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White Light
Parametric Instabilities in Plasmas.

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White Light
Parametric Instabilities in Plasmas

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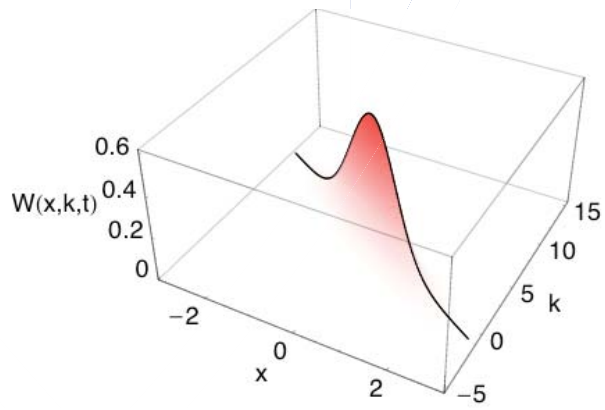


Outline

- ⊙ Why "White Light"? Why "Parametric Instabilities"?
- ⊙ Plan to describe statistical properties of light for arbitrary bandwidths + coupling with plasma - connection with quantum mechanics
    - ✳ The Wigner distribution
    - ✳ Generalization of the Wigner-Moyal formalism to describe broadband radiation processes
- ⊙ White light effects on Stimulated Raman Scattering (coupling with electron plasma waves)
    - ✳ Dependence of growth rates on bandwidth
- ⊙ Future directions and summary

Parametric instabilities

- Class of nonlinear oscillations
  - ★ exponential growth or decay of the oscillations
  - ★ an oscillator equation in which a parameter has an explicit periodic oscillation/modulation
  - ★ e.g. pendulum with externally driven oscillating suspension point

$$\partial_t^2 \xi + \omega_p^2 \left(1 + \epsilon \cos(\omega_0 t)\right) \xi = 0$$

Mathieu equation

- Common in many plasma physics scenarios (e.g. ultra intense radiation in plasmas)

Stimulated Raman Scattering

- ★ electrons oscillate in transverse field of pump light wave such that $\mathbf{v}_\perp = e\mathbf{E}_L/m\omega_0$
- ★ small density oscillation dn associated with electron plasma wave originates transverse current $\delta \mathbf{J}_\perp = -e\mathbf{v}_L \delta n$
- ★ if wavenumbers/frequencies are properly matched transverse current generates scattered wave
- ★ beating of scattered wave with pump light wave increases radiation pressure @ plasma frequency and further enhances density perturbation

Parametric instabilities in plasmas
one example: stimulated Raman Scattering

$$\partial_t^2 \mathbf{a}(\mathbf{r},t) - c^2\nabla_{\mathbf{r}}^2\mathbf{a}(\mathbf{r},t) + \omega_p^2(\mathbf{r},t)\mathbf{a}(\mathbf{r},t) = 0$$

$\mathbf{a}(\mathbf{r},t)$: normalized vector potential of electromagnetic field

$\omega_p^2(\mathbf{r},t)\mathbf{a}(\mathbf{r},t)$: transverse current in the plasma

$$\mathbf{a}(\mathbf{r},t) = \frac{e\mathbf{A}(\mathbf{r},t)}{m_e c^2}$$

$$a_0 = 0.86\lambda_0(\mu m)\sqrt{\frac{I}{10^{18}\mathrm{W/cm^2}}}$$

plasma response

$$c d_t \mathbf{p} = e\left(c\nabla\phi - \partial_t \mathbf{A} + \mathbf{v}\times\nabla\times\mathbf{A}\right) \qquad d_t = \partial_t + \mathbf{v}\cdot\nabla$$

$$\mathbf{v} = \mathbf{v}_q + \tilde{\mathbf{v}}$$

$\tilde{\mathbf{v}}$: longitudinal

$\mathbf{v}_q$: transverse

$$\mathbf{v}_q = c\mathbf{a} \qquad m_e\partial_t\mathbf{v}_q = -e\mathbf{E}$$

in the linear limit $|a| \ll 1$

$$d_t\tilde{\mathbf{p}} = e\nabla\tilde{\phi} - m_e\left[\left(\mathbf{v}_q\cdot\nabla\right)\mathbf{v}_q + c\mathbf{v}_q\times\left(\nabla\times\mathbf{a}\right)\right]$$

$$d_t\tilde{\mathbf{p}} = e\nabla\tilde{\phi} - m_e c^2\nabla(a^2/2)$$

$-m_e c^2\nabla(a^2/2)$: ponderomotive force

+ continuity equation

$\mathbf{a} = \mathbf{a}_0 + \tilde{\mathbf{a}}$ linearization + Fourier transform of r,t ⟹ dispersion relation

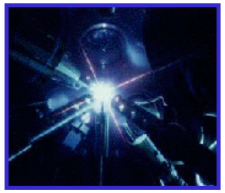
White light and parametric instabilities

- White Light = Intense radiation with large bandwidth
  - in astrophysics

  - nonlinear incoherent optics

  - laser plasma accelerators - short laser pulses

  - Inertial Confinement Fusion



- Bandwidth effects on parametric instabilities

G. E. Vekshtein, G. M. Zaslavsky, Sov. Phys. Dokl. 12, 34 (1967)
G. M. Zaslavsky, V. S. Zakharov, Sov. Phys. Tech. Phys. 12, 7 (1967)
E. Valeo, C. Oberman, Phys. Rev. Lett. 30, 1035 (1973)
J. J. Thomson, W. L. Kruer, S. E. Bodner, J. S. DeGroot, Phys. Fluids 17, 849 (1974)

M. Mitchel, M. Segev, Nature 387, 880 (1997)
D. N. Christodoulies et al, Phys. Rev. Lett. 78, 646 (1997)
H. Buljan et al, Phys. Rev. E 68, 036607 (2003)
D. A. Anderson et al, Phys. Rev. E 70, 026603 (2004)

- Random Phase Approximation (incoherent light) vs Fixed Phase (monochromatic light)

Plan to build a theoretical framework capable of describing white light parametric instabilities

Nonlinear full wave equation for e.m. waves in plasma
[Klein-Gordon like field]

Generalized Wigner-Moyal formalism

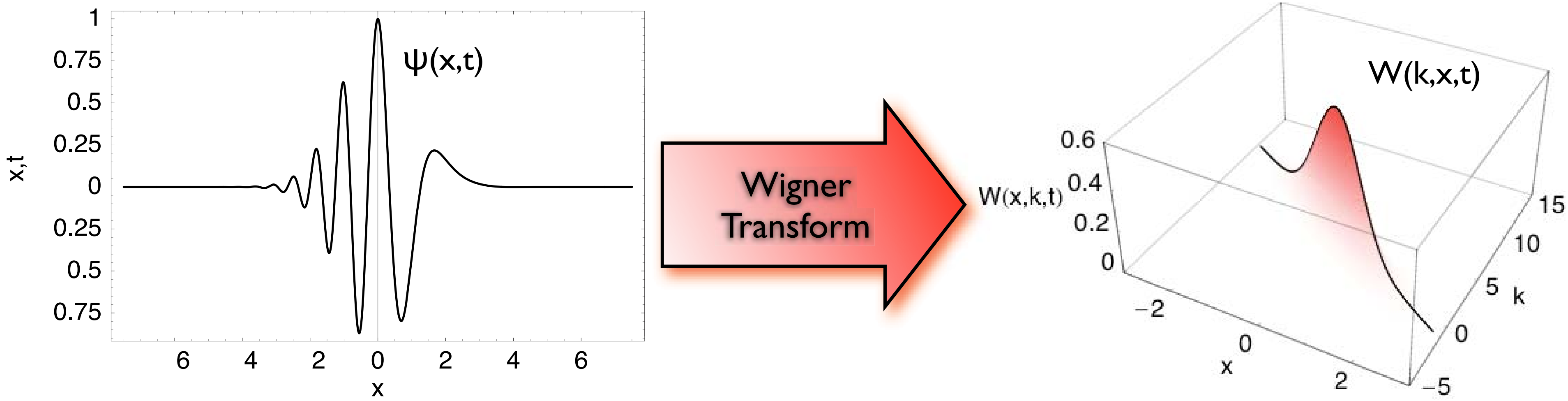
Statistical description of electromagnetic waves
[set of kinetic equations for Wigner functions]

Coupling with plasma through ponderomotive force
[electron plasma waves, ion acoustic waves]

Generalized dispersion relation for white light parametric instabilities in plasmas

Inspiration from Quantum Mechanics

In the 1930's Wigner proposed a statistical representation of Quantum Mechanics formally equivalent to Schrödinger equation



- Fields are represented by distribution of quasi-particles: Wigner distribution
  - ✳ Quasi-particles described by k (momentum) and r (position)
  - ✳ Dynamics described by transport equations

Main properties of Wigner distribution

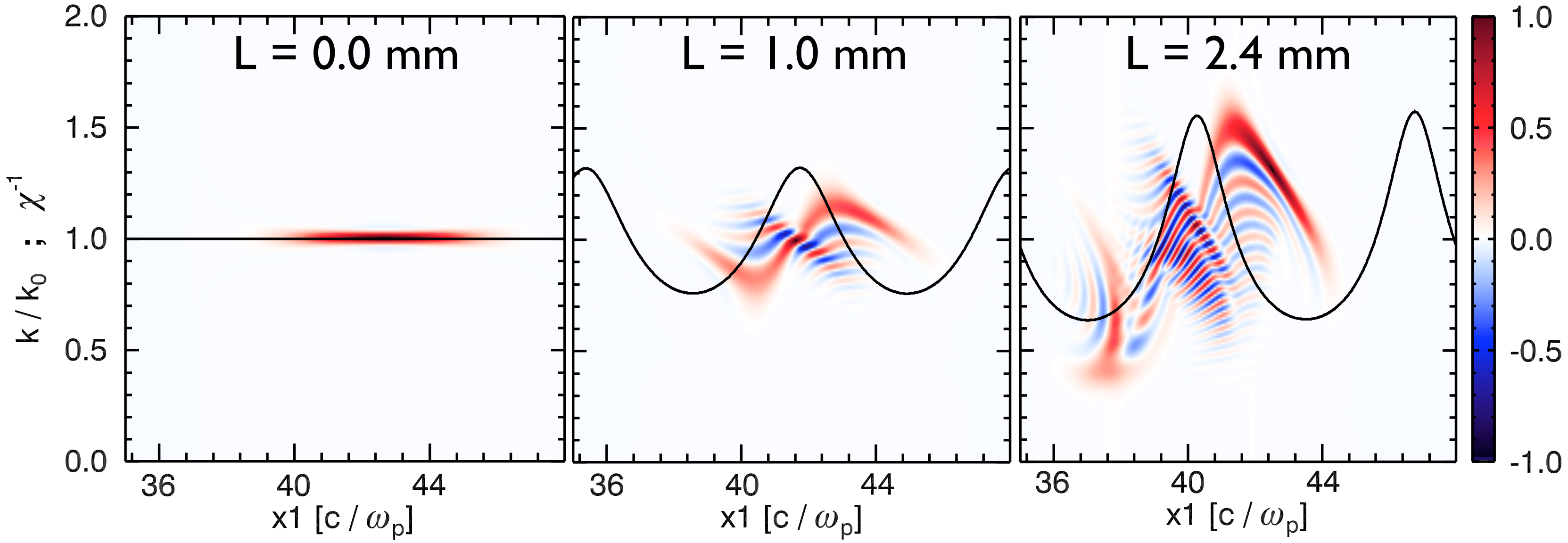
$$W(\mathbf{k},\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d\mathbf{s}\exp(i\mathbf{k}\cdot s)\psi(\mathbf{r}-\mathbf{s}/2,t)\cdot\psi(\mathbf{r}+\mathbf{s}/2,t)^*$$

Wigner distribution

Spectrum of autocorrelation function

$$|\psi(\mathbf{k})^2| = \int d\mathbf{r}W(\mathbf{k},\mathbf{r},t) \qquad |\psi(\mathbf{r})^2| = \int d\mathbf{k}W(\mathbf{k},\mathbf{r},t)$$

Wigner distribution of laser field propagating in electron plasma wave



Schrödinger vs Klein-Gordon

Schrödinger Equation:

$$2i\omega_{p0}\partial_t\tilde{\phi} = -c^2\overrightarrow{\nabla}_{\mathbf{r}}^2\tilde{\phi} + \tilde{\omega}_p^2\tilde{\phi}$$

Valid on the paraxial wave approximation for e.m. waves

Formally equivalent

Wigner-Moyal Equation:

$$\partial_t W(\mathbf{k},\mathbf{r},t) = 2\omega_k(\mathbf{k},\mathbf{r},t)\hat{S}^{cl}W(\mathbf{k},\mathbf{r},t)$$

$$\hat{S}^{cl} = \sin\left[\frac{1}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}}\cdot\overrightarrow{\nabla}_{\mathbf{k}} - \overleftarrow{\nabla}_{\mathbf{k}}\cdot\overrightarrow{\nabla}_{\mathbf{r}}\right)\right]$$

Local dispersion relation for e.m. waves in plasma

Schrödinger, single mode problem

Full wave equation for normalized vector potential of e.m. wave:

$$\partial_t^2\mathbf{a}(\mathbf{r},t) - c^2\nabla_{\mathbf{r}}^2\mathbf{a}(\mathbf{r},t) + \omega_p^2(\mathbf{r},t)\mathbf{a}(\mathbf{r},t) = 0$$

Nonlinear variable mass term

2nd order in time

Formally equivalent

Generalized set of Wigner-Moyal equations

Klein-Gordon, two mode problem

Generalized Wigner-Moyal Equations I

Wave Equation

$$\partial_t^2 \mathbf{a}(\mathbf{r},t) - c^2\nabla_{\mathbf{r}}^2 \mathbf{a}(\mathbf{r},t) + \left[\omega_{p0}^2 + \tilde{\omega}_p^2(\mathbf{r},t)\right]\mathbf{a}(\mathbf{r},t) = 0$$

a (r,t) = +

Feshbach-Villars prescription[1]

$$\boldsymbol{\phi}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{a}(\mathbf{r},t) + i\frac{\partial_t \mathbf{a}(\mathbf{r},t)}{\omega_{p0}}\right], \quad \boldsymbol{\chi}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{a}(\mathbf{r},t) - i\frac{\partial_t \mathbf{a}(\mathbf{r},t)}{\omega_{p0}}\right]$$

Two first order equations in time

$$\begin{cases} i\partial_t\boldsymbol{\phi} = -\dfrac{c^2\overrightarrow{\nabla}_{\mathbf{r}}^2}{2\omega_{p0}}(\boldsymbol{\phi}+\boldsymbol{\chi}) + \dfrac{\tilde{\omega}_p^2}{2\omega_{p0}}(\boldsymbol{\phi}+\boldsymbol{\chi}) + \omega_{p0}\boldsymbol{\phi} \\ i\partial_t\boldsymbol{\chi} = \dfrac{c^2\overrightarrow{\nabla}_{\mathbf{r}}^2}{2\omega_{p0}}(\boldsymbol{\phi}+\boldsymbol{\chi}) - \dfrac{\tilde{\omega}_p^2}{2\omega_{p0}}(\boldsymbol{\phi}+\boldsymbol{\chi}) - \omega_{p0}\boldsymbol{\chi} \end{cases}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\chi} \end{bmatrix}$$

Matrix equation for Ψ

$$i\partial_t\boldsymbol{\Psi} = \frac{(\tau_3 + i\tau_2)}{2\omega_{p0}}\left(-c^2\overrightarrow{\nabla}_{\mathbf{r}}^2 + \tilde{\omega}_p^2\right)\boldsymbol{\Psi} + \omega_{p0}\tau_3\boldsymbol{\Psi}$$

τ_i are the Pauli matrices

[1] H. Feshbach, F. Villars, Rev. Mod. Phys. 30, 24 (1958).

Generalized Wigner-Moyal Equations II

Wigner Transform

$$W_{\mathbf{f}\cdot\mathbf{g}}(\mathbf{k},\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \mathrm{d}\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}} \mathbf{f}^*\left(\mathbf{r}+\frac{\mathbf{y}}{2}\right)\cdot\mathbf{g}\left(\mathbf{r}-\frac{\mathbf{y}}{2}\right)$$

Wigner Matrix

$$W = \begin{bmatrix} W_{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} & -W^*_{\boldsymbol{\phi}\cdot\boldsymbol{\chi}} \\ W_{\boldsymbol{\phi}\cdot\boldsymbol{\chi}} & -W_{\boldsymbol{\chi}\cdot\boldsymbol{\chi}} \end{bmatrix}$$

$W_0 = W_{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} - W_{\boldsymbol{\chi}\cdot\boldsymbol{\chi}}$ $\quad W_1 = 2\mathrm{Im}\left[W_{\boldsymbol{\phi}\cdot\boldsymbol{\chi}}\right]$

$W_2 = 2\mathrm{Re}\left[W_{\boldsymbol{\phi}\cdot\boldsymbol{\chi}}\right]$ $\quad W_3 = W_{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} + W_{\boldsymbol{\chi}\cdot\boldsymbol{\chi}}$

Transport Equations

$$\partial_t W(\mathbf{k},\mathbf{r},t) + (\hat{D}-\hat{S})\tfrac{1}{2}\left\{\tau_3 + i\tau_2, W(\mathbf{k},\mathbf{r},t)\right\} +$$
$$i\left[\mathcal{H}_0(\hat{\mathbf{k}}) + \hat{C}\right]\tfrac{1}{2}\left[(\tau_3+i\tau_2), W(\mathbf{k},\mathbf{r},t)\right] + i\omega_{p0}\left[\tau_3, W(\mathbf{k},\mathbf{r},t)\right] = 0$$

Operators

$$\hat{S} = \frac{\tilde{\omega}_p^2(\mathbf{r},t)}{\omega_{p0}}\sin\left(\frac{1}{2}\overleftarrow{\partial}_{\mathbf{r}}\cdot\overrightarrow{\partial}_{\mathbf{k}}\right), \quad \hat{C} = \frac{\tilde{\omega}_p^2(\mathbf{r},t)}{\omega_{p0}}\cos\left(\frac{1}{2}\overleftarrow{\partial}_{\mathbf{r}}\cdot\overrightarrow{\partial}_{\mathbf{k}}\right), \quad \hat{D} = \frac{c^2}{\omega_{p0}}\mathbf{k}\cdot\overrightarrow{\nabla}_{\mathbf{r}}$$

Generalized Wigner-Moyal Equations

Statistical Description of Light

$$\partial_t W_0 + \left(\hat{D}-\hat{S}\right)(W_2+W_3) = 0$$

$$\partial_t W_1 - \left[\frac{c^2}{\omega_{p0}}\left(\mathbf{k}^2 - \frac{\overrightarrow{\nabla}^2_{\mathbf{r}}}{4}\right) + \hat{C}\right](W_2+W_3) - \frac{2}{\omega_{p0}}W_2 = 0$$

$$\partial_t W_2 - \left(\hat{D}-\hat{S}\right)W_0 + \left[\frac{c^2}{\omega_{p0}}\left(\mathbf{k}^2 - \frac{\overrightarrow{\nabla}^2_{\mathbf{r}}}{4}\right) + \hat{C}\right]W_1 + \frac{2}{\omega_{p0}}W_1 = 0$$

$$\partial_t W_3 + \left(\hat{D}-\hat{S}\right)W_0 - \left[\frac{c^2}{\omega_{p0}}\left(\mathbf{k}^2 - \frac{\overrightarrow{\nabla}^2_{\mathbf{r}}}{4}\right) + \hat{C}\right]W_1 = 0$$

J. P. Santos, L. O. Silva, J. Math. Phys, 46, 102901 (2005).

Coupling with electron plasma waves

$$\nabla \cdot \mathbf{E}_{\|}(\mathbf{r},t) = -\frac{4\pi e}{n_0}\left[\frac{n_e(\mathbf{r},t)}{n_0} - 1\right]$$

$$\gamma(\mathbf{r},t) = \sqrt{1 + \mathbf{a}^2(\mathbf{r},t) + \frac{\mathbf{p}_{\|}^2(\mathbf{r},t)}{m_e^2 c^2}}$$

$$\partial_t \mathbf{p}_{\|}(\mathbf{r},t) = -e\mathbf{E}_{\|} - m_e c^2 \nabla\gamma(\mathbf{r},t)$$

$$\partial_t n_e(\mathbf{r},t) + \nabla \cdot \left[\frac{n_e(\mathbf{r},t)\mathbf{p}_{\|}}{m_e \gamma(\mathbf{r},t)}\right] = 0$$

Linearization

$a(r,t) = a_p + \tilde{a}$

$$\left(\partial_t^2 + \frac{\omega_{p0}^2}{\gamma_0}\right)\tilde{n} = \frac{n_0 c^2}{\gamma_0^2}\nabla_{\mathbf{r}}^2 \mathrm{Re}\left[\mathbf{a}_p \cdot \tilde{\mathbf{a}}^*\right]$$

Nonlinear relativistic effect

Ponderomotive force term

Goal

Field evolution determined by statistical equations for light

Why?

Usual analytical formalism cannot take into account arbitrary light bandwidth/statistics

Perturbation theory for photon kinetics

$$W_{\phi\cdot\phi} = \frac{1}{4}\left(1+\frac{\sqrt{\gamma_0}\omega_{k0}(\mathbf{k})}{\omega_{p0}}\right)^2\rho_0(\mathbf{k}) + \delta\tilde{W}_{\phi\cdot\phi}$$

$$W_{\phi\cdot\chi} = \frac{1}{4}\left(1-\frac{\gamma_0\omega_{k0}^2(\mathbf{k})}{\omega_{p0}^2}\right)\rho_0(\mathbf{k}) + \delta\tilde{W}_{\phi\cdot\chi}$$

$$W_{\chi\cdot\chi} = \frac{1}{4}\left(1-\frac{\sqrt{\gamma_0}\omega_{k0}(\mathbf{k})}{\omega_{p0}}\right)^2\rho_0(\mathbf{k}) + \delta\tilde{W}_{\chi\cdot\chi}$$

Incident pump light ≡ 0-order

$$W_0 = \frac{\sqrt{\gamma_0}\omega_{k0}(\mathbf{k})}{\omega_{p0}}\rho_0(\mathbf{k}) + \delta\tilde{W}_0$$

$$W_1 = \delta\tilde{W}_1$$

$$W_2 = -\frac{\gamma_0\mathbf{k}^2c^2}{2\omega_{p0}^2}\rho_0(\mathbf{k}) + \delta\tilde{W}_2$$

$$W_3 = \frac{[\omega_{p0}^2+\omega_{k0}^2(\mathbf{k})]\gamma_0}{2\omega_{p0}^2}\rho_0(\mathbf{k}) + \delta\tilde{W}_3$$

$\mathcal{F}$

$$\mathcal{F}\mathrm{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}^*] = \frac{\tilde{\omega}_p^2(\omega_L,\mathbf{k}_L)}{4\gamma_0^2}\int_{\mathbb{R}^3}\mathrm{d}\mathbf{k}\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D^-(\mathbf{k})}+\frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D^+(\mathbf{k})}\right]$$

$$\tilde{\omega}_p^2(\omega_L,\mathbf{k}_L) = \frac{\omega_{p0}^2}{\gamma_0}\left[\frac{\mathbf{k}_L^2c^2}{\omega_L^2-\frac{\omega_{p0}^2}{\gamma_0}}-1\right]\mathcal{F}\mathrm{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}^*]$$

Generalized dispersion relation I*

$$1 = \frac{\omega_{p0}^2}{4\gamma_0^3}\left[\frac{\mathbf{k}_L^2 c^2}{\omega_L^2 - \frac{\omega_{p0}^2}{\gamma_0}} - 1\right]\int_{\mathbb{R}^3} d\mathbf{k}\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D^-(\mathbf{k})} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D^+(\mathbf{k})}\right]$$

Arbitrary distribution of photons

$$D^\pm = \omega_L^2 \mp 2\left[\mathbf{k}\cdot\mathbf{k}_L c^2 - \omega_L\omega_{k0}\left(\mathbf{k}\mp\frac{\mathbf{k}_L}{2}\right)\right]$$

Incident Monochromatic Light

with wavenumber k_0

$$\rho_0(\mathbf{k}) = a_0^2\delta(\mathbf{k}-\mathbf{k}_0)$$

$$1 = \frac{a_0^2\omega_{p0}^2}{4\gamma_0^3}\left[\frac{\mathbf{k}_L^2 c^2}{\omega_L^2 - \frac{\omega_{p0}^2}{\gamma_0}} - 1\right]\left[\frac{1}{\omega_L^2 + 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2} + \frac{1}{\omega_L^2 - 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2}\right]$$

Identical to
Kruer, The physics of laser plasmas interactions(1988)

*J. P. Santos, L. O. Silva, R. BIngham, Phys. Rev. Lett. 98, 235001 (2007)

Generalized dispersion relation II

$$= \frac{\omega_{p0}^2}{4\gamma_0^3}\left[\frac{\mathbf{k}_L^2 c^2}{\omega_L^2 - \frac{\omega_{p02}}{\gamma_0}} - 1\right] \int d\mathbf{k}\rho_0(\mathbf{k}) \left[\frac{1}{\omega_L^2 + 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2} + \frac{1}{\omega_L^2 - 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2}\right]$$

Weighted average of plane wave plasma response

Incident Monochromatic Light

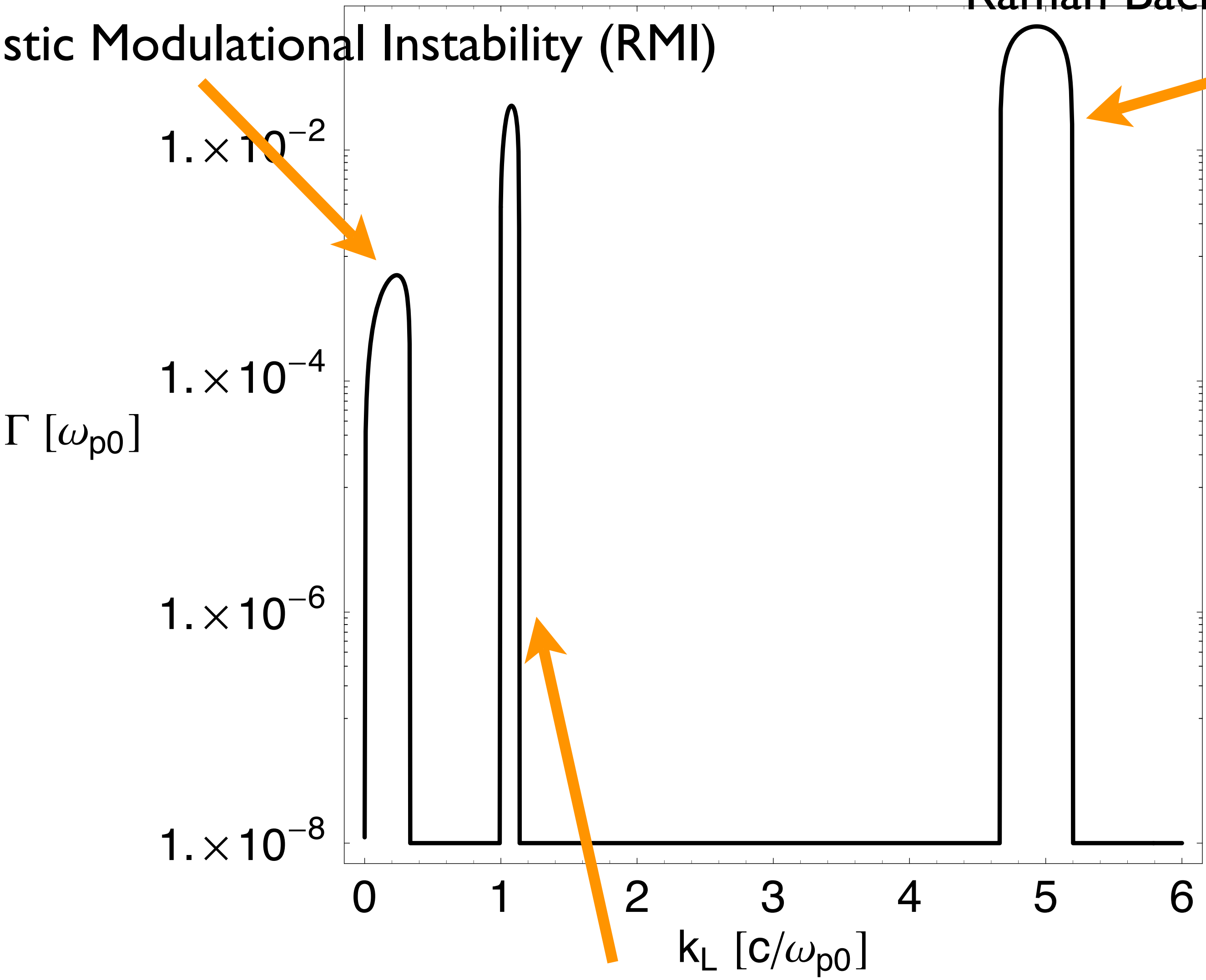
with wavenumber k_0

$$1 = \frac{a_0^2\omega_{p0}^2}{4\gamma_0^3}\left[\frac{\mathbf{k}_L^2 c^2}{\omega_L^2 - \frac{\omega_{p0}^2}{\gamma_0}} - 1\right] \left[\frac{1}{\omega_L^2 + 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2} + \frac{1}{\omega_L^2 - 2(\mathbf{k}_0\cdot\mathbf{k}_L c^2 - \omega_0\omega_L) - \mathbf{k}_L^2 c^2}\right]$$

Standard monochromatic pump results

Raman Backscattering (RBS)

Relativistic Modulational Instability (RMI)



$a_0 = 0.1$

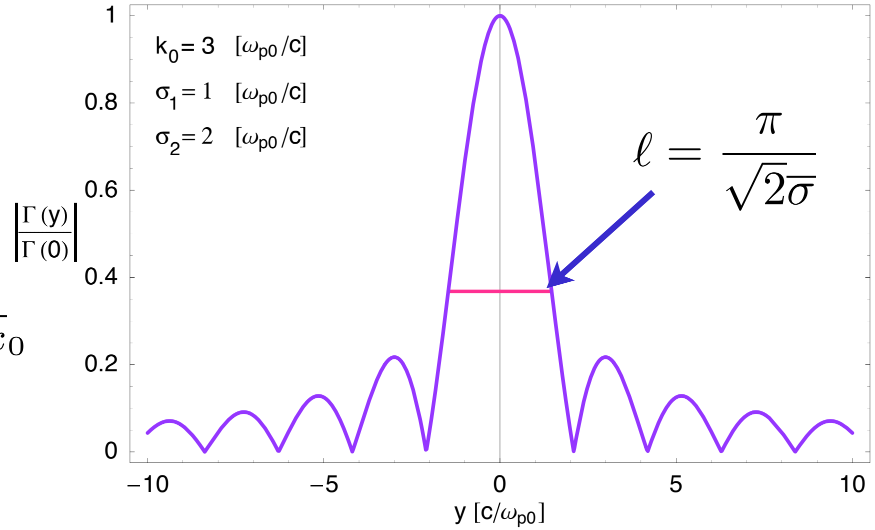
$k_0/k_p = 3$

Raman Forward Scattering (RFS)

Statistics of white light

Incident light with stochastic phase Ψ(x), described by autocorrelation function

$$\Gamma(y) = \left\langle e^{-i\psi\left(x+\frac{y}{2}\right)+i\psi\left(x-\frac{y}{2}\right)} \right\rangle = \frac{a_0^2 \sin y\overline{\sigma}}{y\overline{\sigma}} e^{-iy\overline{k}_0}$$

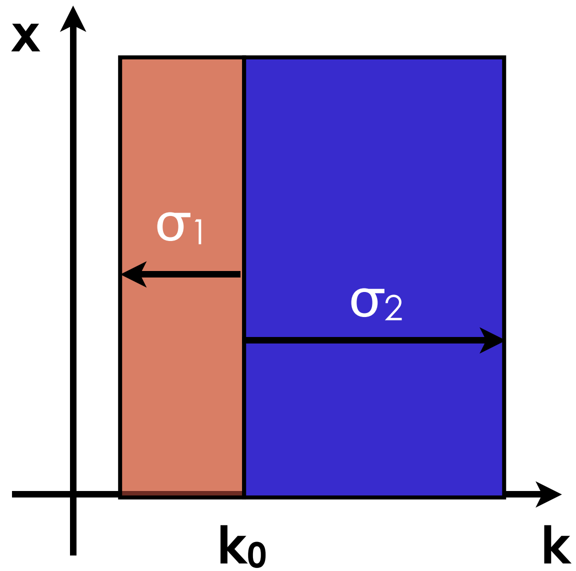


The corresponding zero-order Wigner distribution is a waterbag distribution function:

$$\rho_0(\mathbf{k}) = \frac{a_0^2}{\sigma_1 + \sigma_2}\left[\Theta(k - k_0 + \sigma_1) - \Theta(k - k_0 - \sigma_2)\right]$$

$$\overline{\sigma} = \frac{\sigma_1 + \sigma_2}{2}$$

$$\overline{k}_0 = k_0 + \frac{\sigma_2 - \sigma_1}{2}$$



Analytical results I

For a waterbag distribution function, analytical dispersion relation can be calculated

$$1 = \frac{a0^2\omega_{p0}^2}{4\gamma_0^3}\frac{1}{c^2k_L(\sigma_1+\sigma_2)}\left[\frac{k_L^2c^2}{\omega_L^2-\frac{\omega_{p0}^2}{\gamma_0}}-1\right]$$

$$\left[\frac{2ck_L\omega_L}{\sqrt{Q_0}}\left(\text{arctanh}\ b^+ + \text{arctanh}\ b^-\right)+\frac{k_L^2c^2}{k_L^2c^2-\omega_L^2}\log\left(\frac{D_1^-D_2^+}{D_1^+D_2^-}\right)\right]$$

Term contributing for instability

$$b^\pm = \frac{2\sqrt{Q_0}\left[c(\sigma_1+\sigma_2)+\omega_{01}-\omega_{02}\right]}{\frac{Q_0}{\omega_L+ck_L}-\frac{(\omega_L+ck_L)Q^\pm}{c^2k_L^2}}$$

$$\omega_{0i} = \sqrt{\left[k_0+(-1)^i\sigma_i\right]^2c^2+\frac{\omega_{p0}^2}{\gamma_0}}$$

$$D_i^\pm = \omega_L^2 \mp 2\left[(k_0+(-1)^i\sigma_i)k_Lc^2-\omega_{0i}\omega_L\right]-k_L^2c^2$$

$$Q^\pm = \left[D_1^\pm+(ck_L-\omega_L)(\omega_L-2\omega_{01})\right]\left[D_2^\pm+(ck_L-\omega_L)(\omega_L-2\omega_{02})\right]$$

$$Q^0 = (k_L^2c^2-\omega_L^2)\left(k_L^2c^2-\omega_L^2+\frac{4\omega_{p0}^2}{\gamma_0}\right)$$

Analytical results II

Asymmetric waterbag

RFS

$k_L = \omega_{p0}/c\sqrt{\gamma_0}$

$\omega_L = \omega_{p0}/\sqrt{\gamma_0} + \delta$

$$\Gamma_{\mathrm{RFS}} = \frac{a_0}{2\sqrt{2}\gamma_0^2\sqrt{(k_0-\sigma_1)(k_0+\sigma_2)}}$$

RBS

$k_L = 2(k_0+\sigma_2) - \omega_{p0}/c\sqrt{\gamma_0}.$

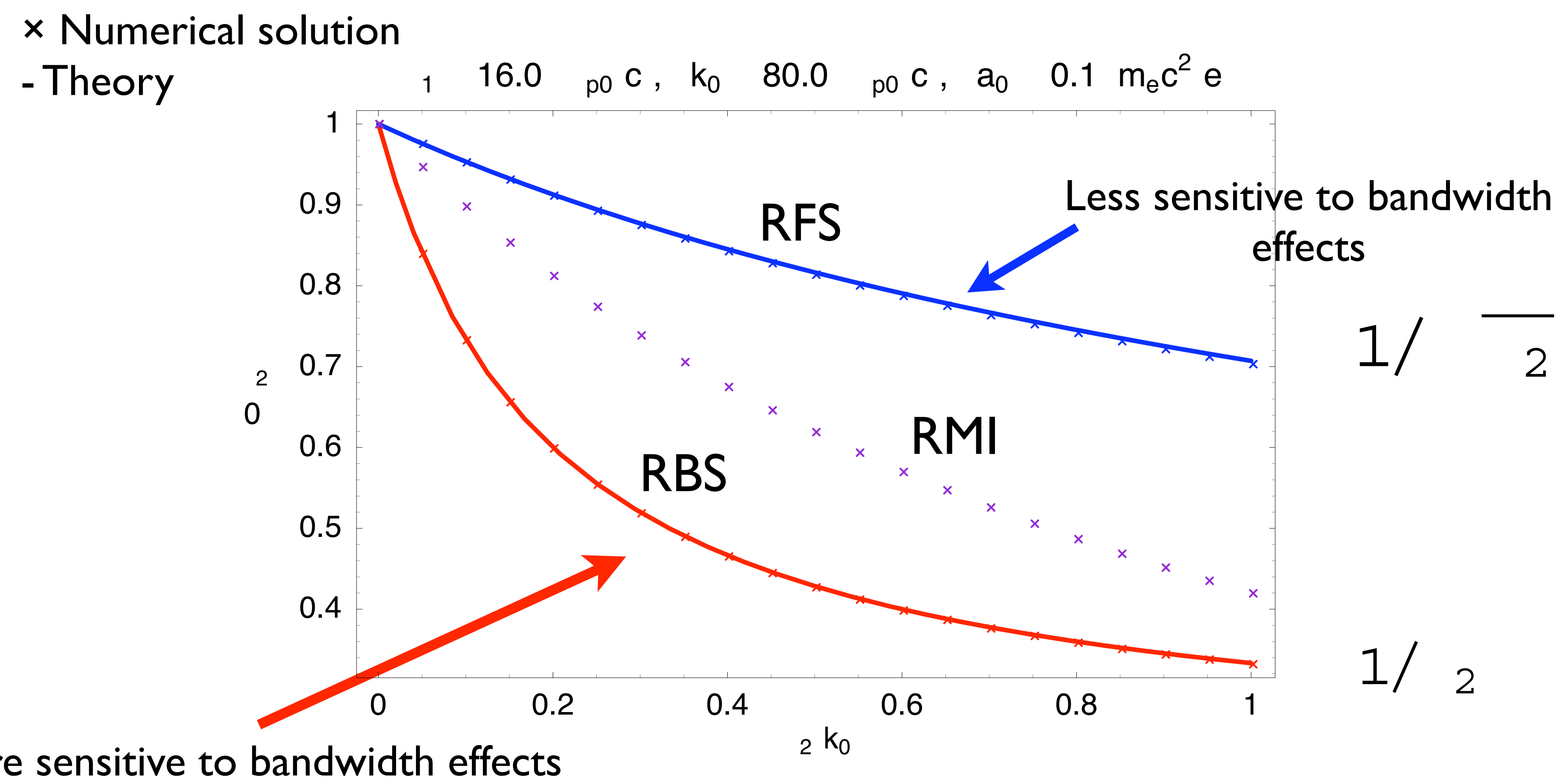
$\omega_L = \omega_{p0}/\sqrt{\gamma_0} + \delta$

$$\Gamma_{\mathrm{RBS}} = \frac{\pi a_0^2}{8\gamma_0^{5/2}} \frac{k_0+\sigma_2}{\sigma_1+\sigma_2} \frac{1}{1+\frac{a_0^2}{8\gamma_0^{5/2}}\frac{k_0+\sigma_2}{(\sigma_1+\sigma_2)^2}}$$

For Lorentzian, similar dependences with bandwidth

Comparison between different instability regimes I

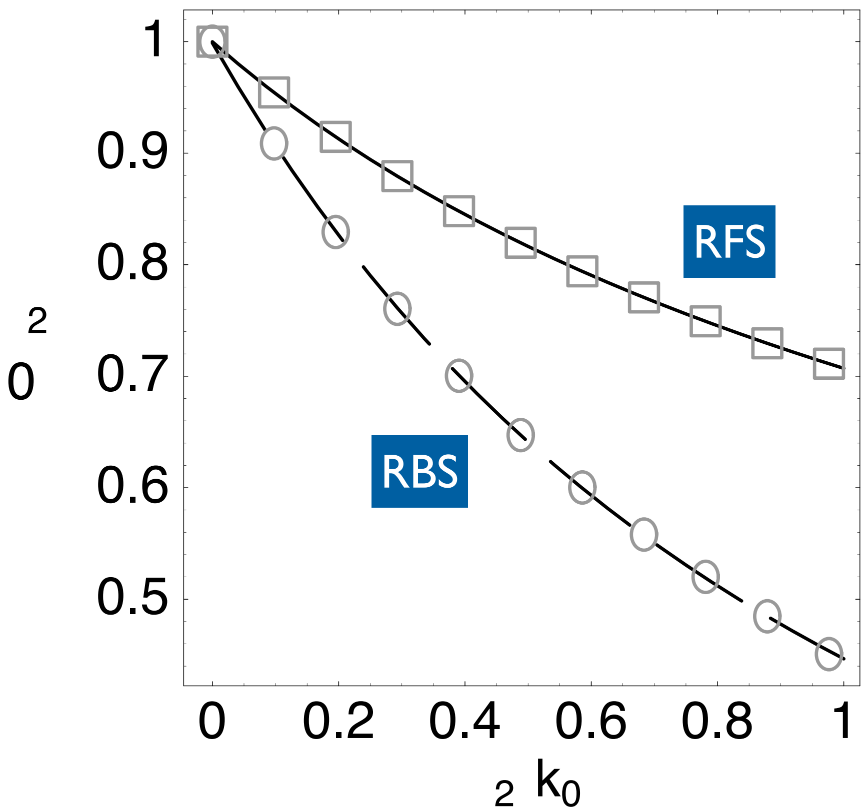
Asymmetric waterbag



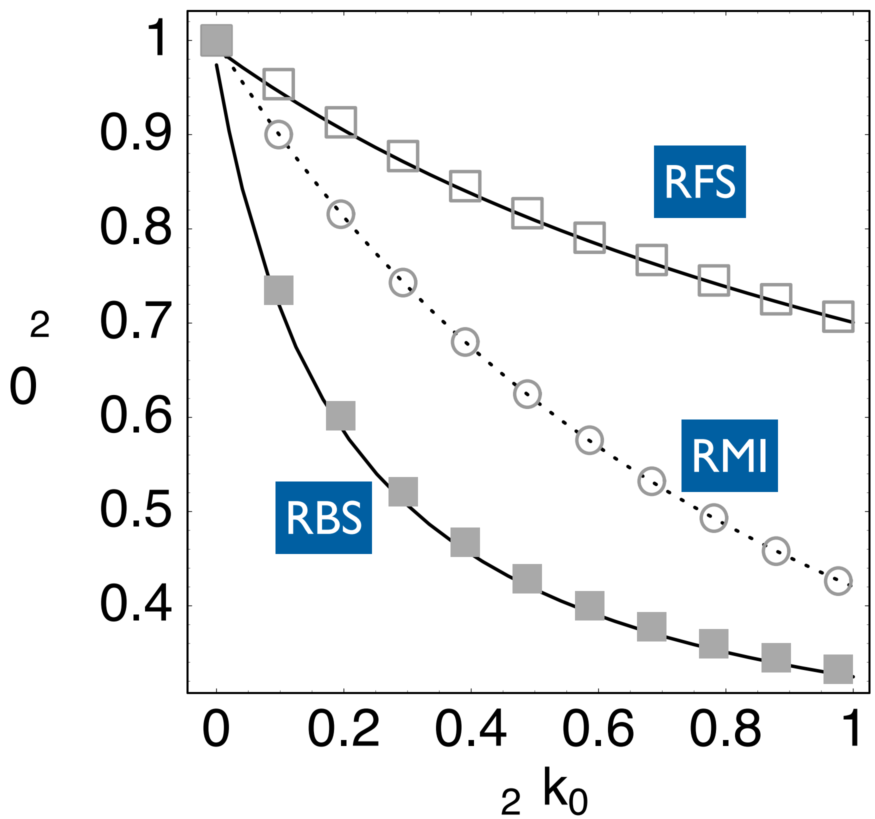
Comparison between different instability regimes II

Asymmetric waterbag

a0 = 3



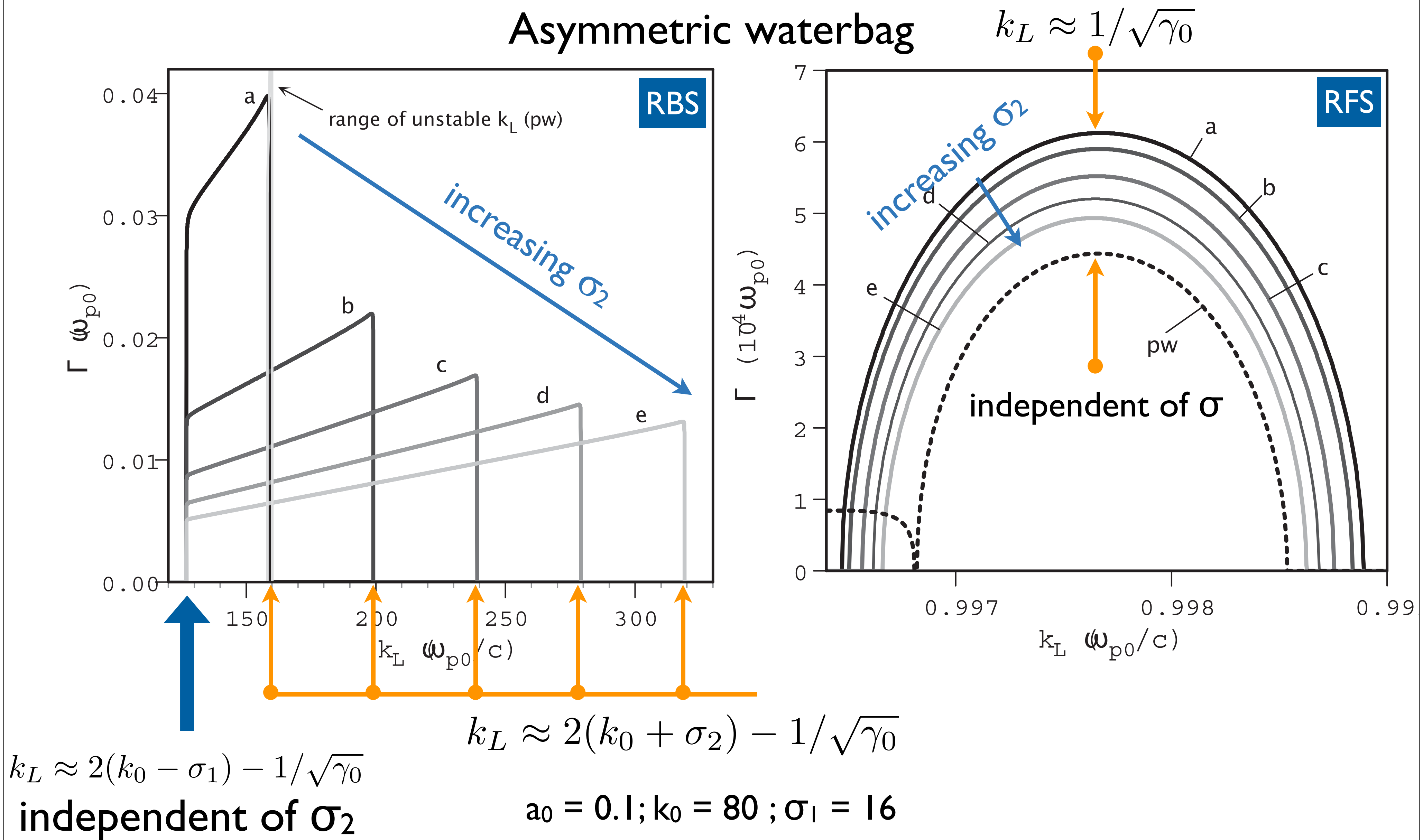
a0 = 0.1



$k_0 = 80$; $\sigma_l = 16$

Range of unstable wavenumbers

Asymmetric waterbag



$a_0 = 0.1$; $k_0 = 80$; $\sigma_1 = 16$

Self-focusing or transverse modulation instability
- monochromatic beam -

The generalized dispersion relation (valid for all scattering angles) is employed

Incident Monochromatic Light
with wavenumber k_0

$$\rho_0(\mathbf{k}) = a_0^2 \delta(\mathbf{k} - \mathbf{k}_0)$$

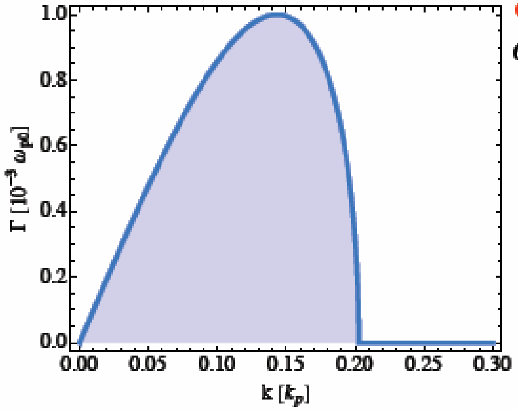
self-focusing

$$\mathbf{k}_L \cdot \mathbf{k}_0 = 0$$

Dispersion relation for monochromatic self-focusing instability

$$-2\alpha \left(\omega_L^2 - \mathbf{k}_L^2\right)\left(1 - \gamma_0\left(\omega_L^2 - \mathbf{k}_L^2\right)\right) + \left(\omega_L^2 \gamma_0 - 1\right)\left(\left(\omega_L^2 - \mathbf{k}_L^2\right)^2 - 4\omega_0^2\omega_L^2\right) = 0$$

$$\alpha = 2a_0^2/\gamma_0^3$$



$$\gamma_0(\mathbf{k}_L - \omega_L^2) \ll 1$$ long wavelengths

$$\omega_L^2 \gamma_0 \ll 1$$ underdense plasma

$$-2\alpha \left(\omega_L^2 - \mathbf{k}_L^2\right) - \left(\omega_L^2 - \mathbf{k}_L^2\right)^2 + 4\omega_0^2\omega_L^2 = 0$$

Identical to
C. E. Max et al, Phys. Rev. lett. 33, 209 (1974)

Transverse modulation instability broadband dispersion relation

Incident Light

propagation direction is x

with transverse spread in wavenumbers along k_z

$$\rho(\mathbf{k}) = a_0^2 \delta(k_x - k_0)\delta(k_y) f(k_z)$$

$$\omega_0^2 = k_0^2 + \frac{1}{\gamma_0}$$

$$1 = -\alpha \left(\frac{\mathbf{k}_L^2 \gamma_0}{\omega_L^2 \gamma_0 - 1} - 1 \right) \int dk_z f(k_z) \left(\frac{1}{D_\perp^+} + \frac{1}{D_\perp^-} \right)$$

$$D_\perp^\pm = (\mathbf{k}_L^2 - \omega_L^2) \mp 2\omega_L \sqrt{\omega_0^2 + k_z^2} \pm 2k_L k_z$$

$\gamma_0 (\mathbf{k}_L - \omega_L^2) \ll 1$ long wavelengths

$\omega_L^2 \gamma_0 \ll 1$ underdense plasma

$k_z^2 \ll \omega_0^2$ "small" transverse spread

$$1 = \alpha \frac{\omega_0}{\omega_L} \int dk_z f(k_z) \left(\frac{1}{d_1 + d_2 + (k_z - d_3)^2} + \frac{1}{d_1 - d_2 + (k_z - d_3)^2} \right)$$

$$d_1 = \frac{\omega_0}{\omega_L} \left(2\omega_L \omega_0 - \frac{\mathbf{k}_L^2 \omega_0}{\omega_L} \right) \qquad d_2 = \frac{\omega_0}{\omega_L} (\mathbf{k}_L^2 - \omega_L^2) \qquad d_3 = \frac{\omega_0}{\omega_L} k_L$$

Transverse modulation instability broadband dispersion relation - waterbag

Waterbag distribution in k_z (width = 2 k_{z0})

$$f(k_z) = \frac{1}{2k_{z0}}\left(\Theta(k_z + k_{z0} - \Theta(k_z - k_{z0})\right)$$

$$1 = \frac{\alpha}{2k_{z0}}\frac{\omega_0}{\omega_L}\left(\frac{1}{d_+}\left(\arctan\frac{d_3 + k_{z0}}{d_+} - \arctan\frac{d_3 - k_{z0}}{d_+}\right) + \frac{1}{d_-}\left(\arctan\frac{d_3 + k_{z0}}{d_-} - \arctan\frac{d_3 - k_{z0}}{d_-}\right)\right)$$

$$d_1 = \frac{\omega_0}{\omega_L}\left(2\omega_L\omega_0 - \frac{\mathbf{k}_L^2\omega_0}{\omega_L}\right) \qquad d_2 = \frac{\omega_0}{\omega_L}\left(\mathbf{k}_L^2 - \omega_L^2\right) \qquad d_3 = \frac{\omega_0}{\omega_L}k_L \qquad d_+ = \sqrt{d_1 + d_2} \quad d_- = \sqrt{d_1 - d_2}$$

Summary

⊙ Generalized Wigner-Moyal statistical theory for photons formally equivalent to full nonlinear wave equation for e.m. waves (no paraxial wave approximation)

⊙ Generalized dispersion relation for coupling with electron plasma waves for light with arbitrary statistics (valid for all angles, all intensities)

⊙ White Light Effects

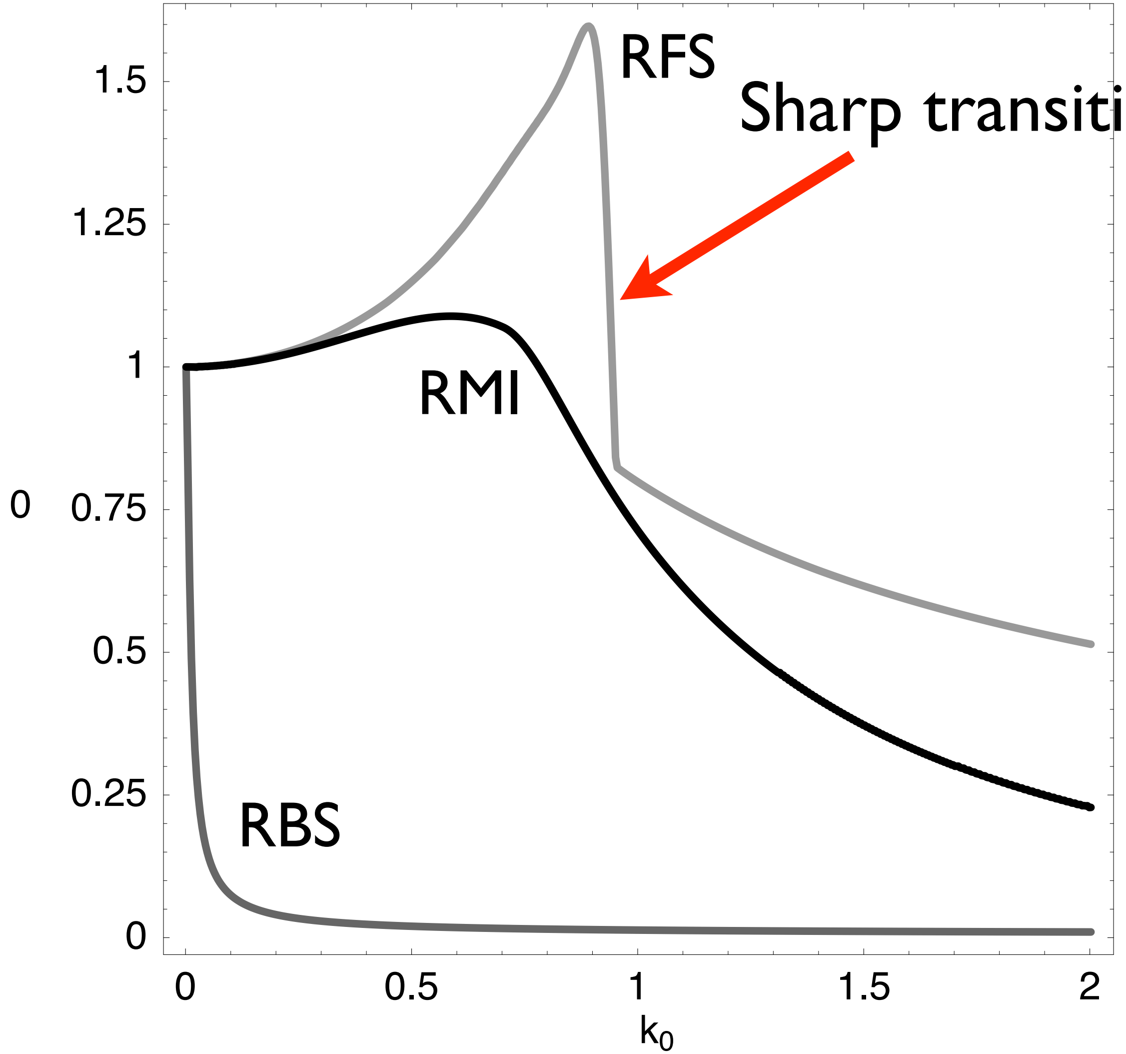
- ✳ Growth rates for Raman Forward Scattering and Relativistic Modulational Instability can increase with increasing bandwidth
- ✳ Range of unstable wavenumbers increases with increasing bandwidth for Raman Forward/Backscattering
- ✳ Strong qualitative variations due to merging of Raman Forward Scattering and Raman Backscattering when bandwidth increases

Future directions

- Multidimensional analysis of coupling with electron dynamics (e.g. transverse modulation instability - relativistic effects)
- Generalize formalism to describe coupling with ion acoustic waves (Stimulated Brillouin Scattering)
- Describe fluctuations of the plasma via Wigner-Moyal equation with the goal of including zero-order statistics of plasma fluctuations
- Identify absolute vs convective nature of the instabilities derived from the generalized dispersion relation
- Spatial-temporal theory
- Comparison with particle-in-cell simulations and experiments

Growth rate goes up with increasing bandwidth

Symmetric waterbag



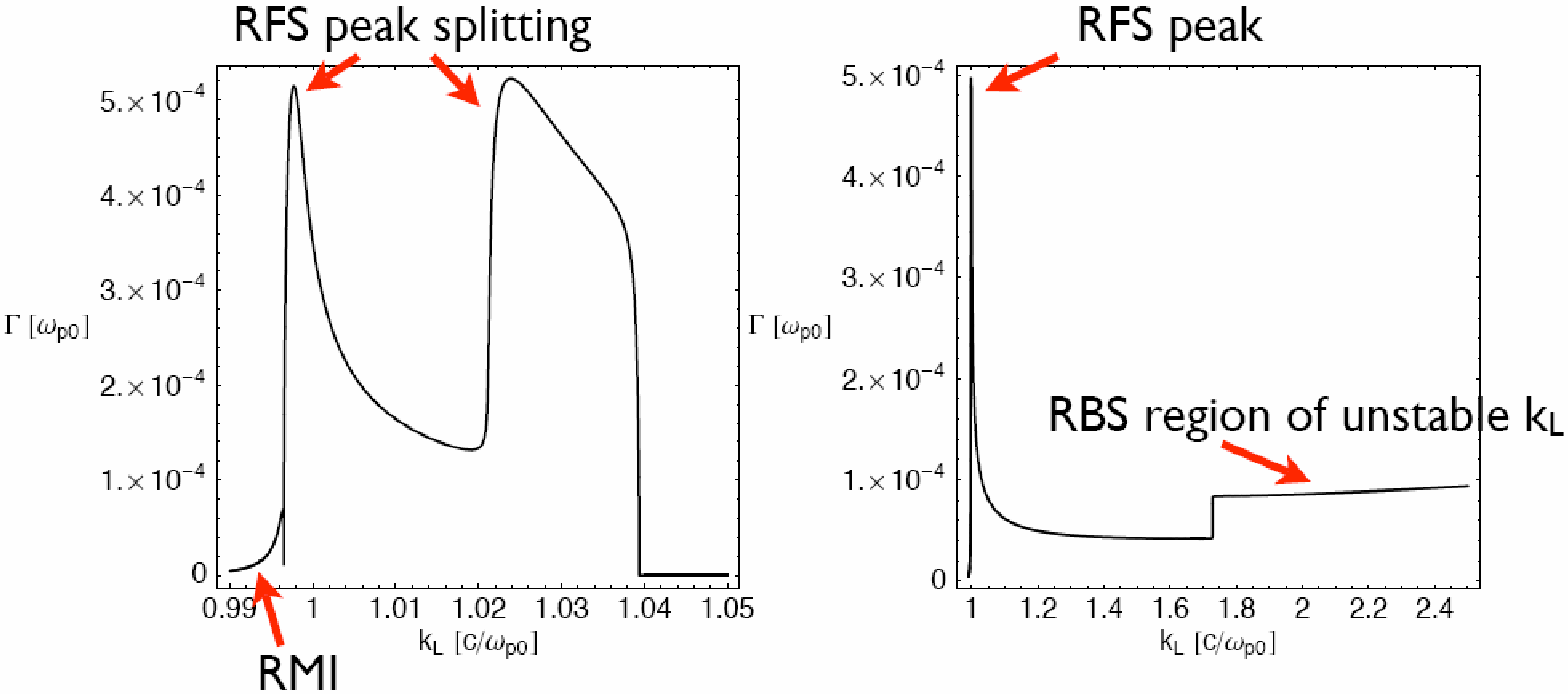
Sharp transition @ k_0

$a_0 = 0.1$

$k_0/k_p = 80$

RFS peak splitting and merging with RBS

$a_0 = 0.1 \quad k_0/k_p = 80$



$\sigma < k_0$ $\sigma > k_0$

Range of unstable k_L in RBS increases with bandwidth $\approx [2(k_0 - \sigma), 2(k_0 + \sigma)]$