

$$J^{\mu} = \frac{b}{2\pi} \epsilon^{\mu\nu} (\partial_{\nu} \varphi)$$

$$\partial_{\mu} J^{\mu} = 0$$

$$Q = \int_{-\infty}^{+\infty} J^{0} dx = \frac{b}{2\pi} \int \frac{\partial \varphi}{dx} dx$$

$$= \frac{b}{2\pi} \left[\varphi(\infty) - \varphi(-\infty) \right] = N$$

But J^{μ} was not derived from a symmetry — It is not a Noether current.

Quantum Numbers : Labels of a State