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Summer School on Novel Quantum Phases and Non-Equilibrium Phenomena in Cold Atomic Gases

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Nonlinear Quantum Dynamics of BEC.

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Nonlinear Dynamics of Bose-Einstein Condensates

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Outline

- Nonlinear interference of BEC PRL 00, with Liu and Wu

  inverse scattering method
- Nonlinear Quantum Chaos PRL 04, 05 with Zhang, Liu, Raizen

  Kicked BEC

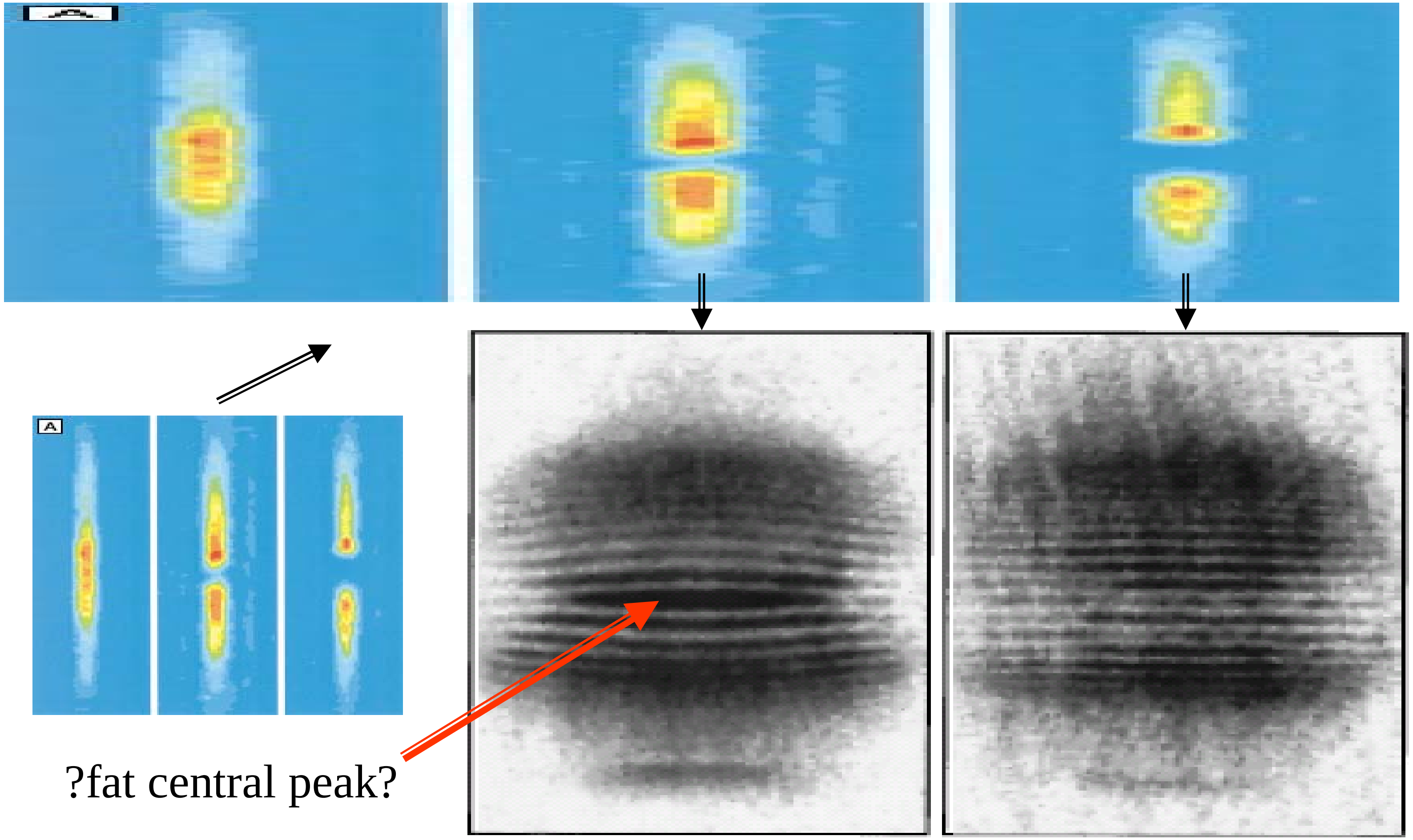
  BEC in a Billiard
- Quasi-particle momentum PRL 06 with Zhang

  Thermal momentum distribution

  Transverse force on a vortex

BEC Interference

- Andrews et al, Science 1997



?fat central peak?

Basic considerations

- Relative phase is well defined
  - two packets are produced coherently from one
  - relative phase=0 because of symmetry
  - Interference pattern repeatable
- GP equation works
  - Numerical solutions by Wallis et al PRA 1997
  - The fat central peak remains to be explained

Inverse scattering method

- 1D GP is exactly soluble
- Scattering problem
  - Reflection coefficient

    r(k,t)=r(k,0) exp(iαt)

    known

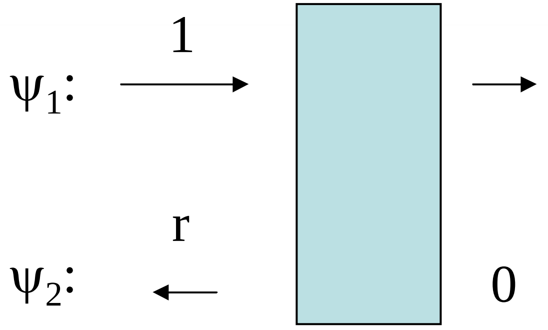
  - Calculate r(k,0) from ϕ(x,0)

- Inverse scattering problem
  - Construct ϕ(x,t) from r(k,t).

GP wavefunction

$$i\,\frac{\partial\psi_1}{\partial x}+\sqrt{g}\,\phi\,\psi_2 - \frac{k}{2}\,\psi_1\,,$$

$$i\,\frac{\partial\psi_2}{\partial x}-\sqrt{g}\,\phi^{*}\psi_1 - -\frac{k}{2}\,\psi_2\,, \qquad (2)$$



Interference pattern

- Asymptotic momentum distribution

  $n(x, t) = |\alpha(k=x/t)|^2$ at large t.
- Inverse scattering theory:

$$|\alpha(k)|^2 - -\frac{1}{2\pi g}\log(1 - |r(k)|^2)\,.$$

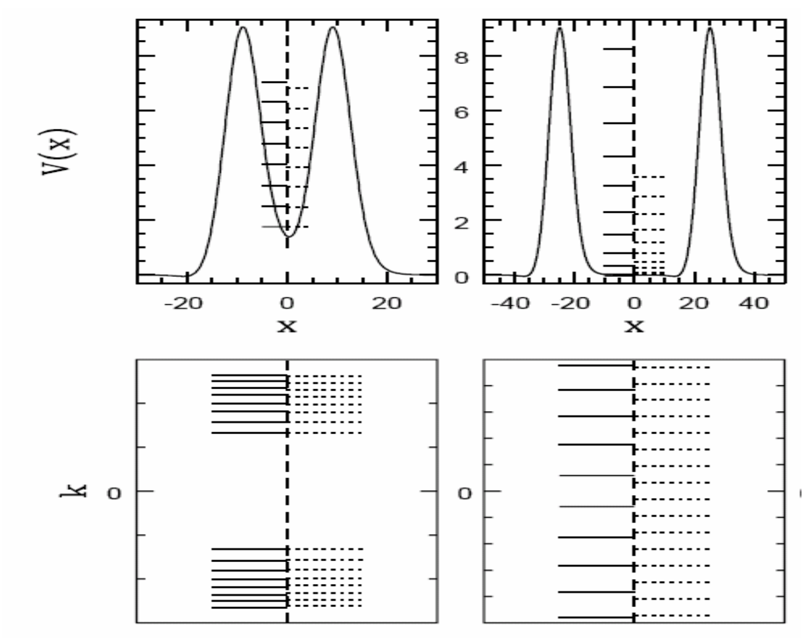
large r gives peaks, small r gives valleys

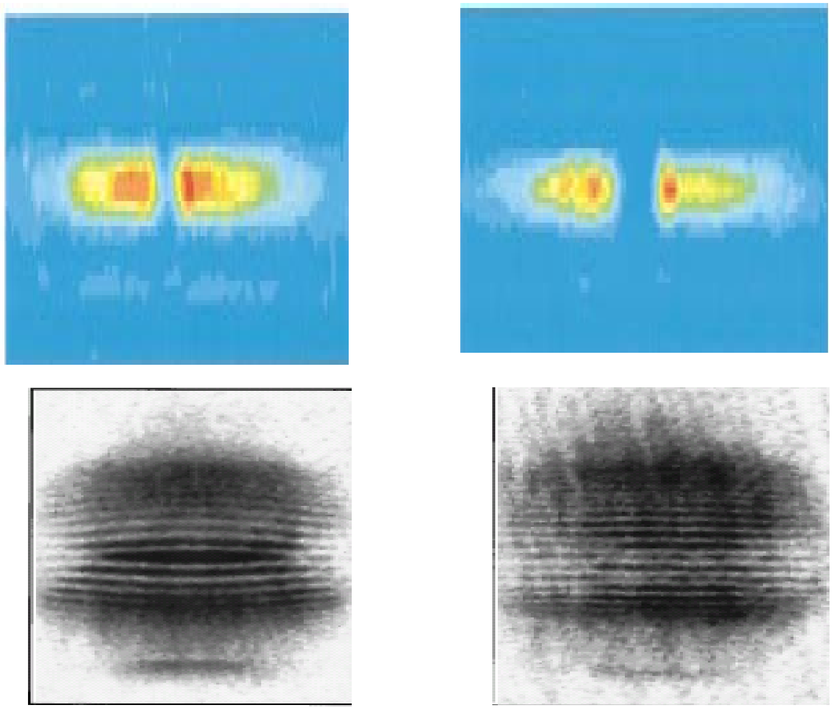
- Interference pattern is given by r(k,0) based on the initial GP wavefunction

Schroedinger scattering

- If initial GP wavefunction is real
  - equivalent Schroedinger problem with potential

$$V(x) = g\phi_0^2 - \sqrt{g}\,\phi_0'$$





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Classical and quantum chaos

- Classical Mechanics:
  - ➢ Regular motion: periodic or quasi-periodic motion
  - ➢ Chaotic motion: exponential divergence of nearby orbits

- Quantum Chaos:
  - ✓ Wave function dynamics is always regular due to linearity.
  - ✓ Quantum signatures in systems which are classically chaotic:

1. Time-independent systems (two or more degrees of freedom):
   ---Energy spectrum and energy eigenvectors; (Billiards)

2. Time-dependent systems (one degree of freedom + time):
   ---Temporal evolution of energy; (Kicked rotor)

Kicked Rotor

- Classical dynamics: the standard mapping

Hamiltonian $H = \frac{p^2}{2} + K\cos x \sum_n \delta(t - nT)$ Energy $\overline{E}_n = K^2 n/4$

- Chaotic motion leads to diffusive growth in the kinetic energy

- Quantum dynamics: Quantum kicked rotor

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2} + K\cos(\theta)\sum_n \delta(t - nT)\psi \qquad T = 4\pi\alpha$$

✓ Quantum resonance

α is a rational number $\alpha \neq 1/2$ Energy $\overline{E}_n \propto n^2$

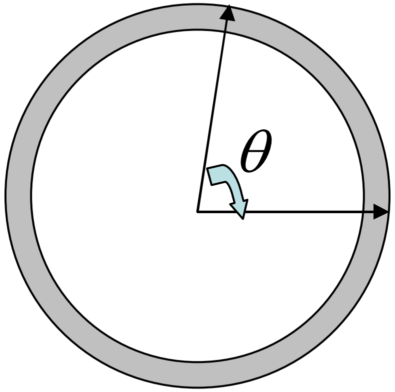
✓ Quantum Anti-Resonance

$\alpha = 1/2; T = 2\pi$ Periodic motion

✓ Quantum suppression of classical diffusion

α is an irrational number, Quasi-periodic motion

Kicked BEC



$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2}+g|\psi|^2\psi+K\cos(\theta)\sum_n\delta(t-nT)\psi$$

✓ How both quantum mechanics and nonlinearity affect the dynamics

Classical: chaotic motion leads to diffusion

Quantum: no chaos because of the linearity of quantum mechanics

Quantum with nonlinearity: new opportunity for chaos?

- Focus: Quantum anti-resonance $T=2\pi$

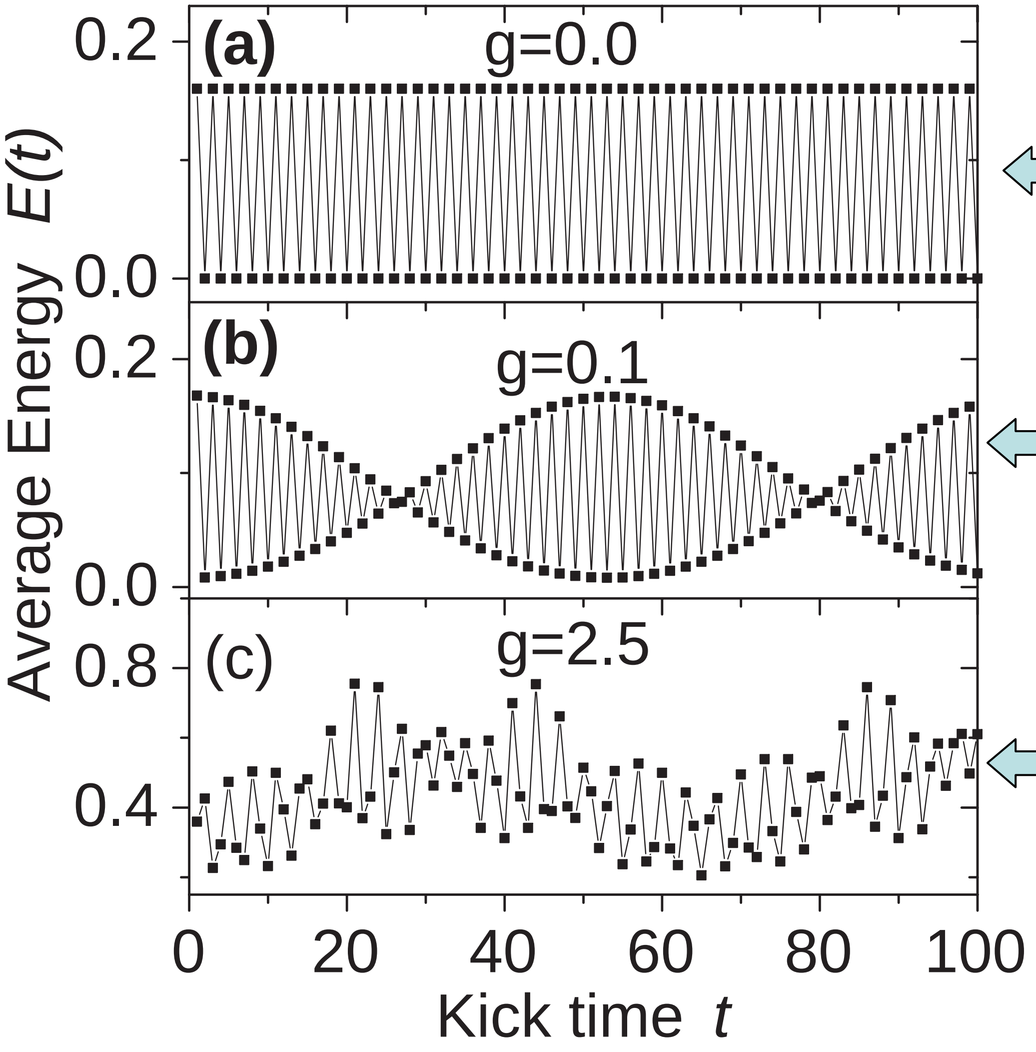
Periodic motion; Pure quantum mechanics; Relatively simple

- Results: (also for dynamically localized states)
  - No or weak interaction: Periodic or quasi-periodic motion, no instability
  - Strong interaction: The destroy of quasi-periodic motion and instability

Evolution of the mean energy of each particle

Initial state $\psi(\theta) = 1/\sqrt{2\pi}$

$$E(t) = \int_0^{2\pi} d\theta \psi^* \left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + \frac{g}{2}|\psi|^2 \right) \psi$$



Anti-Resonance
Periodic

Quasiperiodic

Chaotic

Kick period $T = 2\pi$

Kick strength $K = 0.8$

Dynamical instability of BEC

- Chaos: exponential sensitivity to initial condition

  Exponential divergence of nearby trajectories

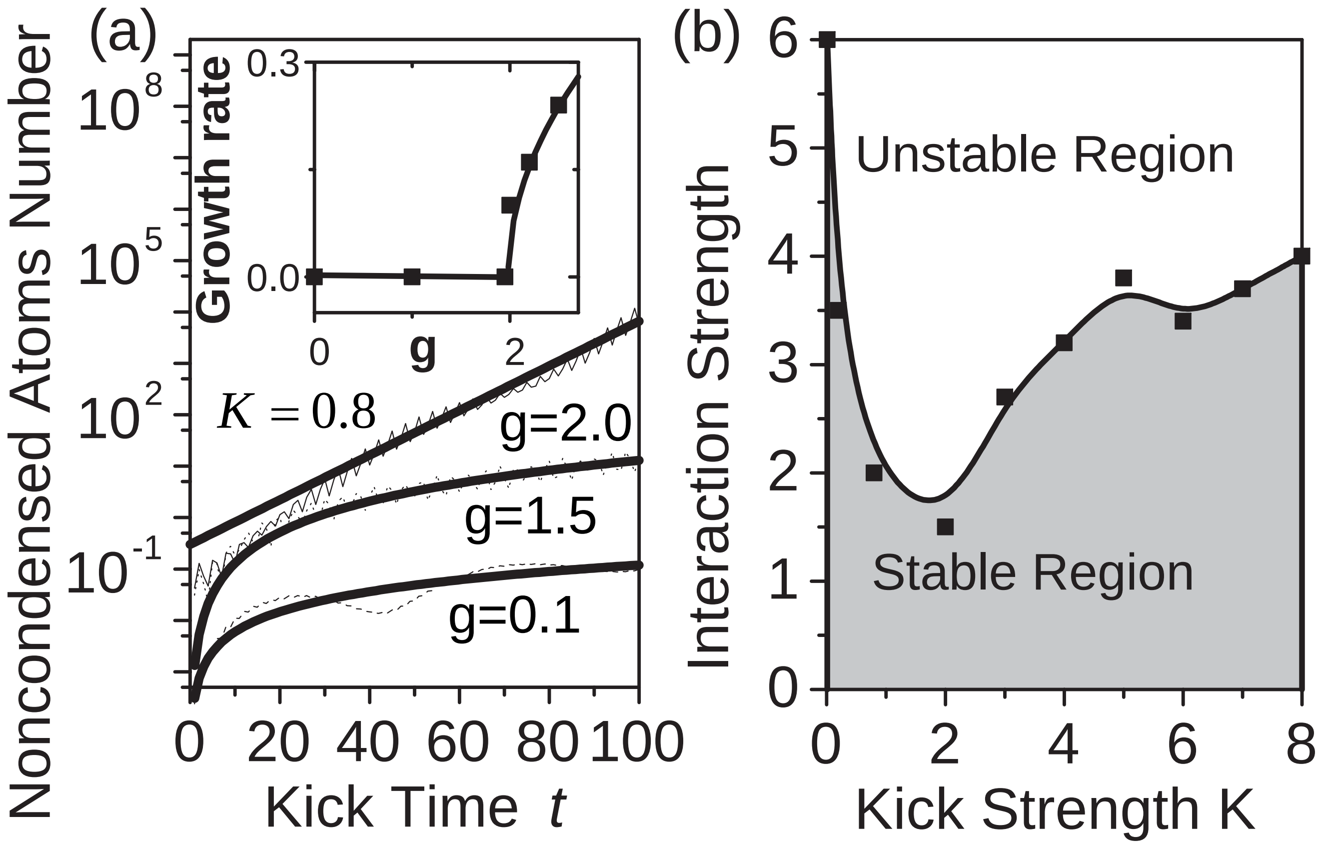
- Exponential growth of Bogoliubov amplitudes

  u_k, v_k -----modes of deviation from the condensate wavefunction

- Exponential growth of non-condensed atoms

$$\langle \delta N(t) \rangle = \sum_{k=1}^{\infty} \langle v_k(t) | v_k(t) \rangle$$

Stability phase diagram

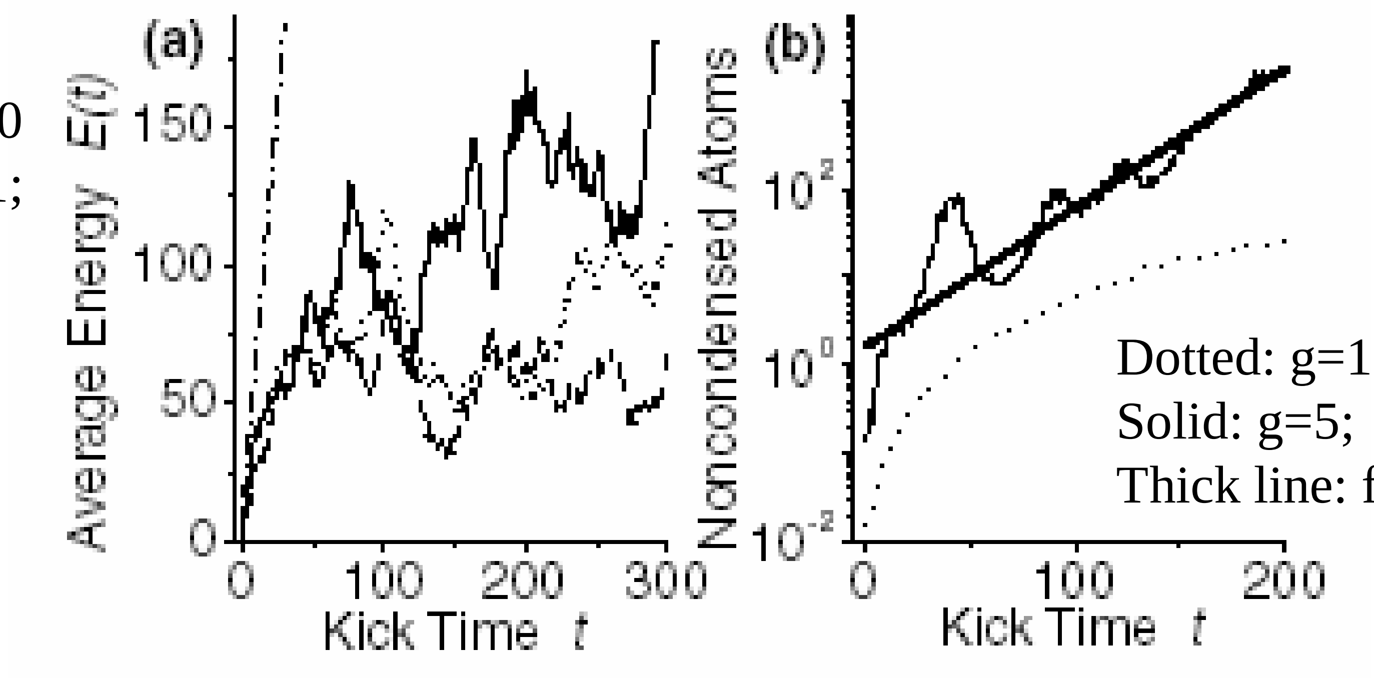


Noncondensed atoms number: $N \sim \exp(\sigma t)$

Growth rate $$\sigma \sim \begin{cases} 0 & g \le g_c \\ (g - g_c)^{1/2} & g > g_c \end{cases}$$

Transition to Instability for a dynamically localized state

Dashed: g=0
Dotted: g=1;
Solid: g=5



Dotted: g=1;
Solid: g=5;
Thick line: fitting

- Noninteaction: Quasi-periodic motion
- Weak interaction: Still quasi-periodic motion,
  Slow growth in the number of noncondensed atoms
- Strong interaction: Quasi-periodic motion is destroyed;
  Diffusive increase of energy;
  Exponential growth of noncondensed atoms.

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Superfluid and quasiparticles

- The description of a superfluid
  - ✓ Condensate
  - ✓ Elementary excitations: quasiparticles

- Quasiparticle energy in a uniform superfluid

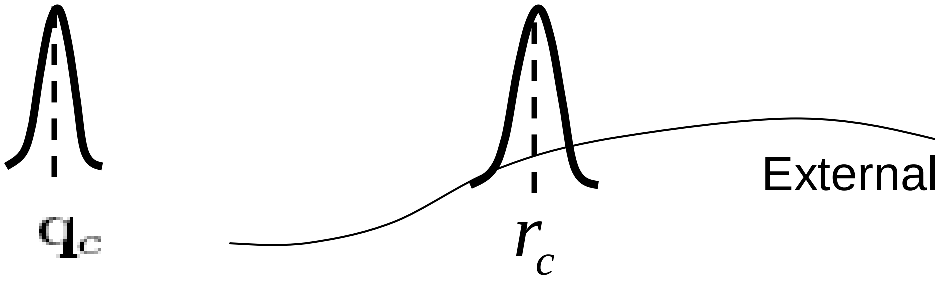
In static superfluid: $$\varepsilon_p = \sqrt{\frac{p^2}{2}\left(\frac{p^2}{2}+2gn\right)}$$

In a moving superfluid: $$E_{ex} = \varepsilon_{\vec{p}} + \vec{p} \bullet \vec{\mathrm{v}}$$

Quasiparticle momentum in the co-moving frame!

What is the quasiparticle momentum in lab frame?

Quasiparticle Wavepacket



External potential

Atomic mass $\rho = \langle \Phi | \Phi \rangle$

Momentum: $\mathbf{q}_c = \mathbf{p}_0 + \rho \mathbf{v}_s$

Energy $\omega = \varepsilon \left(\mathbf{p}_0 \right) + \mathbf{p}_0 \cdot \mathbf{v}_s$

Quasiparticle dynamics

$$\dot{\mathbf{r}}_c = \frac{\partial \omega}{\partial \mathbf{q}_c} + \frac{\partial \rho}{\partial \mathbf{q}_c} \left(\dot{\mathbf{r}}_c \cdot \mathbf{v}_s \right),$$

$$\dot{\mathbf{q}}_c = -\frac{\partial \omega}{\partial \mathbf{r}_c} - \dot{\mathbf{r}}_c \times \left(\frac{\partial \rho}{\partial \mathbf{r}_c} \times \mathbf{v}_s \right) + \left(\dot{\mathbf{q}}_c \cdot \frac{\partial \rho}{\partial \mathbf{q}_c} \right) \mathbf{v}_s$$

Momentum
is not
canonical

Vector potential $\mathbf{A} = i \langle \phi | \sigma_z | \partial \phi / \partial \mathbf{r}_c \rangle = -(\rho - 1) \nabla \alpha (\mathbf{r}_c)$

Berry phase around a vortex

$$\Gamma(\mathcal{C}) = \oint_{\mathcal{C}} d\mathbf{r}_c \cdot \mathbf{A} = -\oint_{\mathcal{C}} (\rho - 1)\, d\alpha(\mathbf{r}_c) = -2\pi (\rho - 1)$$

Canonical dynamics

- Canonical momentum:

$$\mathbf{k}_c = \mathbf{q}_c + (1-\rho)\,\mathbf{v}_s$$

- Equations of motion:

$$\dot{\mathbf{r}}_c = \frac{\partial\omega}{\partial\mathbf{k}_c}, \dot{\mathbf{k}}_c = -\frac{\partial\omega}{\partial\mathbf{r}_c},$$

Transverse force on a Vortex

- Magnus force $\mathbf{F} = \kappa\left(\mathcal{C}\right)\hat{\mathbf{z}} \times \mathbf{v}_L$

- controversy

$\kappa\left(\mathcal{C}\right) = 2\pi\hbar n_{tot}/m$ or

$\downarrow$

- Contributions: Superfluid + Normal fluid

$n_{tot} = n_s + \rho_n \qquad \rho_n = \frac{2\pi^2}{45}\frac{(k_B T)^4}{\hbar^3 s^5}$

Total density (from n_{tot}) · Superfluid density (from n_s) · Normal fluid density (from ρ_n)

$\kappa\left(\mathcal{C}\right) = 2\pi\hbar n_s/m$

- Thouless,Ao,Niu: momentum circulation comes only from superfluid.
- No contribution from normal fruid.

No Iordanskii force.

Confirmed by calculating quasi-particle momentum distribution, provided Berry phase effect is taken care of.

Quasiparticle momentum distribution around a vortex

$$N = 5 \times 10^5$$

In region S

Condensate atoms: 9740
Thermal atoms: 1650

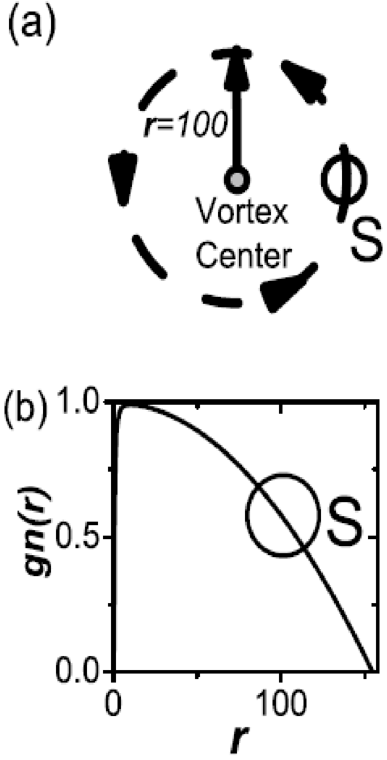
$$\omega_z = 2\pi \times 800 \ \text{Hz}$$

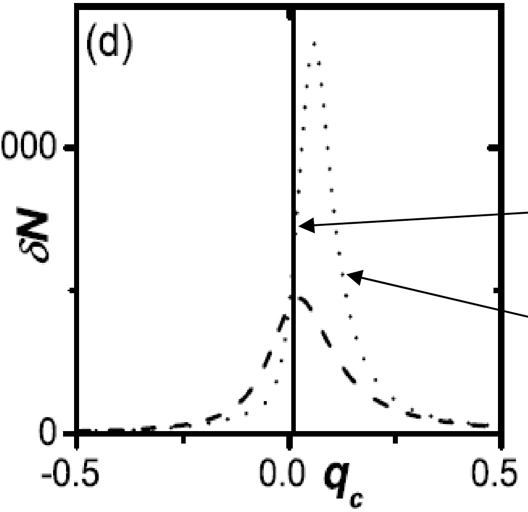
$$\omega_r = 2\pi \times 2 \ \text{Hz}$$

$$\Omega = 0.4\omega_r$$

$$T \approx 21.1 \ \text{nK}$$

$$T_c \approx 52.8 \ \text{nK}$$





Condensate

thermal atoms

Conclusion

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- **Nonlinear Dynamics of BEC is Exciting!**

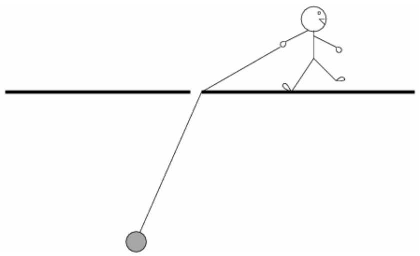
Classical Adiabatic Theorem

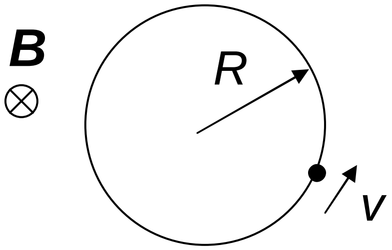
In an integrable system, its actions are conserved during adiabatic evolution

actions: $I_i = \dfrac{1}{2\pi}\oint_{\gamma_i} \mathbf{p}\cdot d\mathbf{q}$ angles: $\theta_1, \theta_2, \ldots, \theta_{\mathrm{n}}$

Examples: 1. Pendulum, $I = E/\omega$

2. charge in magnetic field $I = \dfrac{3qBR^2}{2}$





Quantum adiabatic theory

Initially,

$$\psi(0) = \sum_n a_n \varphi_n(\mathbf{R}_0)$$

Later,

$$\psi(t) = \sum_n a_n e^{-i\lambda_n(t)} \varphi_n(\mathbf{R}(t))$$

The occupation probabilities

$$I_n = |a_n|^2$$

are adiabatic invariants.

$$\lambda_n(t) = \tfrac{1}{\hbar}\int_0^t E_n(\mathbf{R}(\tau))\,d\tau - \gamma_n(t)$$

Adiabatic theory for nonlinear quantum systems

1. **How to generalize the quantum adiabatic theorem**
   - what is the adiabatic condition?
   - what are the adiabatic invariants?

2. **How to generalize the Berry phase**

   what will be its use?

Nonlinear Schrödinger equation

Expand over a set of orthonormal basis

$$\psi = \sum_{n=1}^{N} \psi_n \varphi_n$$

nonlinear Schrödinger equation in matrix form

$$i \frac{d}{dt} |\psi\rangle = H(\psi^*, \psi; \mathbf{R}) |\psi\rangle \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$

Global gauge symmetry:

total prabability is conserved

over-all phase is separated out of dynamics

Map to a classical problem

We can write the nonlinear Schrödinger equation as a set of Hamilton’s equations of motion with Poisson brackets

$$\{\psi^*_j, \psi_k\} = i\delta_{jk}$$

$$i\frac{d}{dt}|\psi\rangle = H(\psi^*, \psi; \mathbf{R})|\psi\rangle$$

$$\Rightarrow \quad \frac{d}{dt}\psi_j = \{\psi_j, \mathrm{H}(\psi^*, \psi; \mathbf{R})\} = -i\frac{\partial}{\partial\psi^*_j}\mathrm{H}(\psi^*, \psi; \mathbf{R})$$

NOTE: This is different from the usual classical-quantum correspondence at $\hbar \to 0$

Nonlinear two level problem

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = H(\gamma)\begin{pmatrix} a \\ b \end{pmatrix}$$

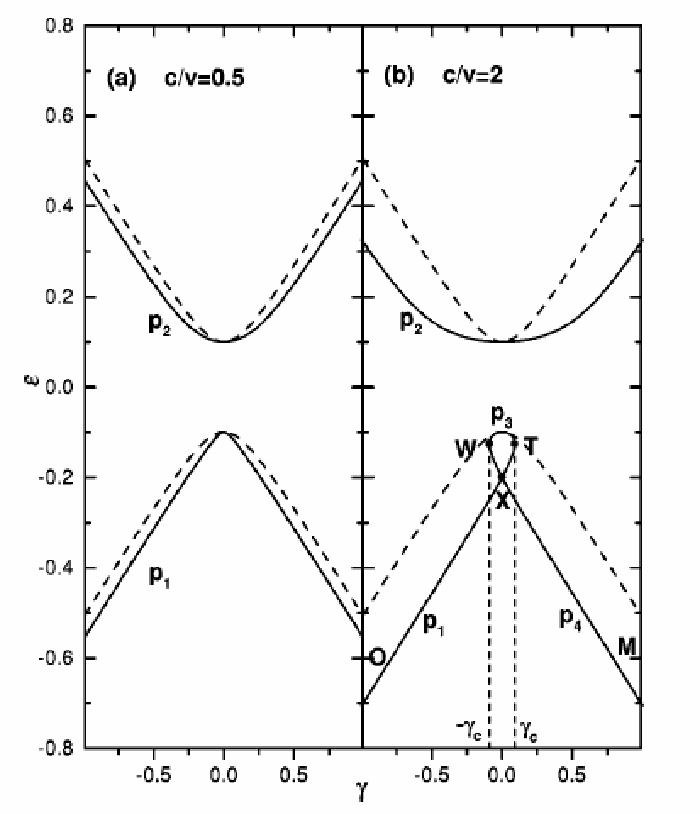
$$H(\gamma) = \begin{pmatrix} \frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2) & \frac{V}{2} \\ \frac{V}{2} & -\frac{\gamma}{2} - \frac{C}{2}(|b|^2 - |a|^2) \end{pmatrix}$$

$$H_e(s, \theta, \gamma) = \frac{C}{2}s^2 + \gamma s - V\sqrt{1 - s^2}\cos\theta$$

S, relative population.
θ, relative phase.

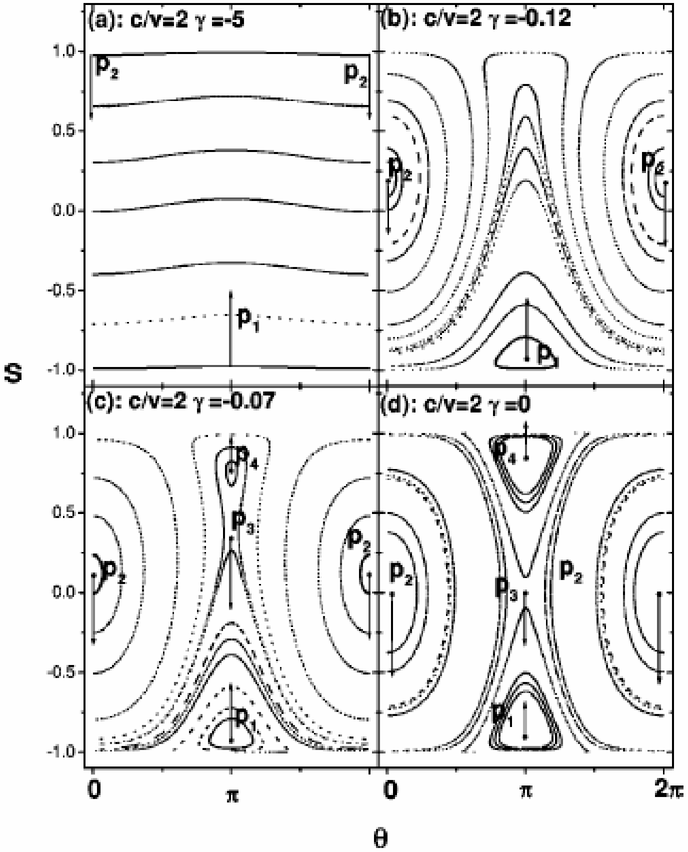
Breakdown of adiabaticity

- **appearance of additional eigenstates**



Classical Portraits of Quantum State Evolution

Fix control parameter γ,
plot evolution of quantum states for each initial condition



Fixed points: eigenstates

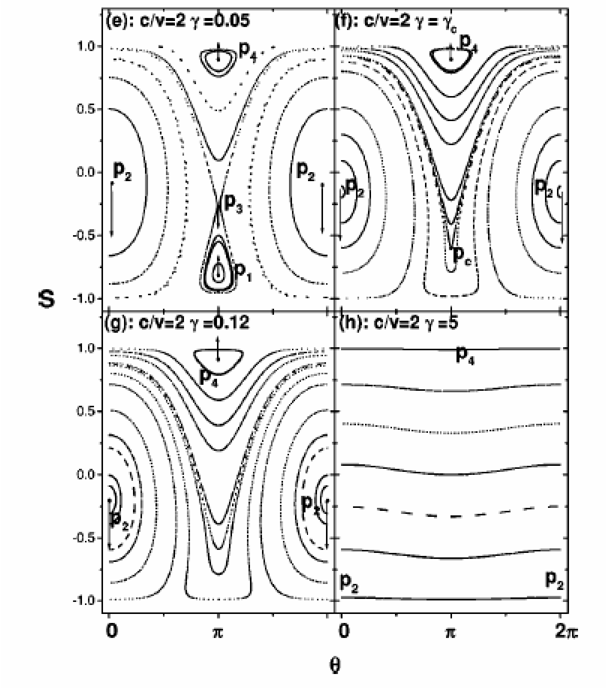
Orbits: time-dependent states

Adiabatic problem:
how these states change
when γ changes slowly

Adiabatic theorem

- Condition for adiabaticity
  - Stability of fixed points or orbits

- Adiabatic invariant
  (conserved quantity)



Classical action

$$\gamma^{j}_{\mathrm{AA}}(R)=\frac{1}{2\pi}\oint dt\left\langle \phi^{j}\middle| i\frac{d}{dt}\middle|\phi^{j}\right\rangle=\oint dt\sum_{m=1}^{N-1}p_{m}dq_{m}=I_{j}$$

Aharonov-Anandan phase