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Pairwise entanglement in 1D spin chains

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Superconductivity
Mesoscopics
Theory group

Materials and Technologies
for Information and communication Sciences

Outline

- **General ideas.**
- **Part I: Bipartite entanglement in the quantum XY chain.**
  - Entanglement & QPT

    Summary at T=0

    Thermal Entanglement close to QPT
  - **Entanglement & separable states in low dimensional systems**
- **Part II: Multipartite entanglement in spin chains.**

General Ideas: Entangled Vs Not entangled states.

Example: 2 spins (or qubits)

- **Pure states:** $|\Psi\rangle$

Separable: $|\uparrow\rangle \otimes |\downarrow\rangle = |\uparrow\downarrow\rangle$

$$(|\uparrow\rangle + |\downarrow\rangle) \otimes |\uparrow\rangle = |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle$$

Entangled: $|\Phi\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$

$$|GHZ\rangle = |\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle$$

- **Mixed states:** $\sum_i p_i |\psi_i\rangle\langle\psi_i|$

Separable: $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$

Entangled: $(1-p)|\Phi\rangle\langle\Phi| + p|\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ $\quad 0 \le p < 1$

Classical Vs quantum correlations

- A separable state (not entangled) may contain classical correlation:

$$
\begin{aligned}
\rho &= \textstyle\sum_i p_i(|A,B\rangle\langle B,A|)_i \\
&= \tfrac{2}{4}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \tfrac{1}{4}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + \tfrac{1}{4}|\uparrow\uparrow\rangle\langle\uparrow\uparrow|
\end{aligned}
$$

Measure of B $B = |\downarrow\rangle$

$$\rho = \frac{2}{3}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \frac{1}{3}|\uparrow\downarrow\rangle\langle\uparrow\downarrow|$$

Once $B = |\downarrow\rangle$, the probability to find $A = |\downarrow\rangle$ is double than the probability to find $A = |\uparrow\rangle$: **A and B are classically correlated.**

General aim:

Quantify Entanglement in many body systems.

Possible questions:

- Entanglement as a resource (q-compuation....)
- Correlation Vs Entanglement
  - Entanglement and Critical phenomena ?

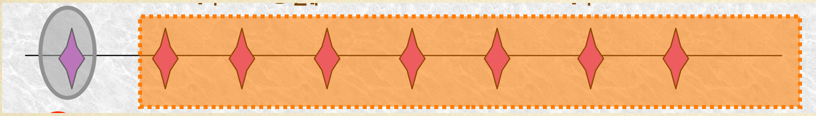
Entanglement measures

Review: Horodecki's family quant-ph/0702225

Rationale: how many Bell states are 'contained ' in a given state? **(Bennet et al., PRA 1996.)**

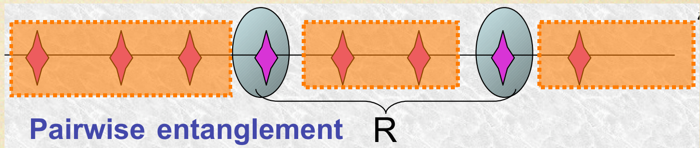
- **Bipartite entanglement for pure qubit states: One-tangle**

von Neumann Entropy: $E = Tr\rho_1 \log_2 \rho_1 \quad \leftrightarrow \quad 4det\rho_1$



Coffman, Kundu,Wootters 2000.
Osterloh, Siewert 2004.

- **Bipartite entanglement for mixed states.**



- **For qubits: Concurrence.**

$$C(R) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$$

Wootters 1998.

λ_α are eigenvalues of $S = \rho_2 \sigma_y \otimes \sigma_y \rho_2^* \sigma_y \otimes \sigma_y$

- **For higher dimensional local Hilbert space:**

The Peres criterium $\longleftrightarrow$ **Negativity:** sum of negative eigenvalues of partial transpose density matrix.

Horodecki's family 1996; Peres 1996.

QPT in 1d-Anisotropic XY models

$$H = J\sum_i (1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y - h\sigma_i^z$$

- Completely integrable.
- Quantum phase transition at $a_c = .1$

$$a = \frac{2h}{J}$$

Lieb, Schulz, Mattis Ann. Phys.NY 16, 407 (1961);
Barouch, McCoy, Dresden PRA 2, 1075 (1970);
Barouch, McCoy PRA 3, 786 (1971);
Pfeuty Ann. Phys.NY 57, 79 (1970).

Spectrum

Jordan-Wigner:

$$\sigma_l^x = A_l \prod_{s=1}^{l-1} A_s B_s \qquad \sigma_l^y = -iB_l \prod_{s=1}^{l-1} A_s B_s \qquad \sigma_l^z = -A_l B_l$$

where $A_l \doteq c_l^\dagger + c_l\,, \quad B_l \doteq c_l^\dagger - c_l$

$$H = \sum_{ij} c_i^\dagger \mathcal{V}_{ij} c_j + \frac{1}{2}\left(c_i^\dagger \mathcal{W}_{ij} c_j^\dagger + h.c\right) + \sum_i V_{ii}$$

$$\mathcal{V}_{ij} = -J\left(\delta_{i,j+1} + \delta_{j,i+1}\right) - h\delta_{i,j}$$

$$\mathcal{W}_{ij} = -\gamma J\left(\delta_{i,j+1} - \delta_{j,i+1}\right)$$

$$H = \sum_k \Lambda_k \eta_k^\dagger \eta_k - \frac{1}{2}\sum_k \Lambda_k \qquad \Lambda_k = \sqrt{\left(1 + \lambda\cos k\right)^2 + \lambda^2\gamma^2 \sin^2 k}$$

Correlation functions.

$$g_{\nu,\nu}(l,m) = \langle \sigma_l^\nu \sigma_m^\nu \rangle \ , \quad \nu = x, y, z$$

$$\langle \sigma_l^z \sigma_m^z \rangle = \langle A_l B_l A_m B_m \rangle$$
$$\langle \sigma_l^x \sigma_m^x \rangle = \langle A_l A_{l+1} B_{l+1} \dots A_{m-1} B_{m-1} A_m \rangle$$

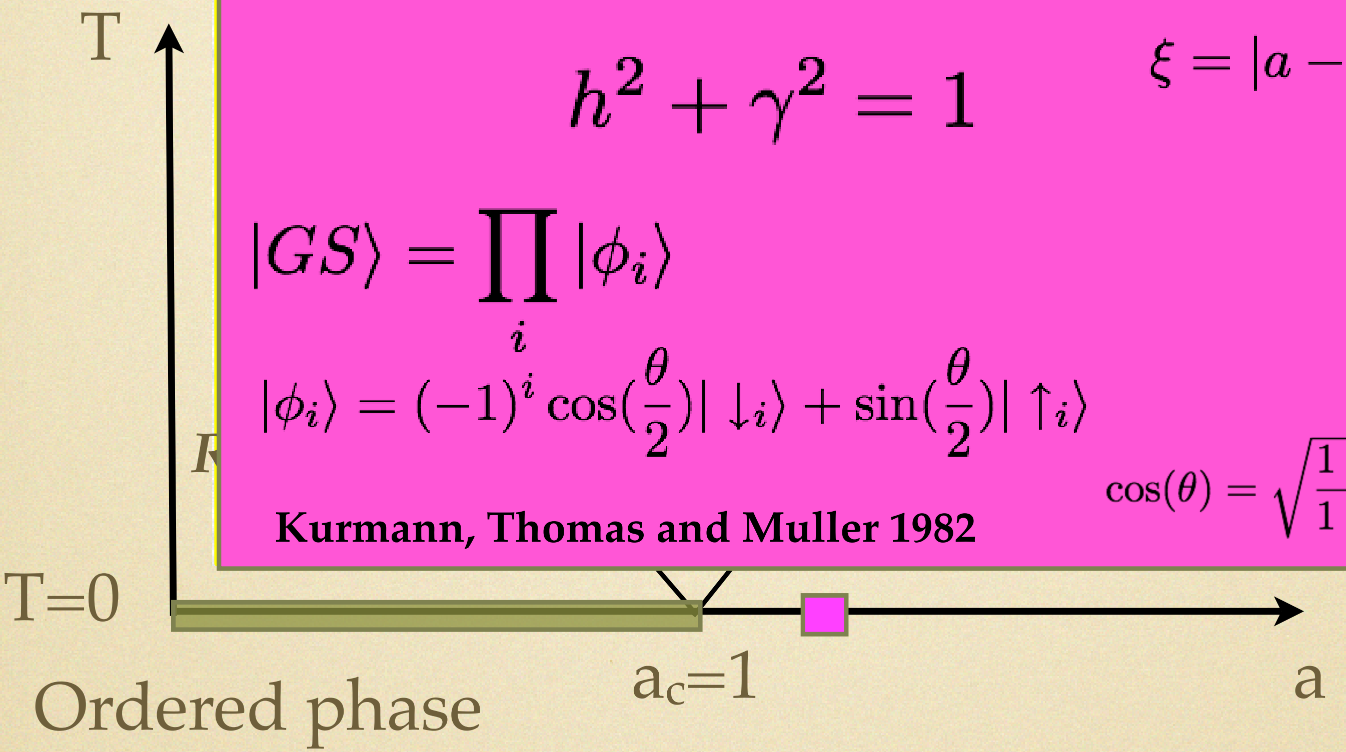
Wick theorem:

$$g_{zz}(R) = m_z^2 - G_R G_{-R}$$

$$g_{xx}(R) = \begin{vmatrix} G_1 & G_{-2} & \dots & G_{-R} \\ G_0 & G_{-1} & \dots & G_{-R+1} \\ \dots & \dots & \dots & \dots \\ G_{R-2} & G_{R-3} & \dots & G_{-1} \end{vmatrix}$$

$$G_R = \frac{1}{\pi} \int_0^\pi d\phi \left[\gamma \sin(\phi R) \sin\phi - \cos(\phi R) \left(\cos\phi - a \right) \right] \frac{\tanh(\beta\Lambda)}{\Lambda}$$
$$m_z = \frac{1}{2\pi} \int_0^\pi d\phi \left(\cos\phi - a \right) \frac{\tanh(\beta\Lambda)}{\Lambda}$$

Cross-over phase diagram for the quantum Ising models



$$\xi = |a - a_c|^{z\nu}$$

$$h^2 + \gamma^2 = 1$$

$$|GS\rangle = \prod_i |\phi_i\rangle$$

$$|\phi_i\rangle = (-1)^i \cos(\frac{\theta}{2})|\downarrow_i\rangle + \sin(\frac{\theta}{2})|\uparrow_i\rangle$$

$$\cos(\theta) = \sqrt{\frac{1-\gamma}{1+\gamma}}$$

Kurmann, Thomas and Muller 1982

Chakravarty, Halperin, Nelson 1989;
Chubukov, Sachdev, Ye (1994);
Sachdev 1996;
Kopp, Chakravarty (2005).

Part I: Entanglement close to QPT

Possible questions:

- Correlation Vs Entanglement ?
- Critical properties ?
- Universality ?

Preskill 2000
Arnesen, Bose, Vedral 2001
Gunlicke, Bose, Kendon, Vedral 2001

Selected results at T=0

Pairwise entanglement close to QPT:

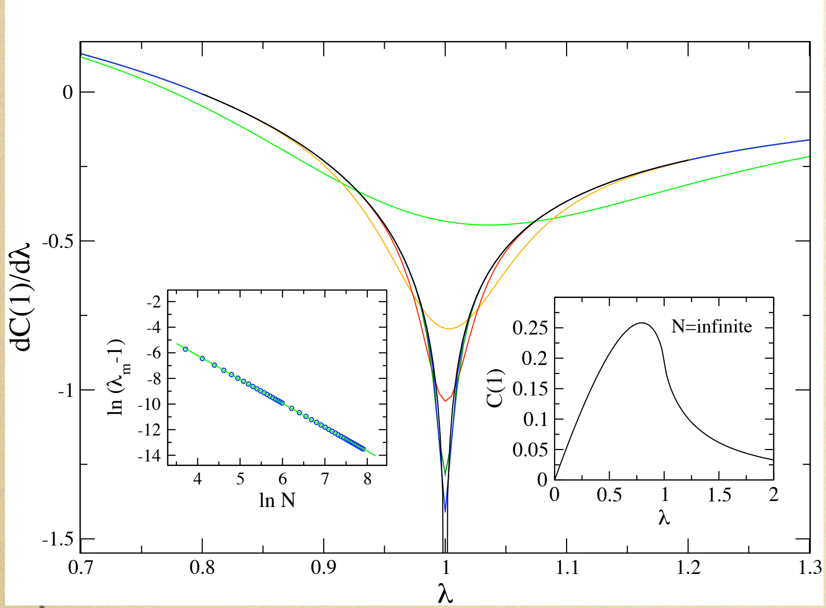
Critical change of Concurrence at the quantum critical point.

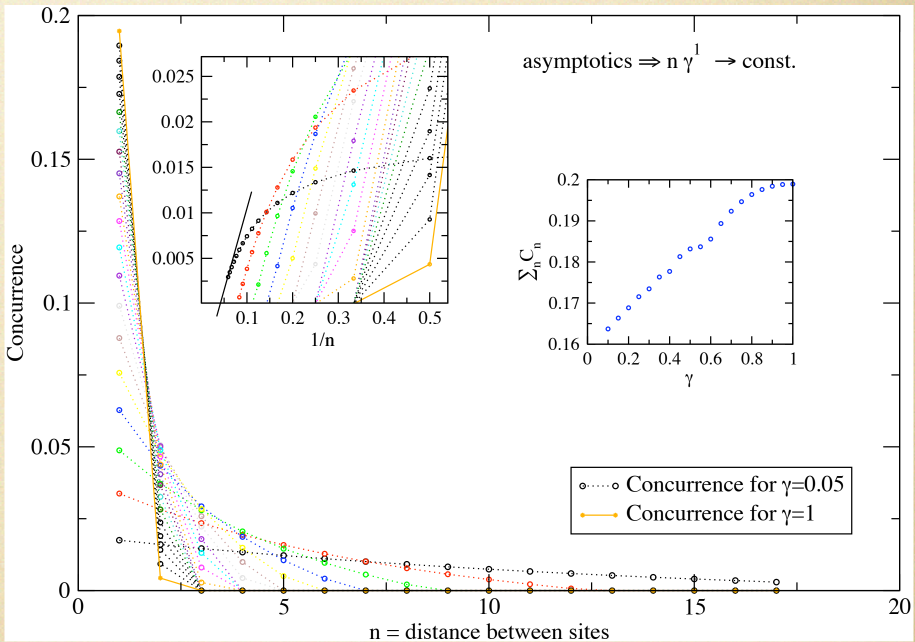
Finite size scaling.

At criticality: In general NO long range Concurrence.

Osterloh, Amico, Falci and Fazio, Nature (2002); Osborne, Nielsen PRA 2002.

Fig.1 Osterloh et al.





$$\partial_a C(1) = \frac{8}{3\pi}|a - a_c|$$

$$R \sim \gamma^{-1}$$

Selected results at T=0

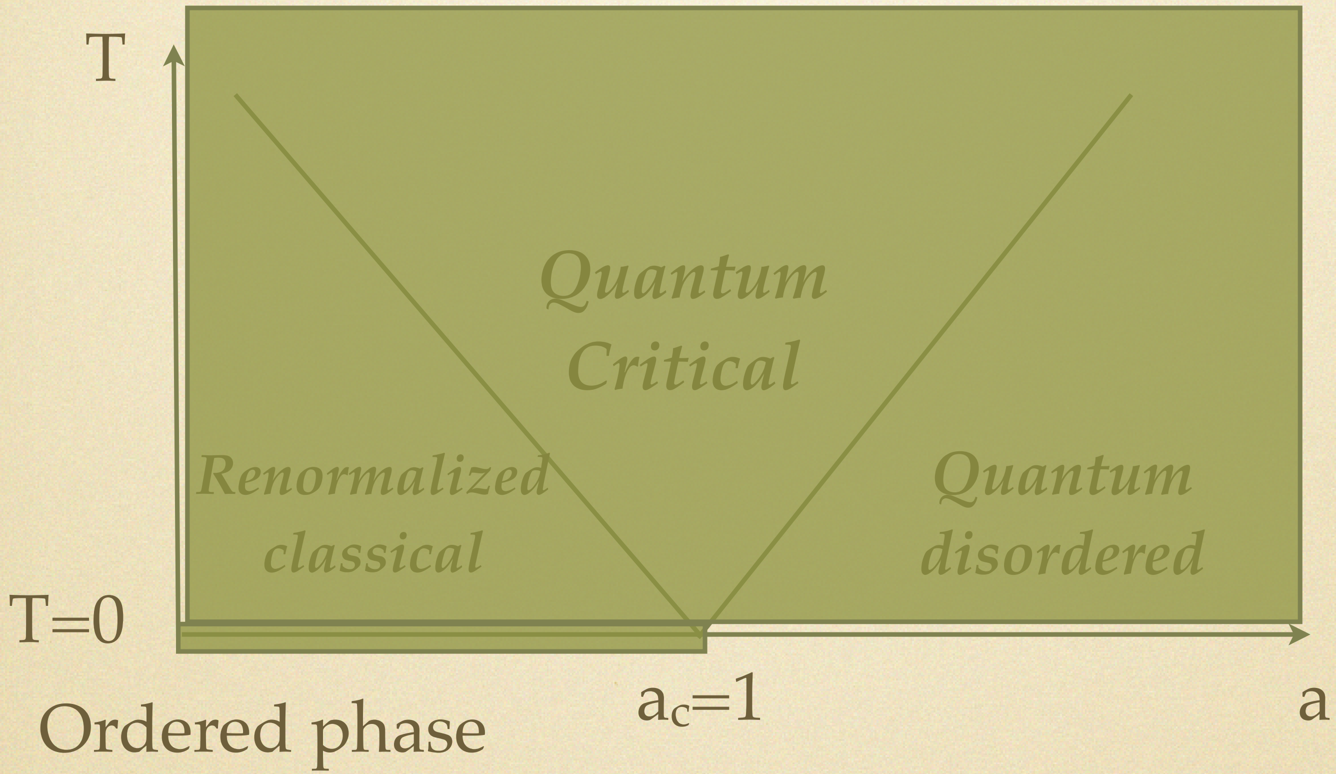
Second order QPT are marked by singularities in the first derivative of a given entanglement measure; first order QPT are marked by anomalies in the entanglement itself.

Wu, Sarandy, Lidar Phys. Rev. Lett. 2004; Wu, Sarandy, Lidar, Sham 2005. Yang 2005; Gu, Tian, Lin 2005. See also: Verstrate, Popp, Cirac 2004; Jin, Korepin 2004.

$H = H_0 + \sum_l \lambda_l A_l$, **The reduced density operator is a function of correlators. Then**

$$M(\langle A_l \rangle) = M\left(\frac{\partial E_{gs}}{\partial \lambda_l}\right)$$

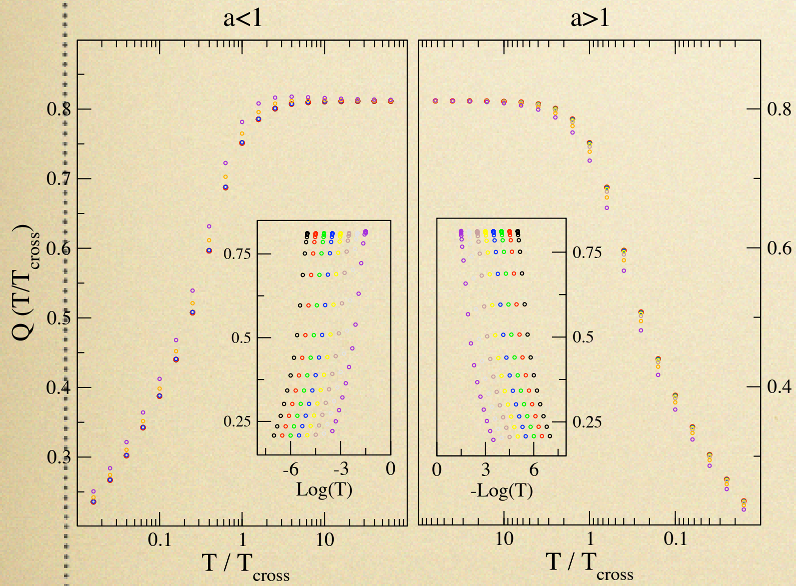
Thermal entanglement close to a QPT



Scaling

Sensitivity to quantum fluctuations:

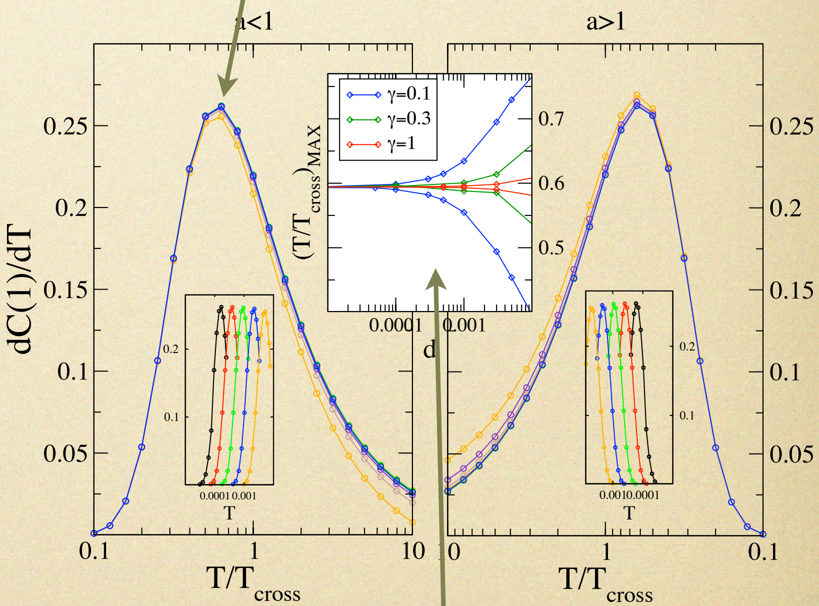
$$\partial_a C(1) \approx \ln \left[T^{\Upsilon} Q \left(\frac{T}{T_{cross}} \right) \right]$$



Sensitivity to thermal fluctuations:

$$\partial_T C(1) \approx P \left(\frac{T}{T_{cross}} \right)$$

$T^* = \alpha T_{cross}$ $\alpha \sim 0.595$



$$T_{cross} = |a - a_c|^{z\nu}$$

Universal T^*

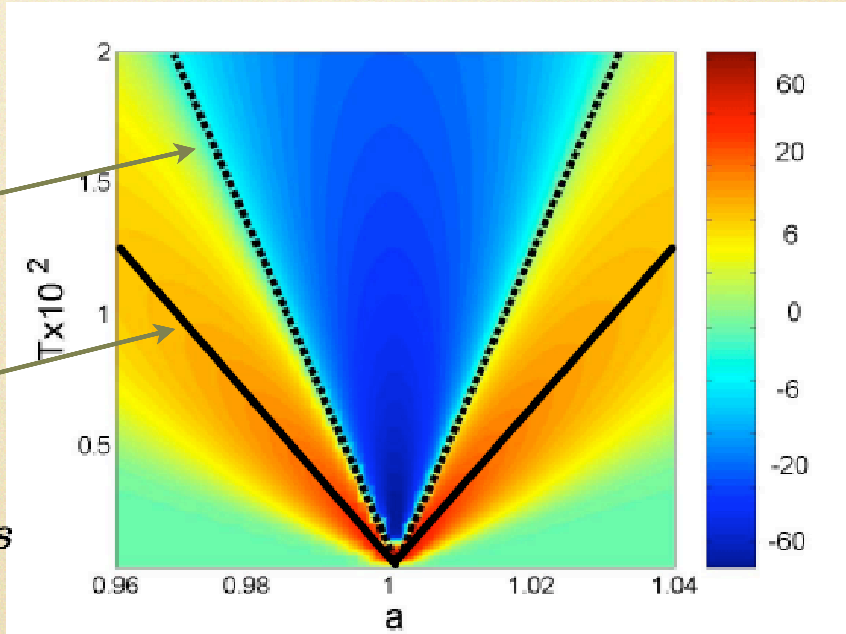
Amico and Patane' 2006

Entanglement crossover

- **How is the entanglement affected by the combinations of thermal and quantum fluctuations?**

$$\partial_T\left[\partial_a C(R)\right]$$

T^*



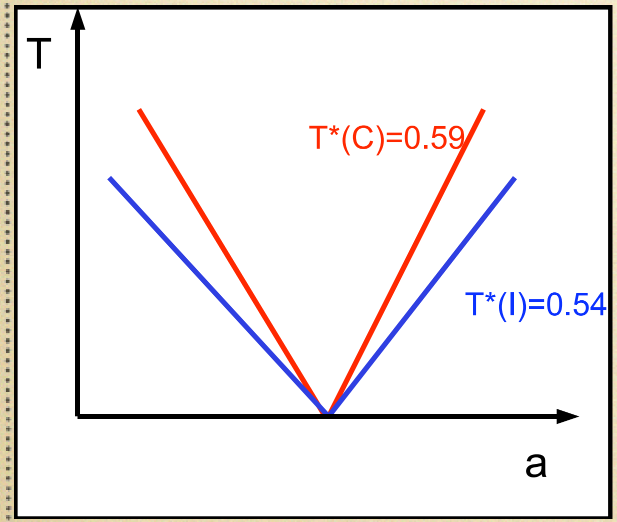
$T_M = 0.290 T_{cross}$

<u>Temperature is a strong effect in the quantum critical region and around T_M</u>

Entanglement Vs Classical correlations

How typical is the behaviour of Concurrence?

Finite Temperature Correlations ⇨ Classical (thermal ensemble) average & Quantum entanglement



Total mutual information:

$$I_{ij} = S_i + S_j - S_{ij}$$

Vedral 2002; Groisman, Popescu, and Winter 2004; Anfossi, Giorda, Montorsi 2005.

Results:

Same "phenomenology" of C_{ij}

Crossover Temp. altered by < 10%

Separable ground states in low dimensional systems

$$H(j_x, j_y, j_z) = \sum_i j_x S_i^x S_{i+1}^x + j_y S_i^y S_{i+1}^y + j_z S_i^z S_{i+1}^z - h S_i^z$$

The ground state is factorized in real space at

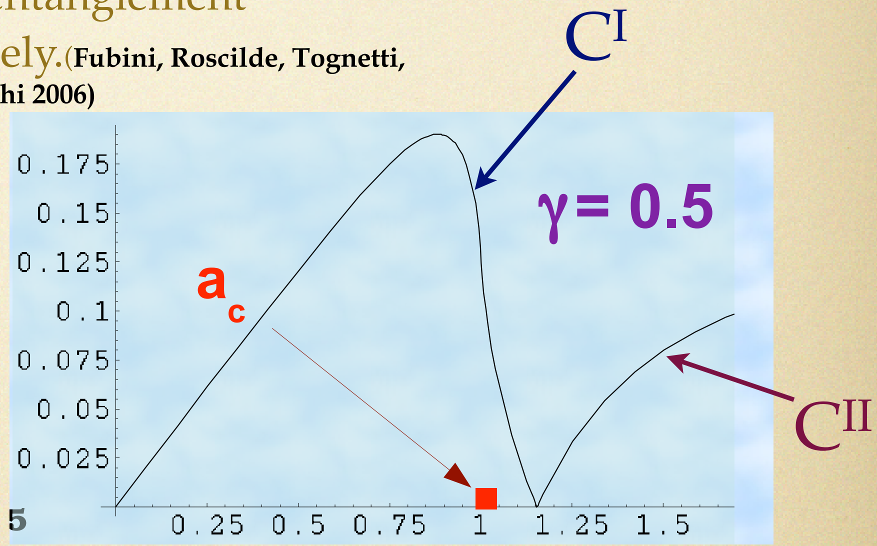
$$h = h_f = \sqrt{(j_x + j_z)(j_y + j_z)}$$

Kurman, Thomas, and Muller (1982);
2d: Roscilde, Verrucchi, Fubini, Haas, Tognetti 2005

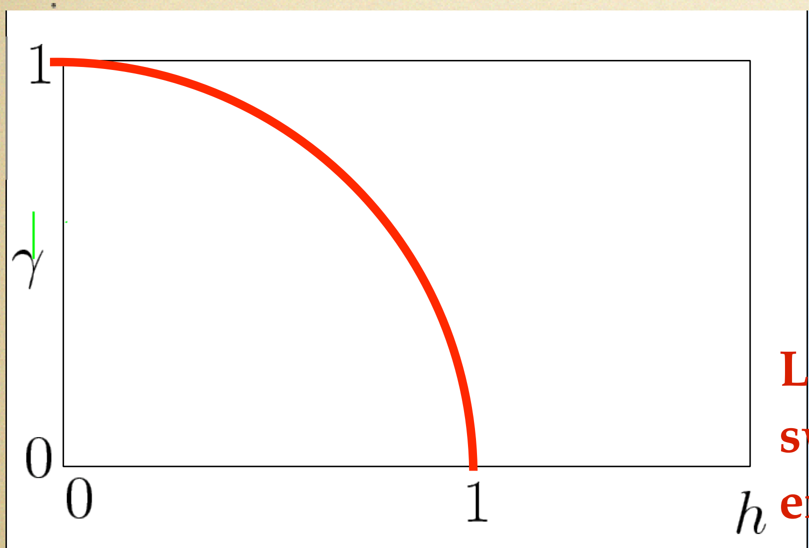
Explanation of Kurman's result

$$C_r = 2\max\{0, C_r^I, C_r^{II}\}$$
$$C_r^I = |g_r^{xx} + g_r^{yy}| - \sqrt{\left(\frac{1}{4} + g_r^{zz}\right)^2 - M_z^2}$$
$$C_r^{II} = |g_r^{xx} - g_r^{yy}| + g_r^{zz} - \frac{1}{4}$$

- C^I and C^{II} reflect antiparallel and parallel entanglement respectively. **(Fubini, Roscilde, Tognetti, Tusa, Verrucchi 2006)**



$$R \propto \left(\ln\frac{1-\gamma}{1+\gamma}\right)^{-1} \ln|1-h_f|^{-1}$$



Long range reshuffling of the ground states switching from parallel to antiparallel entanglement.

Amico, Baroni, Fubini, Patane', Tognetti, Verrucchi 2006.

XY model

Divergence of entanglement range at:

$$h_f = \sqrt{1-\gamma^2}$$

$$R^{XY} \propto \left(\ln\frac{1-\gamma}{1+\gamma}\right)^{-1} \ln|h-h_{\mathrm{f}}|^{-1}.$$

Used exact results by Barouch, McCoy 1971.

XXZ model: $h_f = 1 + j_z$

$$R^{XXZ} \propto (h - h_{\mathrm{s}})^{-\theta/4}$$

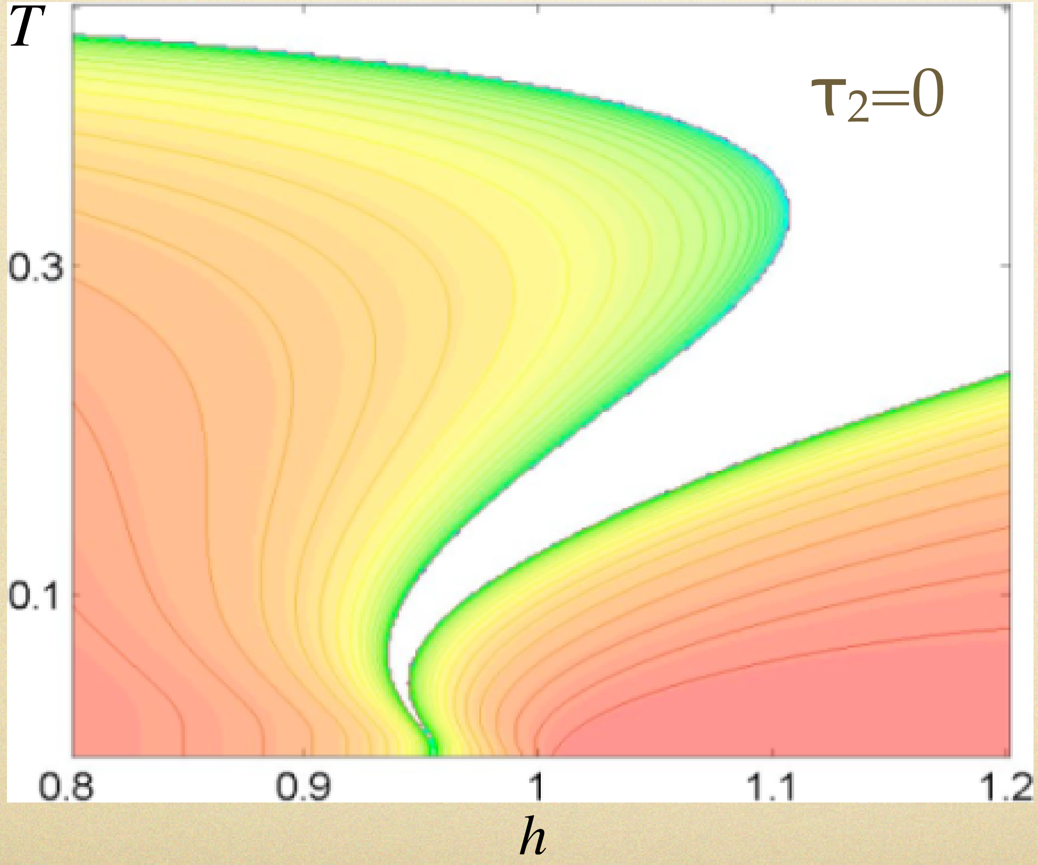
$$\theta = 2 + \frac{4\sqrt{h-h_{\mathrm{s}}}}{\pi \tan(\pi\eta/2)\tan(\pi\eta)}$$

$$\eta = \frac{1}{\pi}\mathrm{arcos}(j_z)$$

Used exact results of Lukyanov 1997-98;
Hikihara, Furusaki 2004.
summariezed in Jin, Korepin 2004.

Amico, Baroni, Fubini, Patane', Tognetti, Verrucchi PRA (2006).

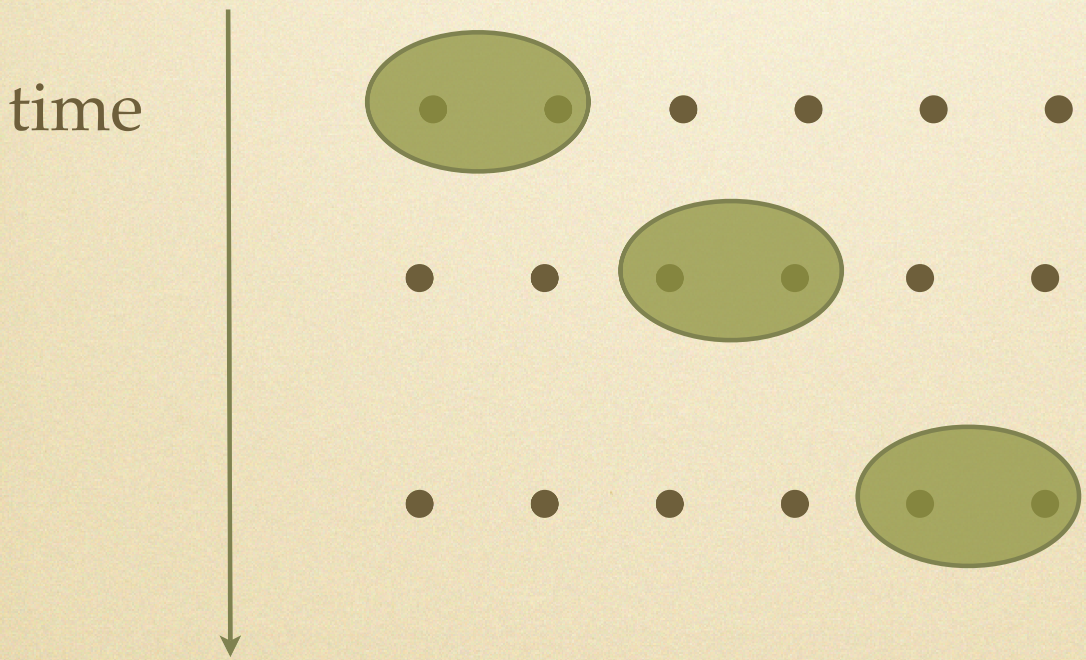
Factorization at finite temperature



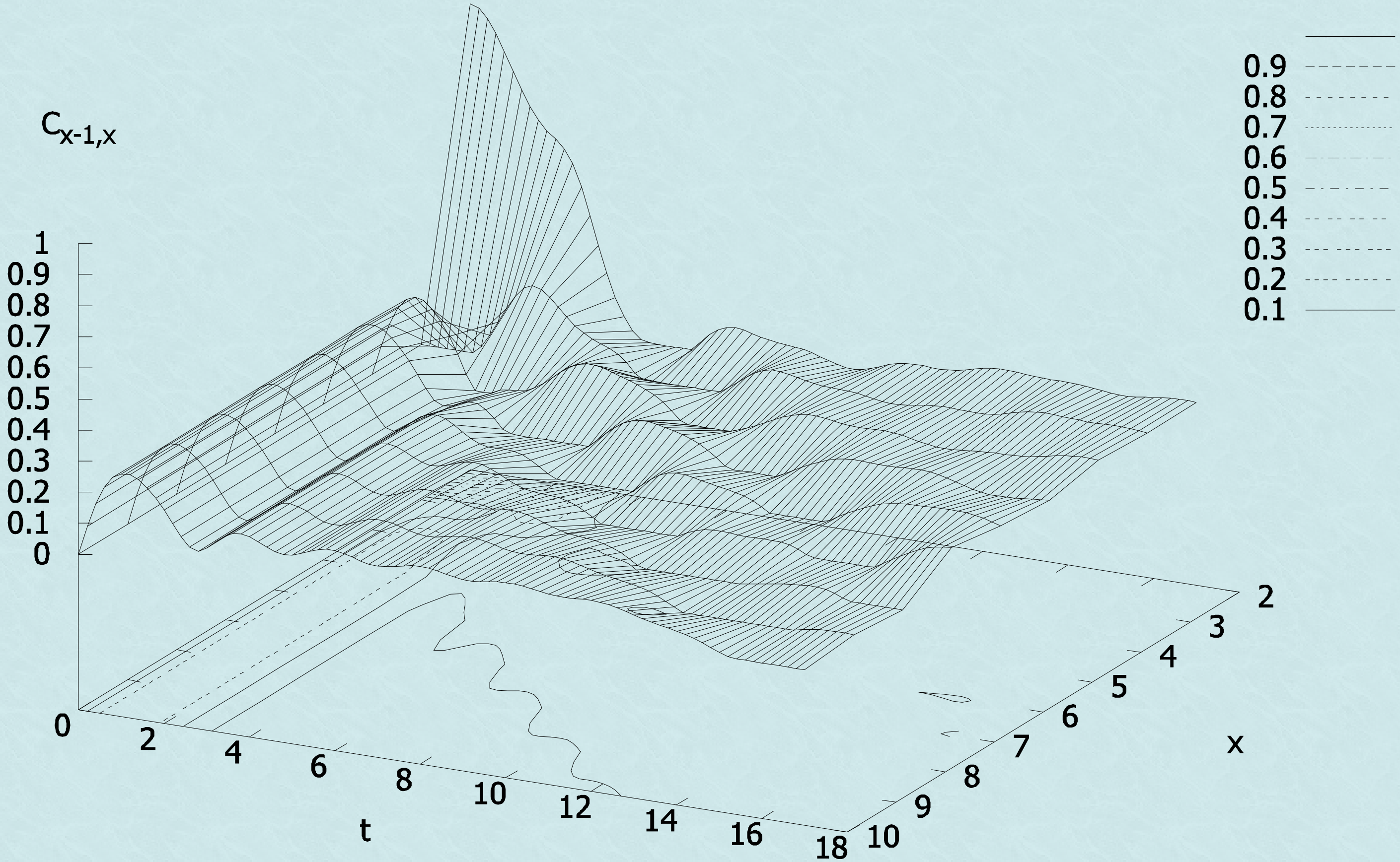
γ=0.3

Part III: Dynamics (Singlet Propagation)

Initial state is a Bell state on a fully polarized state $|\downarrow \ldots \downarrow\rangle$



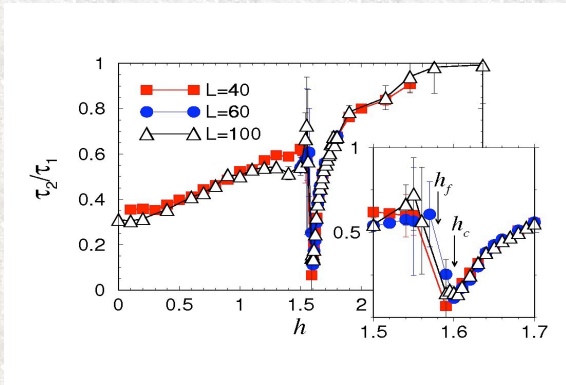
C_1 for $\gamma=1$, $\lambda=0.5$



JPA 2005; PRA 2005; NJP 2005

Part II: Multipartite entanglement in spin chains

- **Quantum Phase Transition:**
  Enhancement of multiparticle entanglement
  G. Vidal et al. , Phys.Rev.Lett. 90, 227902 (2003); T. Roscilde et al., Phys. Rev. Lett. 93, 167203 (2004).



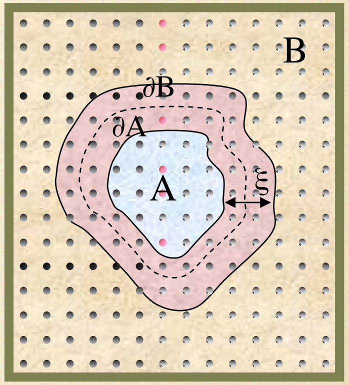
- **Simulations of strongly correlated systems:**

  - Failure DMRG for unbounded block entangl. (i.e. 1D critical systems or D>2).

  - New algorithms taking into account long range entanglement (i.e. PEPS, graph states..)
    G. Vidal Phys. Rev. Lett. 93, 040502 (2004); F. Verstraete and J. I. Cirac; cond-mat/0407066; G. Vidal, cond-mat/0512165; S. Anders, Phys. Rev. Lett. 97, 107206 (2006).

AREA LAW

The information of a subsystem about the rest can be quantified by $S = -tr\left(\rho_A \ln \rho_A\right)$

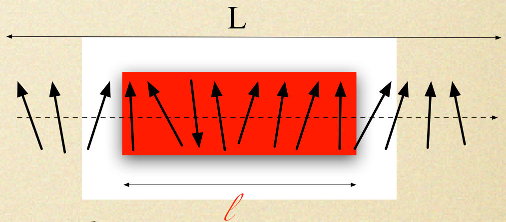


When the area law holds:

$$S \sim \ell^{d-1}$$

Sredniki 1993; Fiola, Preskill, Strominger, and Trivedi 1994; Holzhey, Larsen and Wilczek 1994.

In d=1 S is independent on the block size



For short range interactions this holds for gapped systems.

$$S = \frac{c}{3}\log_2\frac{\xi}{a} \qquad \text{for} \quad \ell \to \infty$$

Violation of the area law for gapless systems: log corrections appear with a universal prefactor

$$S = \frac{c}{3}\log_2\left[\frac{L}{\pi a}\sin\frac{\pi}{L}\ell\right] + A$$

(Vidal, Latorre, Rico, Kitaev 2003; Calabrese and Cardy 2004-06; Its, Jin, Korepin 2005).

Analysis of the block entanglement

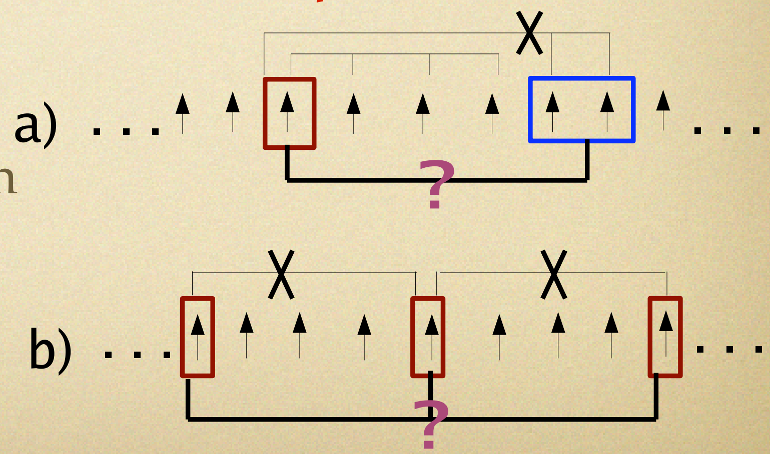
Motivations: analysis of one-tangle and block entropies reveal multipartite entanglement....BUT: ***How is it shared?***

- **Strategy**: Search for configurations/regimes where the two-particle entanglement is known to vanish.

Example for three spin entanglement:

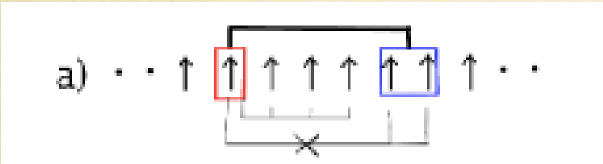
i) If the spin are distant enough then there is ***no two-spin*** entanglement....

ii) Then, pairwise entanglement between ***suitably distant blocks is a measure of genuine multiparticle entanglement.***

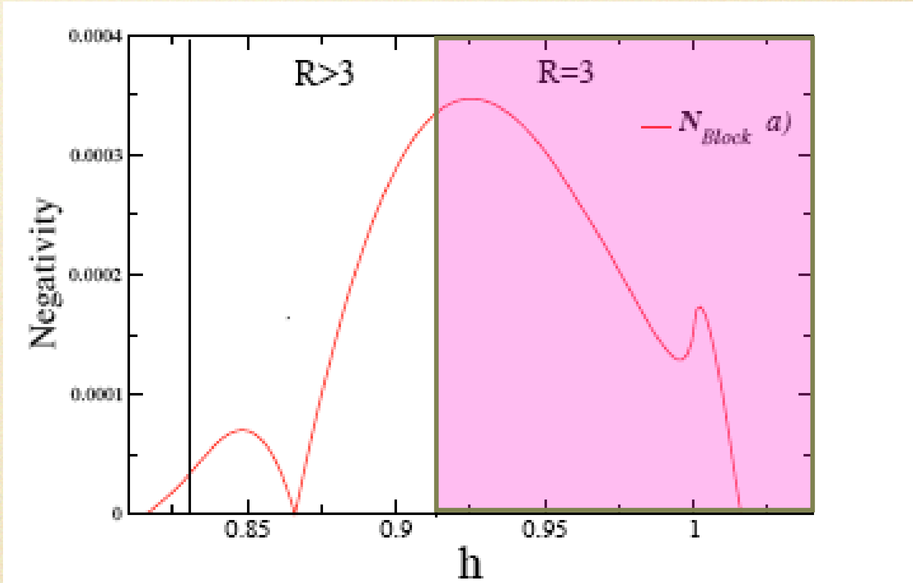


Patane', Fazio, Amico, 2007

Multi spin entanglement in the ground state of the quantum XY models



γ=0.5



Spin/block entanglement without spin-spin entanglement for distance d=4.

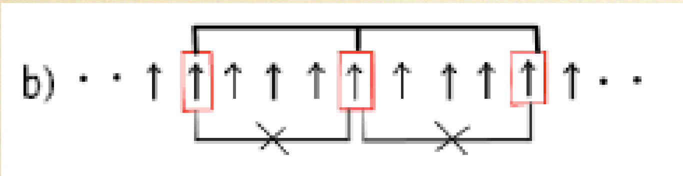
Entanglement survives following a 'microscopic' to 'macroscopic' hierarchy.

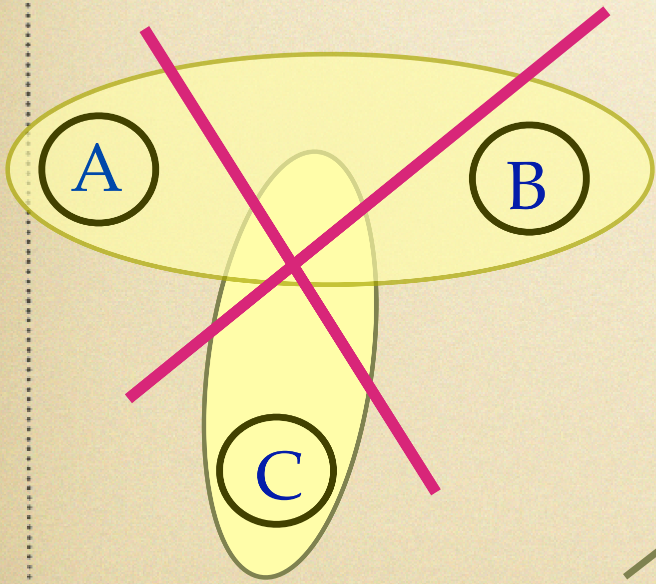
Amico, Fazio, Patanè 2007

Bound Entanglement in quantum XY models

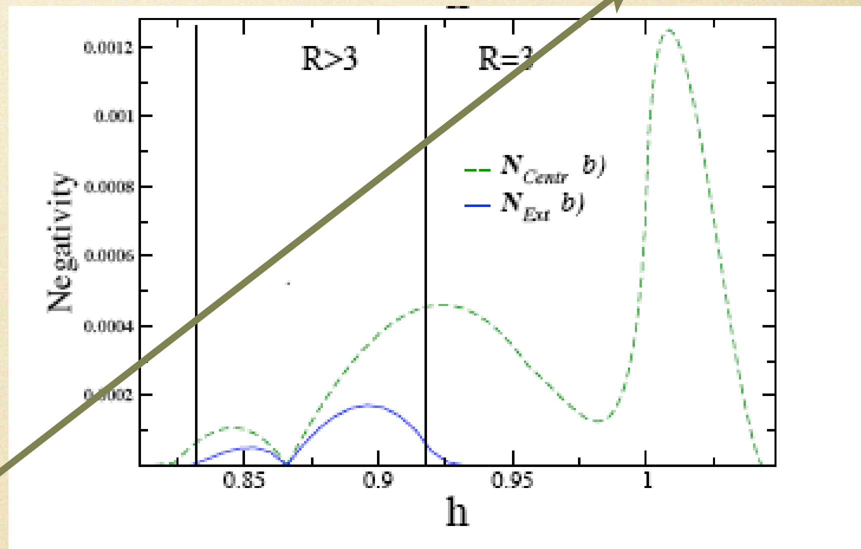
Bound entanglement: entanglement that cannot be distilled.

- Important to understand the interplay between entanglement and local realism
- Useful for certain protocols, but difficult to construct.





GROUND STATE



Incomplete separability: Bound Entangled states.

Horodecki et. al PRL 1998; Dur and Cirac PRA 2000; Dur 2001.

Patane', Fazio, Amico, NJP 2007

Bound entanglement in quantum spin chains at finite temperature

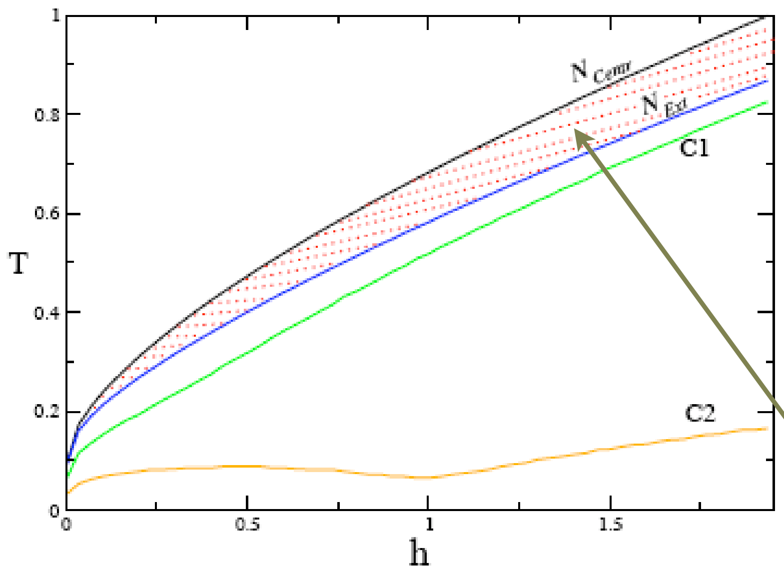
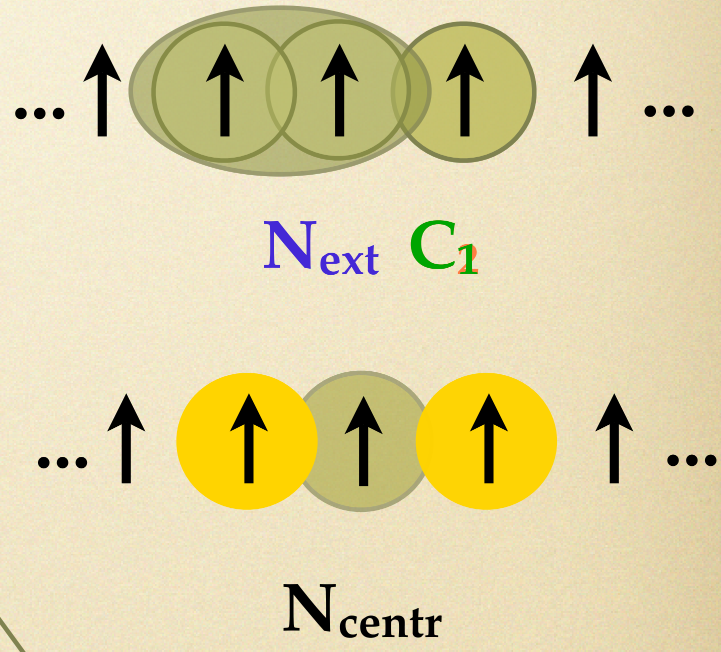


FIG. 3: Entanglement shared in a block of the three adjacent spins. We consider $\gamma = 1$. In this case only nearest neighbor and next nearest neighbor spins are entangled (hence $R = 2$) at $T = 0$ [3, 4]. The lines in the $T - h$ plane indicate the temperatures at which the corresponding type of entanglement disappears. In the marked region $T_{N_{Ext}} < T < T_{N_{Centr}}$ BE is present.



Bound Entanglement.

Conclusions

- Entanglement is sensitive to quantum criticality: similarities & differences with the ordinary correlators.
- Entanglement Crossover: quantum distilled of the mechanism bringing quantum effects up to finite temperatures.
- New light on traditional problems in condensed matter (Es: Kurman factorization; string-order parameter...).
- Entanglement propagation.
- Bound-entanglement in spin chains.

Amico, Fazio, Osterloh, and Vedral, quant-ph/0703044; to be published in Rev. Mod. Phys.