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Lagrangian for collective matter-field couplings

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Lagrangian for collective matter-field couplings

vortex, topology, scale-hierarchy

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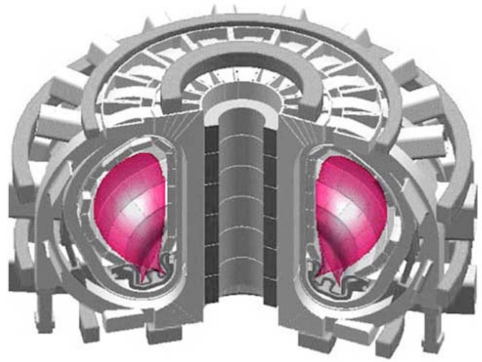
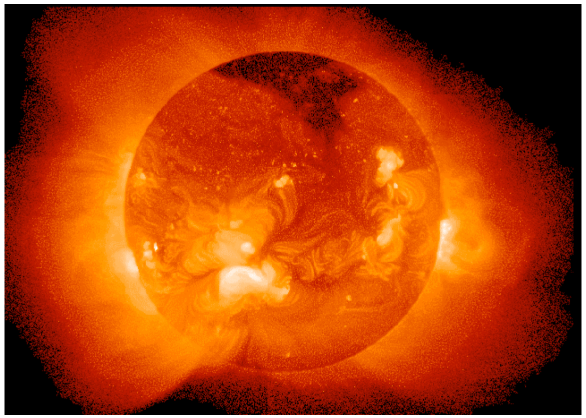
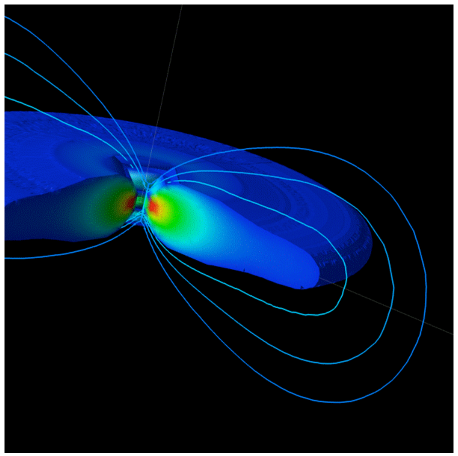
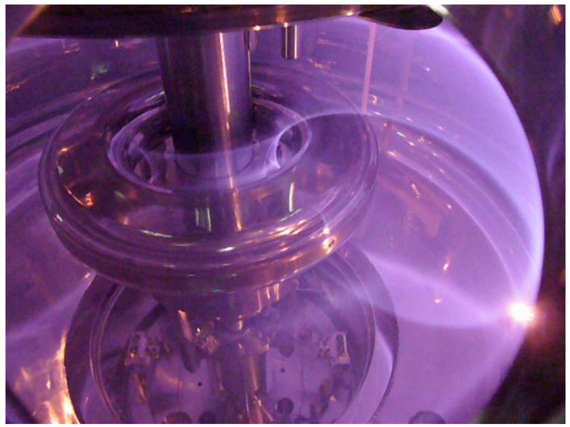
plasma = matter (flow) vortex + EM vortex

- “<u>topological charge</u>” → diversity of structures
- “<u>multi-scale</u>” → encompassing scale-hierarchy
- “<u>Beltrami fields</u>” = simplest model of vortex

some images



How can vortex exist?

- Remember the Hamilton-Jacobi equations:

$$\partial_t S = -H, \quad \boxed{\nabla S = \boldsymbol{P}}$$ ← momentum is irrorational

- Fields constructed from the action cannot have a volticity of the phase (ex. super-fluid).
- How can a fluid (plasma) create vorticity?

Lagrangian view & Eulerian view

- Flow-field coupling: $L = L_F + L_{EM}$

$$L_F = \int \left(\boldsymbol{P} \cdot \boldsymbol{V} - H_F \right) \rho dx, \quad L_{EM} = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} dx$$

- Lagrangian view:

$$\boldsymbol{V} = \dot{\boldsymbol{Q}} \Big|_{t, \boldsymbol{x} = Q(\boldsymbol{x}_0, t)} \quad (\boldsymbol{Q} : \text{diffeomorphism})$$

- Eulerian view:

$\boldsymbol{V}$ with constraints

history

- Eulerian formulation (Serrin,1959;Lin,1963)
  → how can vorticity be created?
- Non-canonical Hamiltonian system （Morrison,1980）
- Vortex as a SO(3) gauge field （Kambe,2003)
- Vortex as a non-Abelian field theory（Jackiw,2002）
- Casimir (helicity) and variational principle（DB field), scale hierarchy, Lyapunov stability
  (Yoshida, Mahajan, Hirota, Ohsaki)

Lagrangian (in Eulerian view)

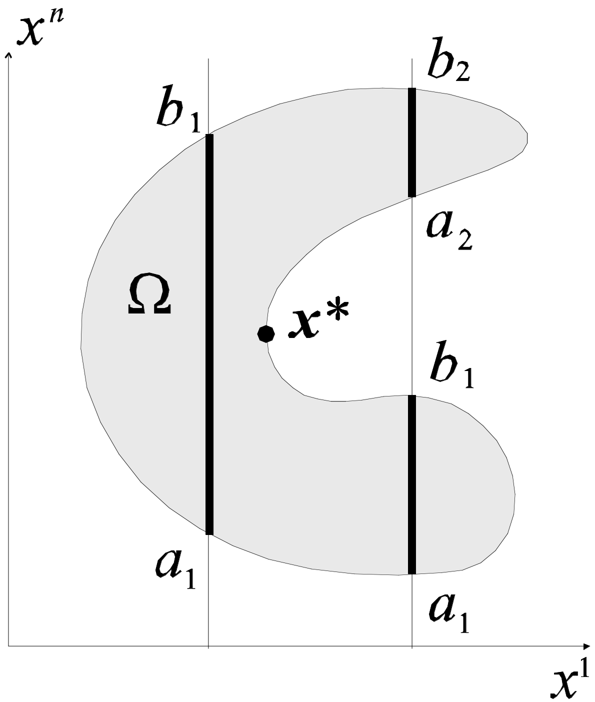
$$L_F = \int \left(\boldsymbol{P} \cdot \boldsymbol{V} - H_F - D_t S - \lambda^j D_t \sigma_j \right) \rho dx$$

Serrin's term: $\left(S D_t^* \rho \right)$ constraint for mass conservation

$\rightarrow irrotational \;\; flow: \quad \boldsymbol{P} = \nabla S$

$We \; need \quad \sum_{j=1}^{3} \lambda^j D_t \sigma_j \quad \Rightarrow \quad P = \nabla S + \sum_{j=1}^{3} \lambda^j \sigma_j$

Why 3 (not 2)?



We cannot control the boundary vale in the variational principle.

$$\varphi(x^1,\ldots,x^n) = \int_{a_m}^{x^n} u_n(x^1,\ldots,x^{n-1},y)\,dy$$

$$+\,C(x^1,\ldots,x^{n-1};m)$$

Lagrangian / Eulerian duality

- Equivalent form: $\boldsymbol{î} = \boldsymbol{x} - \boldsymbol{ó}, \quad \boldsymbol{ì} = \boldsymbol{\ddot{e}} - \boldsymbol{P}$

$$L_F = \int \left(\boldsymbol{P} \cdot D_t \boldsymbol{î} - D_t S - \mu^j D_t \sigma_j - H_F \right) \rho dx$$

- Lagrangian form:

$$\boldsymbol{ó} = \boldsymbol{x}_0 \ \ fix \quad \longrightarrow \quad D_t \boldsymbol{î} = \dot{\boldsymbol{Q}} \Big|_{t, \boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{x}_0(\boldsymbol{x}, t), t)}$$

- Eulerian form:

$$\boldsymbol{ó} = \boldsymbol{x}_0 \ \ fix \quad \longrightarrow \quad \rho, \ A^\mu \Big|_{t, \boldsymbol{x} = \boldsymbol{î} + \boldsymbol{ó}}$$

non-canonical Hamiltonian mechanics

- Generalied Hamilton-Jacobi eqs.:

$$\partial_t S = -(H + h) - \lambda^j \partial_t \sigma_j$$

$$\nabla S = \boldsymbol{P} - \lambda^j \nabla \sigma_j$$

- Singular Lie-Poisson bracket $\rightarrow$ Casimirs

a review of Hamiltonian mechanics

- classical mechanics ⇔ symplectic 2-form

$$L = a_i(u)\dot{u}^i - H(u) \quad \Leftrightarrow \quad dS = a_i du^i - Hdt \text{ (canonical 1 form)}$$

$$\Rightarrow \quad A_{ij}(u)\dot{u}^i = \partial_{u^i} H(u) \quad (A_{ij} = \partial_{u^i} a_j - \partial_{u^j} a_i)$$

$$\omega = d(a_i du^i) = A_{ij} du^i \wedge du^j$$

- If A_{ij} has a unique inverse A^{ij}, we have

$$\dot{u}^i = A^{ij}(u)\partial_{u^j} H(u)$$

$$= \{H, u^i\} = \{u^j, u^i\}\partial_{u^j} H(u)$$

Topological defects (*Casimir invariants*)

- A "non-canonical" Hamiltonian system admits the Poisson bracket (operator A) to have a kernel, i.e.,

$$\exists C \; s.t. \{G, C\} = 0 \; (\forall G).$$

- Such C is called a "Casimir invariant".
- Casimir invariants poses topological constraints.

Equilibrium (stationary points)

- Characterized by $\delta H = 0$ (but, often trivial)
- Relaxation process (cf. Dirichlet's principle) drives the system towards the equilibrium.
- Constraints may yield non-trivial class of equilibria characterized by

$$\delta \tilde{H} = 0 \quad (\tilde{H} = H + \sum_j \mu_j C_j).$$

Helicity (*Casimir*)

- Vortex dynamics system:

$$\partial_t \boldsymbol{u} = -\grave{\boldsymbol{U}} \times \boldsymbol{u} - \nabla\theta \quad \left(\grave{\boldsymbol{U}} = \nabla \times \boldsymbol{u}\right)$$

$$\Leftrightarrow \quad \dot{\boldsymbol{u}} = A\partial_{\boldsymbol{u}} H \quad \left(H = \tfrac{1}{2}\|\boldsymbol{u}\|^2,\ A\boldsymbol{u} = -P\grave{\boldsymbol{U}} \times \boldsymbol{u}\right)$$

(We assume incompressible $\boldsymbol{u}$. P is the “projection” onto the function space of incompressible fields.)

- Helicity:

$$C = \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{u}\, dx \quad \Rightarrow \quad A\partial_{\boldsymbol{u}} C = 0$$

helicity (topological degree)

- Gauss' liking number:

$$l(C_1, C_2) = \oint_{C_1}\oint_{C_2} d\boldsymbol{x}_1 \times \boldsymbol{K}(\boldsymbol{x}_1, \boldsymbol{x}_2) \cdot d\boldsymbol{x}_2$$

$$\boldsymbol{K}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \frac{1}{4\pi} \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^3} \quad \text{(Biot – Savart's kernel)}$$

- helicity:

$$C = \int curl^{-1}\boldsymbol{w} \cdot \boldsymbol{w}\, dx \quad \longleftarrow$$

Casimir invariant of ideal vortex dynamics

MHD case

- Canonical form of MHD

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{B} \end{pmatrix} = A \begin{pmatrix} \partial_{\boldsymbol{v}} H \\ \partial_{\boldsymbol{B}} H \end{pmatrix}$$

$$H = \frac{1}{2}\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 \right), \quad A = \begin{pmatrix} -P\boldsymbol{\grave{U}}\times\circ & P[(\nabla\times\circ)\times\boldsymbol{B}] \\ \nabla\times(\circ\times\boldsymbol{B}) & 0 \end{pmatrix}$$

- Casimirs (helicities)

$$C_1 = (\boldsymbol{A}, \boldsymbol{B}), \quad C_2 = (\boldsymbol{v}, \boldsymbol{B})$$

Structured equilibria

- Parameterized Hamiltonian (Lyapunov function):

$$\delta\widetilde{H}(\boldsymbol{u}) = \delta\left(H(\boldsymbol{u}) + \sum \alpha_j C_j(\boldsymbol{u})\right) = 0$$

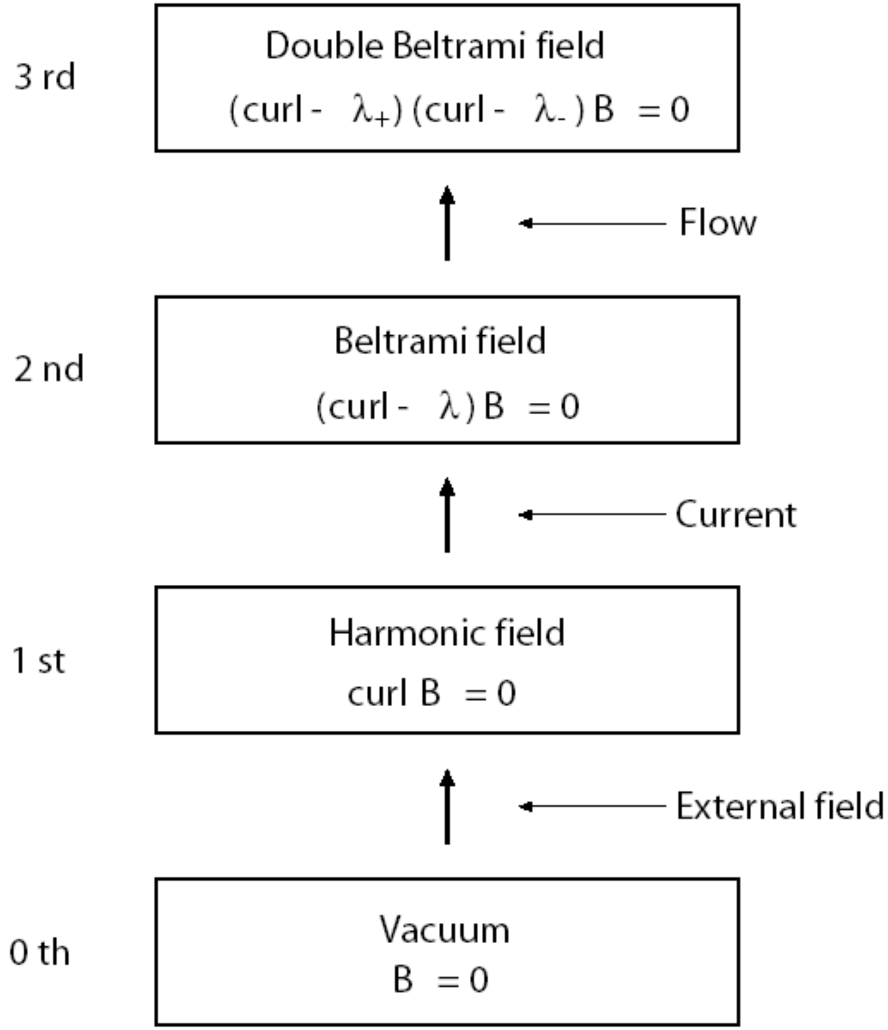
- Parameterized stationary points:

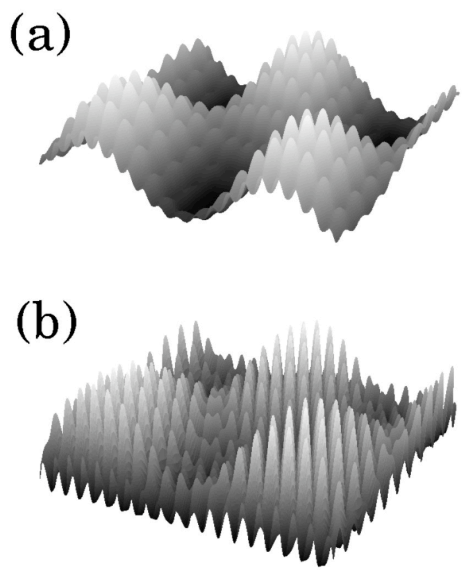
$$(\mathrm{curl} - \lambda_1)\cdots(\mathrm{curl} - \lambda_N)\boldsymbol{u} = 0$$

- Beltrami-class of equilibria (multi-scale):

$$\boldsymbol{u} = \sum a_j \boldsymbol{G}_j \quad \left(\nabla \times \boldsymbol{G}_j = \lambda_j \boldsymbol{G}_j\right)$$

flow(matter) –field(EM) coupling and scale hierarchy





S.M. Mahajan and Z.Yoshida, Phys. Rev. Lett. **81** (1998) 4863.
Z. Yoshida and S.M. Mahajan, Phys. Rev. Lett. **88** (2002) 095001.

Summary

- generalized Clebsch form (n=3)
  - → unification of the Lagrangian & Eulerian views
- vortex
  - → singularity (non-canonical bracket)
  - → Casimirs (helicity)
  - → Beltrami fields (quantized vortex)
  - → multi-scale flow-field couplings
- variational principle
  - → "self-organization"
  - → Lyapunov stability (need "coercivity")