Quantum Field Theory in Non-Stationary States
(*QFT is Inexhaustible*)

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Statement of the problem

- prepare a system at time $t = 0$ in the ground state $|\psi_0\rangle$ of a (regularised) QFT with hamiltonian H_0
- for time $t > 0$ evolve *unitarily* with a different hamiltonian H, where $[H, H_0] \neq 0$, *e.g.* by suddenly changing a parameter – a *quantum quench*, relevant to experiments on cold atoms in optical lattices
- how do the correlation functions of local operators evolve?
- for fixed separations, do they become stationary as $t \to \infty$?
- do the reduced density matrices of large but finite regions become stationary? If so what is their form?

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- for fixed separations, do they become stationary as $t \to \infty$?
- do the reduced density matrices of large but finite regions become stationary? If so what is their form?
- these questions involve understanding properties of QFT in a pure state which is not an eigenstate

Simple harmonic oscillator

$$H_0 = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 q^2 \qquad\qquad H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2$$

Heisenberg equation of motion has solution

$$q(t) = q(0)\cos\omega t + (p(0)/\omega)\sin\omega t$$

Using $\langle q(0)^2\rangle = 1/2\omega_0$, $\langle p(0)^2\rangle = \omega_0/2$, $\langle q(0)p(0) + p(0)q(0)\rangle = 0$, and $[q(0), p(0)] = i$ we get the propagator

$$\begin{aligned}\langle T(q(t_1)q(t_2))\rangle &= \frac{1}{4}\left(\frac{1}{\omega_0} + \frac{\omega_0}{\omega^2}\right)\cos\omega(t_1 - t_2) - \frac{i}{2\omega}\sin\omega|t_1 - t_2| \\ &\quad + \frac{1}{4}\left(\frac{1}{\omega_0} - \frac{\omega_0}{\omega^2}\right)\cos\omega(t_1 + t_2)\end{aligned}$$

Deep quench limit, $\omega_0 \gg \omega$

$$\langle q(t_1)q(t_2)\rangle \sim \frac{\omega_0}{2}\,\frac{\sin\omega t_1}{\omega}\,\frac{\sin\omega t_2}{\omega}$$

and similarly for higher correlations

- $q(t)$ behaves classically, with a random initial velocity

Imaginary time

$$\begin{aligned}\langle T(q(\tau_1)q(\tau_2))\rangle &= \frac{1}{4}\left(\frac{1}{\omega_0}+\frac{\omega_0}{\omega^2}\right)\cosh\omega(\tau_1-\tau_2)-\frac{2}{\omega}\sinh\omega|\tau_1-\tau_2| \\ &\quad +\frac{1}{4}\left(\frac{1}{\omega_0}-\frac{\omega_0}{\omega^2}\right)\cosh\omega(\tau_1+\tau_2)\end{aligned}$$

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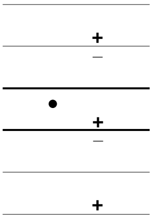
\end{aligned}
$$

- $\langle q(\tau)^2\rangle = 0$ when $\tau = \pm L/2$, where $\omega/\omega_0 = \tanh(L/2)$
- path integral in imaginary time is the same as if the theory were confined to a slab $-\frac{1}{2}L < \tau < \frac{1}{2}L$ with Dirichlet boundary conditions

Method of Images



- dependence on $\tau_1 - \tau_2 \leftrightarrow$ (positive) images at $\tau_1 = \tau_2 + 2nL$
- dependence on $\tau_1 + \tau_2 \leftrightarrow$ (negative) images at $\tau_1 = -\tau_2 + 2nL$

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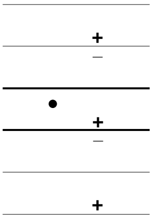

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- dependence on $\tau_1 + \tau_2 \leftrightarrow$ (negative) images at $\tau_1 = -\tau_2 + 2nL$
- if we ignore (or average over) the oscillating term, the propagator is the *same* as that at finite temperature $\beta_{\text{eff}} = 2L$

Free scalar field theory

- a collection of oscillators $H = \int(\frac{1}{2}|\pi_k|^2 + \frac{1}{2}\omega_k^2|\phi_k|^2)d^dk$, $\omega_k = (m^2 + k^2)^{1/2}$
- consider a quench $m_0 \to m$, with $m_0 > m$
- the oscillating part in $\langle T\big(\phi(t_1, x_1)\phi(t_2, x_2)\big)\rangle$ has the form

$$\int \frac{d^dk}{(2\pi)^d} e^{ik(x_1 - x_2)} \left(\frac{1}{\omega_{0k}} - \frac{\omega_{0k}}{\omega_k^2}\right) \cos\big(\omega_k(t_1 + t_2)\big)$$

- if $\omega_k = (k^2 + m^2)^{1/2}$ with $m > 0$ the second term $\sim t_1^{-d/2}\cos(2mt_1) \to 0$ as $t_1 \sim t_2 \to \infty$

- the remainder corresponds to an effective k-dependent temperature

$$\beta_k = (4/\omega_k) \tanh^{-1} \left(\omega_k/\omega_{0k}\right)$$

- if $|x_1 - x_2| \ll t$ the dominant contribution comes from $k \sim 0$, and we can ignore the k-dependence in β_k
- the 2-point function (and the N-point functions) all thermalize (but slowly)

Onset of correlations

- for large m_0

$$\langle\phi(t_1,x_1)\phi(t_2,x_2)\rangle \sim m_0 \int d^dk \frac{e^{ik(x_1-x_2)}}{\omega_k^2}\left(\cos\omega_k(t_1-t_2)-\cos\omega_k(t_1+t_2)\right)$$

$$\begin{aligned}\frac{\partial}{\partial t_1}(\text{this}) \quad \propto \quad & G_F(x_1-x_2,t_1-t_2)-G_F(x_1-x_2,-t_1+t_2)\\ & -G_F(x_1-x_2,t_1+t_2)+G_F(x_1-x_2,-t_1-t_2)\end{aligned}$$

- if $t_1+t_2<|x_1-x_2|$ this vanishes by Lorentz invariance – *horizon effect*
- in general behaviour near horizon is smoothed out over scales $\delta t \sim m_0^{-1}$

Massless case (conformal field theory)

- for $m = 0$ in 1+1 dimensions we find instead

$$\langle \phi(t_1, x_1)\phi(t_2, x_2)\rangle = \begin{cases} 0 & \text{if } t_1 + t_2 < |x_1 - x_2| \\ m_0(t_1 + t_2 - |x_1 - x_2|) & \text{if } t_1 + t_2 > |x_1 - x_2| \end{cases}$$

- many gapless interacting systems in $d = 1$ are equivalent to *conformal field theories*
- local observables

$$\Phi_q(x, t) \sim e^{iq\phi(x,t)}$$

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1-point functions:

$$\langle\Phi_q(x,t)\rangle = e^{-(q^2/2)\langle\phi(x,t)^2\rangle} \sim e^{-m_0 q^2 t}$$

2-point functions:

$$\langle \Phi_q(x_1,t_1)\Phi_{-q}(x_2,t_2)\rangle = e^{-(q^2/2)\langle(\phi(x_1,t_1)-\phi(x_2,t_2))^2\rangle}$$

so, for $t_1 + t_2 < |x_1 - x_2|/c$,

$$\langle \Phi_q(x_1,t_1)\Phi_{-q}(x_2,t_2)\rangle \sim \langle \Phi_q(x_1,t_1)\rangle \langle \Phi_{-q}(x_2,t_2)\rangle$$

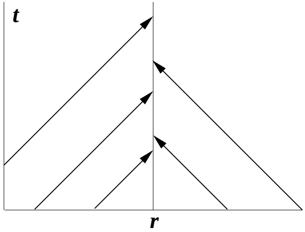
while for $t_1 + t_2 > |x_1 - x_2|/c$

$$\langle \Phi_q(x_1,t_1)\Phi_{-q}(x_2,t_2)\rangle \sim e^{-m_0 q^2\left(t_1+t_2-(t_1+t_2-|x_1-x_2|/c)\right)} = e^{-m_0 q^2|x_1-x_2|/c}$$

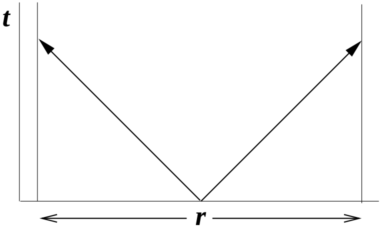
- thermalization of a region of length ℓ takes place exponentially fast (on a time-scale $O(m_0^{-1})$) after the end-points come into mutual causal contact
- these results hold for any CFT in 1+1 dimensions

Physical picture

- $|\psi_0\rangle$ has (extensively) higher energy than the ground state of H
- it acts as a source of (quasi)particles at $t = 0$
- particles emitted from regions size $\sim m_0^{-1}$ are entangled
- subsequently they move classically (at velocity $\pm c$)
- incoherent particles arriving at r from well-separated initial points cause relaxation of local observables (except conserved quantities like the energy) to their ground state values:



- *horizon effect*: local observables with separation r become correlated when left- and right-moving particles originating from the same spatial region $\sim m_0^{-1}$ can first reach them:



- if all particles move at unique speed c correlations are then frozen for $t > r/2c$

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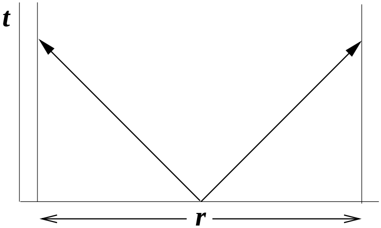


- if all particles move at unique speed c correlations are then frozen for $t > r/2c$
- entanglement entropy of an interval of length ℓ is extensive, and identical to Gibbs-Boltzmann entropy at temperature β_{eff}^{-1}

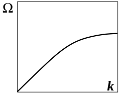
General dispersion relation

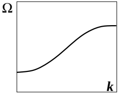
$$(\partial/\partial t)\langle\phi(x_1,t)\phi(x_2,t)\rangle = m_0 \int \frac{d^dk\, e^{ik(x_1-x_2)}}{\omega_k} \sin(2\omega_k t)$$

- large x, t behaviour given by stationary phase approximation

$$|x_1 - x_2|/2t = d\omega_k/dk = \text{group velocity } v_k$$

- correlations begin to form at $t = |x_1 - x_2|/2v_{\max}$
- large t behaviour dominated by slowest moving particles: eg lattice dispersion relation gives a power law approach to asymptotic limit

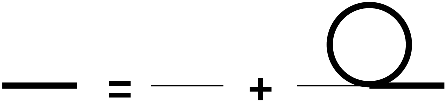




- agrees with exact results for Ising and XY spin chains [Barouch and McCoy, 1970–71]

General interacting QFTs

- can we safely ignore the oscillating terms in the propagator within loops?
- $\lambda\phi^4$ theory in the Hartree (large N) approximation



- even if the renormalized mass is zero, the interactions + the modified propagator generate an effective mass, which means that oscillating terms in the loop are damped $\rightarrow$ thermalisation
- additional renormalisation required in $d = 3$ as $t \rightarrow 0$, in analogy with boundary QFT

Summary and further remarks

- quantum quenches from $m_0 \downarrow m$ appear to lead to thermalisation of finite regions if $m > 0$, and even when $m = 0$ in the presence of interactions
- there is a 'horizon' effect: correlations only begin to change after points come into mutual causal contact
- many interesting questions remain: *eg* quenches from a disordered phase $\rightarrow$ ordered phase – can we drive a phase transition by changing the initial state? $\ldots$