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General properties of the Rayleigh-Taylor instability in different plasma configurations: the plasma foil model.

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First lecture (mathematical properties)

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Rayleigh-Taylor instability in a fluid system

In fluid dynamics the Rayleigh-Taylor instability represents a basic phenomenon that occurs when a lighter fluid is accelerated into a heavier fluid.

For an inhomogeneous fluid system in a gravitational field this instability was first discovered by Lord Rayleigh in the 1880[1].

It is responsible for the fact that, if surface tension effects are neglected, it is not possible to keep water inside an inverted container (i.e., open at its bottom) balanced by atmospheric pressure.

[1] Rayleigh, Lord (John William Strutt), *"Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density,"* Proceedings of the London Mathematical Society, Vol. 14, pages 170 - 177 (1883)

Later the same concept was applied to accelerated fluids by Sir Geoffrey Taylor[2].

Understanding the rate of mixing caused by the Rayleigh-Taylor instability is important to a wide range of applications, that range from inertial confinement fusion, nuclear weapons explosions[3], supernova explosions and supernova remnants, to oceanography and atmospheric physics, to laboratory and space plasmas etc.

[2] Taylor, Sir Geoffrey Ingram, *"The instability of liquid surfaces when accelerated in a direction perpendicular to their planes,"* Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 201, No. 1065, pages 192 - 196 (1950)

[3] It is interesting to look at the dates of these papers: see e.g., *Fermi, E. 1951, "Taylor instability of an incompressible liquid", The Collected Papers of Enrico Fermi (ed. E. Segre), vol. 2, pp. 816, 821.*
>>A discussion is presented in simplified form of the problem of the growth of an initial ripple on the surface of an incompressible liquid in the presence of an acceleration, g, directed from the outside into the liquid. The model is that of a heavy liquid occupying at $t = 0$ the half space above the plane $z = 0$, and a rectangular wave profile is assumed. The theory is found to represent correctly one feature of experimental results, namely the fact that the half wave of the heavy liquid into the vacuum becomes rapidly narrower while the half wave pushing into the heavy liquid becomes more and more blunt. ....

A family of different instabilities can be grouped under the general name of Rayleigh-Taylor instabilities, with e.g., an inhomogeneous pressure playing the role of the inhomogeneous density, or electromagnetic radiation pressure taking the role of the lighter fluid or even, in the case of a magnetized confined plasma, magnetic field pressure[4] and magnetic field line curvature playing the role of the lighter fluid and of gravity, respectively.

In the simple case of two immiscible incompressible fluids with densities ρ_1 and $\rho_2 < \rho_1$ respectively where the denser fluid 1 in initially on top of fluid 2, the instability linear growth rate γ can be written as

$$\gamma^2 = |k|g\, \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \qquad \gamma^2 \sim |k|g, \quad \text{for } \rho_2 \ll \rho, \tag{1}$$

with k the wavenumber in the interface plane and g the gravity acceleration.

[4]Note however that the tensor nature of the magnetic pressure, of the Maxwell stress tensor, makes a magnetic Rayleigh-Taylor instability evolve differently from a fluid Rayleigh-Taylor instability

Equation (1) can be understood with a simple energy argument.

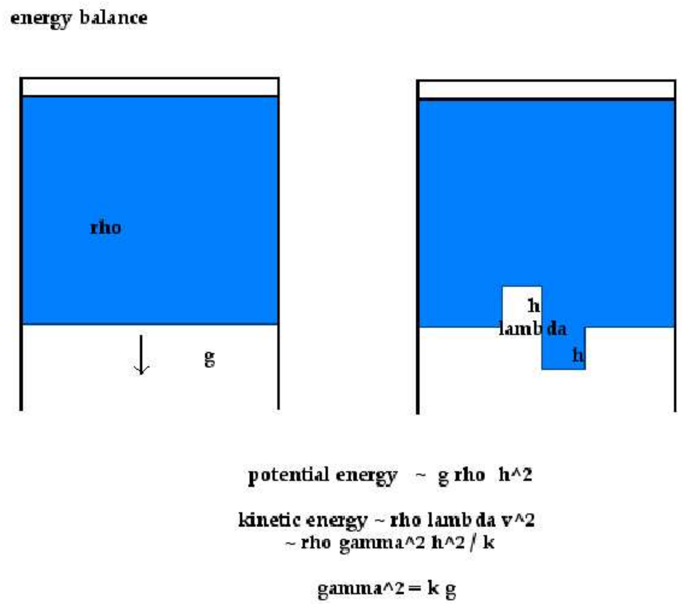

Figure 1: Inverted water air equilibrium

The apparently unbound increase of the growth rate at smaller wavelengths is interrupted by effects, such as surface tension for a real fluid system, that are not included in Eq.(1). At small wavelengths these effects first reduce the mode growth rate and finally stabilize the mode.
Different stabilizing mechanisms have been shown to arise for different forms of Rayleigh-Taylor instabilities, ranging from matter ablation in the pellet compression in inertial fusion, to field line tension in the case of magnetically confined plasmas.

The Rayleigh-Taylor instability is of special importance in the case of inertial fusion where a fuel pellet is compressed by the reaction force exerted by the surface layers of the pellet ablated by the energy deposited by a high intensity laser pulse[5]

[5] S. Atzeni and J. Meyer-ter-Vehn, "The Physics of Inertial Fusion" (Clarendon Press, Oxford, 2004)

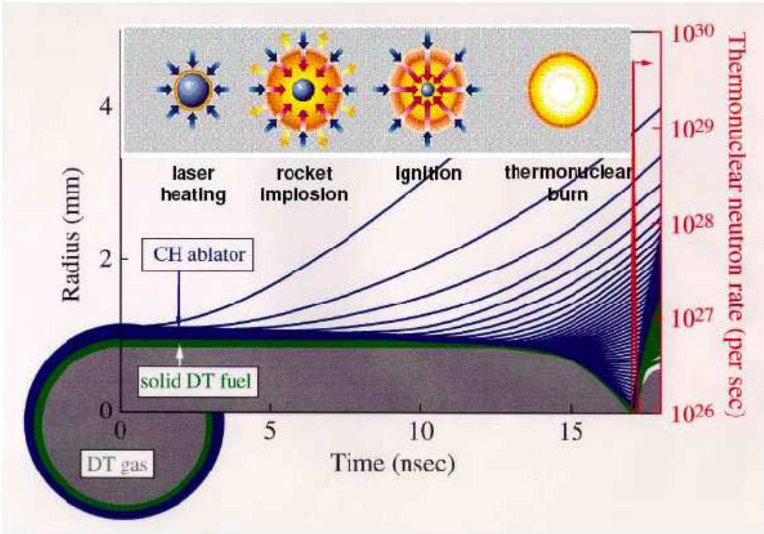

Figure 2: Pellet compression in inertial fusion
from http://www.hiper-laser.org/docs/tdr/HiPERTDR2.pdf

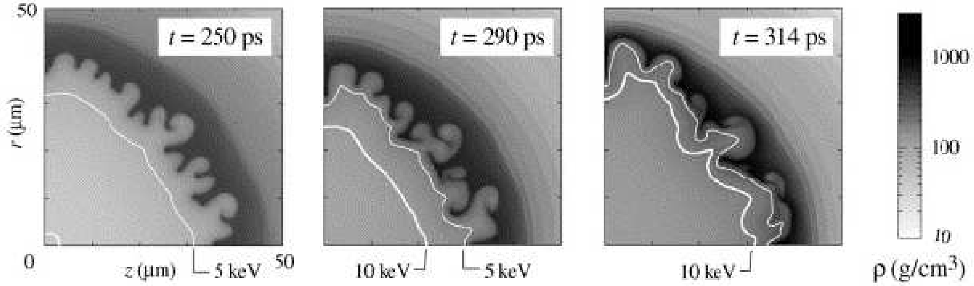

Figure 3: Numerical simulation of the development of the Rayleigh-Taylor instability of a compressed pellet

from" Fluid and kinetic simulation of inertial confinement fusion plasmas", S. Atzeni, A. Schiavi, F. Califano, F.Cattani, F. Cornolti, D. Del Sarto, T. V. Liseykina, A. Macchi, F. Pegoraro, *Computer Physics Communications* **169**, 153-159 (2005)

The foil geometry

In this lecture I will focus on the Rayleigh-Taylor instability of a thin accelerated material foil. This configuration is of both of practical interest in a number of physical conditions and is amenable to analytical solutions.

Indeed, the Rayleigh-Taylor instability of a thin plasma slab[6] provides one of the best examples of the basic nonlinear behavior of a fluid when its equilibrium configuration is unstable against infinitesimal perturbations. In addition, in some simplified limits exact mathematical solutions can be found that make it possible to study the formation and the properties of singularities produced in the nonlinear evolution of the instability.

[6] E. Ott, *Phys. Rev. Lett* **29**, 1429 (1972).

More specifically we will consider an initially planar plasma slab (in the y-z plane)[7] of width d (along x).
We suppose that the characteristic transverse size of the slab in the y-z plane is much larger than d and thus take for the sake of simplicity the slab to be infinitely extended in this plane. We suppose that the foil is acted upon by a spatially uniform pressure difference that is constant in time and that acts at all times along the normal to the foil surface. The validity of this schematic representation and its possible generalization will be rediscussed in the next lecture.

If we restrict ourselves to motions and deformations of the plasma slab that do not involve spatial scales smaller that its width d we can describe the slab as an infinitely thin foil with a given surface density given by the slab density (assumed to be uniform) multiplied times d.

[7] Different geometries, such as e.g., spherical or cylindrical configurations are also easily treated.

This may appear as a restrictive approximation in view of the fact that Eq.(1) would predict that the growth rate of the Rayleigh-Taylor instability increases with its wavenumber[8] k, while d^{-1} can be obviously taken as an upper bound on k for the foil model to be valid.

However, as mentioned before, physical effects non included in Eq.(1) may stabilize short wavelength perturbations and thus the thin foil approximation can be taken as a valid model as long as the slab dynamics is correctly described by "long wavelength" perturbations defined by the condition $kd \ll 1$.

Within this approximation, the basic equations that describe the motion of a foil surface element are its surface mass conservation equation and the momentum equation.

[8] in the foil plane

If we include the presence of a friction force[9], these equations take the form

$$\frac{d}{dt}(\sigma\ \mathbf{d\Sigma}) = 0, \tag{2}$$

and

$$\sigma\left[\frac{d}{dt} + \nu^{(in)}\right]\ \mathbf{v} = \mathcal{P}\ \mathbf{n}. \tag{3}$$

where $\mathbf{v}$ is the velocity of the foil, σ its surface mass density, $\mathcal{P}$ is the pressure jump through the plasma slab with respect to the normal vector $\mathbf{n}$, $\nu^{(in)}$ is an effective friction frequency, d/dt is the Lagrangian time derivative and $\mathbf{d\Sigma}$ is the (oriented) surface element on the shell.

First we consider a general 3-D case where the foil position depends on all three spatial coordinates, x, y, z and on time t.

[9] of interest for example in the case of a plasma foil accelerated through a partially ionized gas

We assume that the foil is initially located on a smooth surface that we parametrize as $x = \mathcal{X}(y,z)$. In order to obtain the equations for the foil evolution, we introduce the Lagrange variables, α, and β, related to the Euler coordinates by

$$x = x(\alpha,\beta,t), \quad y = y(\alpha,\beta,t), \quad \text{and} \quad z = z(\alpha,\beta,t), \tag{4}$$

where α and β are a set of variables marking the foil elements.
A convenient choice of α and β is given e.g., by a set of (local) orthogonal coordinates on the surface where the shell is located at $t = 0$. In the simple case where the shell is initially planar we can choose $\mathcal{X} \equiv 0$ and $y = \alpha$, $z = \beta$ at $t = 0$.
Then the surface density conservation gives $\sigma_0\, d\Sigma_0 = \sigma d\Sigma$, with $d\Sigma_0 = d\alpha \wedge d\beta$, from which we obtain

$$\sigma_0 \left[\frac{d}{dt} + \nu^{(in)}\right] \mathbf{v} = \mathcal{P}\ \mathbf{d\Sigma}/d\Sigma_0. \tag{5}$$

Two dimensional solutions; equations are linear in Lagrange variables

An important simplification occurs in the case of two dimensional (2D) foil evolutions defined e.g. by the condition $\partial/\partial z = 0$ in which case
$\mathbf{v} \equiv (dx/dt)\,\mathbf{e}_x + (dy/dt)\,\mathbf{e}_y$,
$\mathbf{d\Sigma}/d\Sigma_0 \equiv (\partial y/\partial\alpha)\,\mathbf{e}_x - (\partial x/\partial\alpha)\,\mathbf{e}_y$
with α the element initial position along y, $\mathbf{e}_{x,y}$ the unit vectors in the x-y plane.

In this 2-D case, the evolution equations (5) expressed in Lagrangian variables are linear.

Note that the evolution equations are not linear if expressed in Eulerian variables.

Inviscid solutions

Let us first set $\nu^{(in)} = 0$. The 1-D solution $x_0 = x_0(t)$ (i.e., $\partial y_0/\partial\alpha = 1$, $\partial x_0/\partial\alpha = 0$) for the accelerated foil reads

$$d^2 x_0/dt^2 = \mathcal{P}/\sigma_0, \tag{6}$$

while the perturbations $\tilde{x}(\alpha, t)$, $\tilde{y}(\alpha, t)$ obey the equations

$$d^2\tilde{x}/dt^2 = (\mathcal{P}/\sigma_0)(\partial\tilde{y}/\partial\alpha), \qquad d^2\tilde{y}/dt^2 = -(\mathcal{P}/\sigma_0)(\partial\tilde{x}/\partial\alpha), \tag{7}$$

where no linearization has been performed. Choosing solutions of the form $\exp(ik\alpha + \gamma t)$, with k the Lagrangian wavenumber, we recover the scaling

$$\gamma^4 = k^2(\mathcal{P}/\sigma_0)^2. \tag{8}$$

It is interesting to observe that more general solutions can be obtained by introducing[10] the complex function $w(\alpha, \tau) = x + iy$ which then obeys the equation (in terms of a properly normalized time variable τ)

$$\partial_{\tau\tau} w = -i\partial_\alpha w, \tag{9}$$

while for the complex conjugate function $w^*(\alpha, \tau) = x - iy$ we have

$$\partial_{\tau\tau} w^* = i\partial_\alpha w^*, \tag{10}$$

where w and w^* are considered as independent functions.

The solution $w = i\alpha + \tau^2/2$ corresponds to a uniformly accelerated plane foil,

[10] E. Ott, *Phys. Rev. Lett* **29**, 1429 (1972),
S. V. Bulanov, F. Pegoraro, and J.-I. Sakai *Phys. Rev.*, **E.59** , 2292 (1999).

while the solution $w(\alpha, \tau) = x + iy = i\alpha^3 - i\alpha\tau^4/4 - \tau^6/120 + 3\alpha^2\tau^2/2$ describes the local structure[11] of the wave breaking.

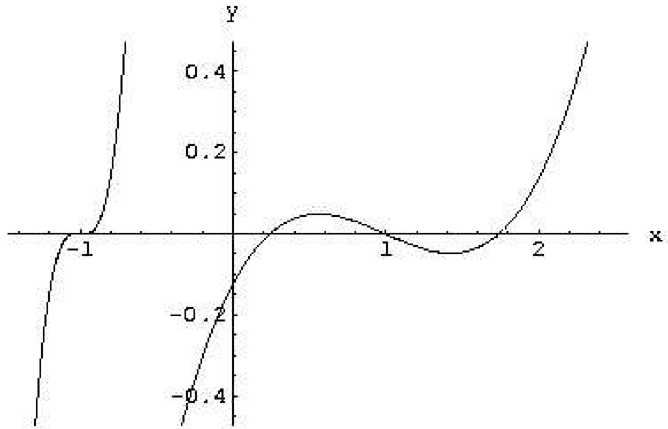

Figure 4: Breaking solution

For $w(\alpha, \tau) \propto \exp(i\alpha q)$ from Eq.(9) we recover exponentially growing and

[11]In this solution a non homogeneous initial surface density is assumed corresponding to $dy(t = 0) = \sigma_0(\alpha) d\alpha$,

decaying modes for $q > 0$ and oscillatory modes with real frequency for $q < 0$([12]).

[12]The intervals in q are interchanged in Eq.(10) for $w^*(\alpha, \tau) \propto \exp(-i\alpha q)$.

Lie symmetries

Eq.(9) admits 7 symmetry transformations[13] represented by the operators:

$$X_\infty = w_1(\alpha, \tau)\partial_w, \tag{11}$$

$$X_1 = i\partial_\alpha, \tag{12}$$

$$X_2 = \partial_\tau, \tag{13}$$

$$X_3 = 2\alpha\partial_\alpha + \tau\partial_\tau, \tag{14}$$

$$X_4 = 2i\alpha\partial_\tau - \tau w\partial_w, \tag{15}$$

$$X_5 = i\tau\alpha\partial_\tau - i\alpha^2\partial_\alpha - (\tau^2 + 2i\alpha)(w/4)w\partial_w, \tag{16}$$

$$X_6 = w\partial_w. \tag{17}$$

The operator X_∞ stems from the fact that Eq.(9) is linear with respect to $w(\alpha, \tau)$ so that, to any solution $w(\alpha, \tau)$, one can add any other solution $w_1(\alpha, \tau)$. The operators X_1 to X_4 and

[13] A discussion of the theory of the Lie group analysis of differential equations is presented e.g., in Ovsiannikov, L.V., *Group Analysis of Differential Equations*, (Academic Press, New York, 1982).

X_6 correspond to time and space translations and to the invariance with respect to stretching of the variables respectively. The operator X_5 represents the transformation

$$(\bar{\alpha}, \bar{\tau}) = (\alpha . \tau)/(1 - a\alpha), \qquad \bar{w} = (1 - a\alpha)^{1/2} \exp\left[ia\tau^2/(4 - 4a\alpha)\right] w. \tag{18}$$

If we choose $w = i/(4\pi)^{1/2}$ and $a = -1/h$, and superpose $w_0 = i\alpha + \tau^2/2$, we obtain a solution of the form

$$\bar{w}(\alpha, \tau) = i\alpha + \frac{\tau^2}{2} + \frac{i(h)^{1/2}}{[4\pi(\alpha + h)]^{1/2}} \exp\left[i\frac{\tau^2}{4(\alpha + h)}\right], \tag{19}$$

where $i(h)^{1/2}$ is the initial perturbation amplitude and h is a complex parameter. The foil is initially a planar with perturbations localized in a region with size of order $|h|$. If h is imaginary and positive, $h = i|h|$, this solution describes perturbations that grow faster than exponential: $\propto \exp(\tau^2/4|h|)$.

Viscous solutions

If viscosity dominates over inertia we obtain[14]

$$\partial_\tau x = \partial_\alpha y, \qquad \partial_\tau y = -\partial_\alpha x, \tag{20}$$

where we have suitably normalized the time variable τ.
These equations are the Cauchy-Riemann conditions for the real and imaginary parts of an analytical function $W(\zeta) = x + iy$ of a complex variable $\zeta = \alpha + i\tau$. Their solutions are thus given by the conformal mapping from the complex plane $\alpha + i\tau$ to the plane $x + iy$. Choosing the analytical function

$$W(\zeta) = -i\zeta + \kappa \frac{1}{1+\zeta^2} \tag{21}$$

[14] S. V. Bulanov, F. Pegoraro, and J.-I. Sakai *Phys. Rev.*, **E.59** , 2292 (1999).

where $\kappa = \kappa_R + i\kappa_I$ is a complex constant, we find

$$x(\alpha,\tau) = \tau + \kappa_R \frac{1-\tau^2+\alpha^2}{(1-\tau^2+\alpha^2)^2+4\alpha^2\tau^2} + \kappa_I \frac{2\alpha\tau}{(1-\tau^2+\alpha^2)^2+4\alpha^2\tau^2}, \tag{22}$$

$$y(\alpha,\tau) = -\alpha + \kappa_I \frac{1+\tau^2+\alpha^2}{(1-\tau^2+\alpha^2)^2+4\alpha^2\tau^2} - \kappa_R \frac{2\alpha\tau}{(1-\tau^2+\alpha^2)^2+4\alpha^2\tau^2}. \tag{23}$$

These expressions describe the growth of perturbations that are faster than exponential. It easy to see that at the finite time $\tau = 1$ the Jacobian $|W'|$ of the transformation becomes infinite at the point $\alpha = 0$.

A more extended description of this method can be found in S. V. Bulanov, F. Pegoraro, and J.-I. Sakai *Phys. Rev.*, **E.59** , 2292 (1999).

Three dimensional viscous solutions; equations remain nonlinear in Lagrange variables

For three dimensional viscosity dominated foil evolutions, we obtain the equations of motion[15]

$$\frac{\partial x}{\partial \tau} = \{y, z\}_{\alpha,\beta} \quad \frac{\partial y}{\partial \tau} = \{z, x\}_{\alpha,\beta} \quad \frac{\partial z}{\partial \tau} = \{x, y\}_{\alpha,\beta} \quad , \tag{24}$$

where the Poisson brackets with respect to the Lagrange variables α and β are defined by

$$\{\ ,\ \}_{\alpha,\beta} \equiv \frac{\partial}{\partial \alpha}\,\frac{\partial}{\partial \beta} - \frac{\partial}{\partial \beta}\,\frac{\partial}{\partial \alpha}. \tag{25}$$

[15] F. Pegoraro, S.V. Bulanov, J-I Sakai , G. Tomassini, Phys Rev E, 64, 016415 , (2001)

These equations are nonlinear but reduce reduce to the linear two-dimensional equations if x and y are independent of β (with $\partial z/\partial\beta = 1$).

Eqs.(24) can be written in the notation of differential forms as the equality of three 3-forms involving exterior products of the 1-forms obtained by differentiating the independent and the dependent variables

$$dx \wedge d\alpha \wedge d\beta = d\tau \wedge dy \wedge dz,$$
$$dy \wedge d\alpha \wedge d\beta = d\tau \wedge dz \wedge dx,$$
$$dz \wedge d\alpha \wedge d\beta = d\tau \wedge dx \wedge dy,$$

where now d denotes the exterior derivative and $\wedge$ the exterior product[16].

[16] for a detailed definition of these operations see e.g., H. Flanders, *Differential Forms with Applications to the Physical Sciences* (Dover Publ.: New York, 1989).

In order to recover Eqs.(24) we express the 1-form dx in terms of the three 1-forms $d\alpha,\ d\beta,\ d\tau$ as

$$dx = \frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta + \frac{\partial x}{\partial \tau} d\tau, \tag{26}$$

and analogously for dy and dz and use the antisymmetric properties of the exterior product.
If we interchange dependent and independent variables (hodograph transformation) and write

$$d\alpha = \frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy + \frac{\partial \alpha}{\partial z} dz, \tag{27}$$

and analogously for $d\beta$ and $d\tau$, we obtain the hodograph transformed equations

$$\frac{\partial \tau}{\partial x} = \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial z} - \frac{\partial \beta}{\partial y}\frac{\partial \alpha}{\partial z}, \tag{28}$$

$$\frac{\partial \tau}{\partial y} = \frac{\partial \alpha}{\partial z}\frac{\partial \beta}{\partial x} - \frac{\partial \beta}{\partial z}\frac{\partial \alpha}{\partial x}, \qquad \frac{\partial \tau}{\partial z} = \frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial \beta}{\partial x}\frac{\partial \alpha}{\partial y}.$$

These can be combined in the vector equation

$$\nabla \tau = \nabla \alpha \times \nabla \beta, \tag{29}$$

where $\nabla \tau$ is the gradient of the function $\tau = \tau(x, y, z)$ in x, y, z space, etc. .

In the generalization from 2-D to 3-D the harmonic property

$$\nabla^2 \tau = 0, \tag{30}$$

where the Laplace operator is taken with respect to x, z in 2-D and to x, y, z in 3-D, is preserved .
From Eq.(29) we also obtain $\nabla\tau \cdot \nabla\alpha = \nabla\tau \cdot \nabla\beta = 0$ and the τ-independent compatibility equation $\nabla \times (\nabla\alpha \times \nabla\beta) = 0$.
Note that, in contrast to the 2-D case where all variables play the same role, in general $\nabla^2\alpha \neq 0$ and $\nabla^2\beta \neq 0$.

The well known systems of orthogonal coordinates that are commonly used in mathematical physics can be rewritten[17] is such a way as to lead to solutions of Eqs.(29).

[17] A number of explicit solutions can be found in F. Pegoraro, S.V. Bulanov, J-I Sakai , G. Tomassini, Phys Rev E, 64, 016415 , (2001)