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Summer School on Novel Quantum Phases and Non-Equilibrium Phenomena in Cold Atomic Gases

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Introduction to the theory of the BEC/BCS crossover

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BCS-BEC CROSSOVER: AN INTRODUCTION

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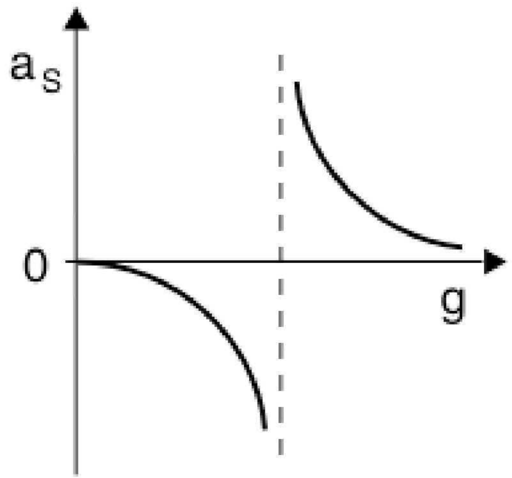
problem: Fermions $\uparrow\downarrow$ with density $n = k_F^3/3\pi^2$ and

attractive two particle interaction $V_{\uparrow\downarrow}(\vec{x}) = -\tilde{g} \cdot \delta(\vec{x})$

two-particle bound state : $g > g_c$

binding energy $\epsilon_b = \hbar^2/ma_s^2$



Superfluidity of Fermions

1911 conventional SC's Hg, Al, ... $T_c \approx 1 - 23\,K$

1960 pairing in nuclei, $\Delta \approx MeV$

1972 superfluid ^3He, $T_c = 1\,mK$ (p-wave)

1975 neutron stars

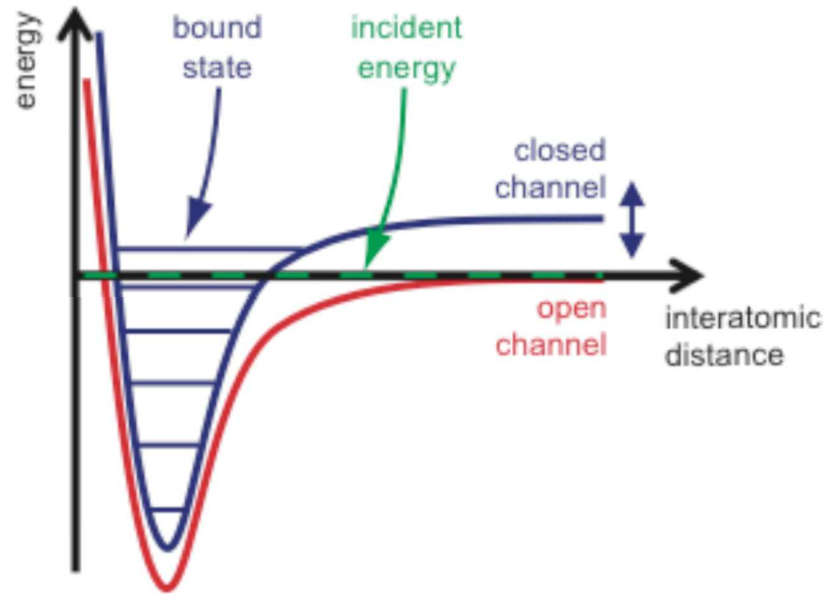
1986 high-T_c SC's $La_2Cu\ O_4, \ldots$ $T_c = 35 - 138$ K

1991 Alkali-doped C_{60} $T_c \approx 30$ K

1994 p-wave SC's in $Sr_2Ru\ O_4$ $T_c = 1.5$ K

1998 color SC's $\langle qq \rangle \neq 0$, $\Delta \approx GeV$

FESHBACH-RESONANCES



closed channel bound state

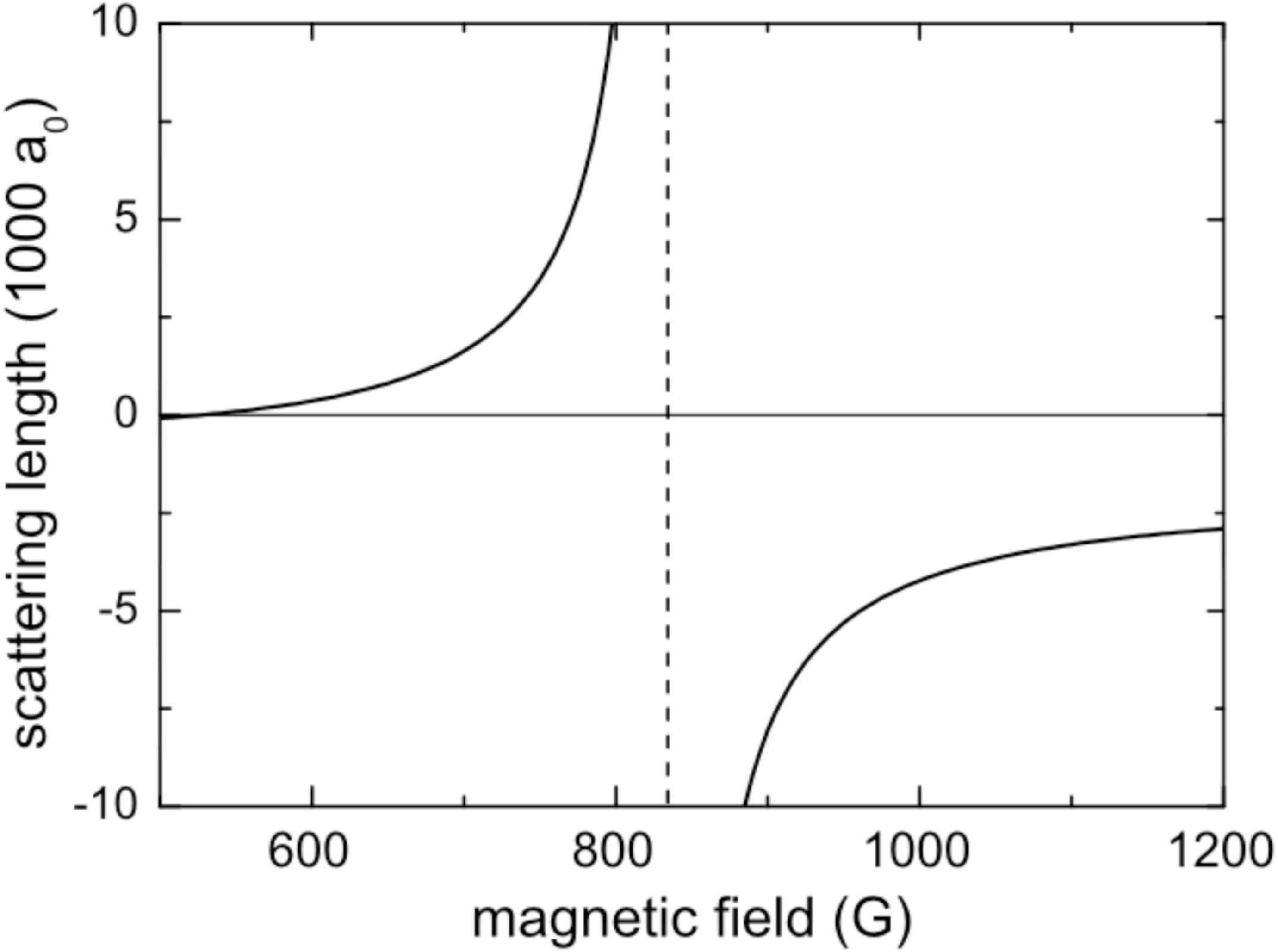
couples resonantly

$$a_s = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$$

scatt. ampl. $f(k) = \frac{-1}{1/a_s + r^\star k^2 + ik} \rightarrow \frac{i}{k}$ at $a_s = \infty$

effective range $r^\star \ll \lambda_F$, bound state for $a_s > 0$

scattering length in ^{6}Li



BCS-LIMIT: weak coupling if $k_F|a_s| < 0.5$ (in practice)

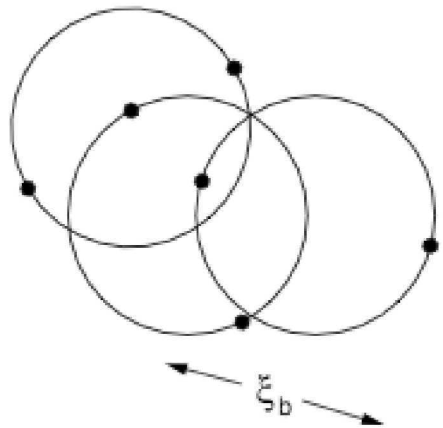
pairs form and condense at

$T_c \approx 0.3\, T_F \exp -\frac{\pi}{2k_F|a_s|} \ll T_F$ $2\Delta_0 = 3.52\, k_B T_c$

superfluidity is destroyed by pair- breaking

pair size $\xi_b = \xi_0 = \frac{\hbar v_F}{\pi \Delta_0}$ is large $\boxed{k_F \xi_b \gg 1}$

$$k_F \xi_0 \approx \begin{cases} 10^3 & \text{supercond.} \\ 10^2 & {}^3\text{He} \\ 10^1 & \text{high-}T_c \end{cases}$$



BCS-transition in a harmonic trap

local density approx. $\xi_b \ll R_{TF} \quad (\Delta_0 \gg \hbar\omega)$ valid

if $N \gg N^* \approx \exp 3\pi/2k_F|a_s| = 10^5$ at $k_F|a_s| = 0.4$

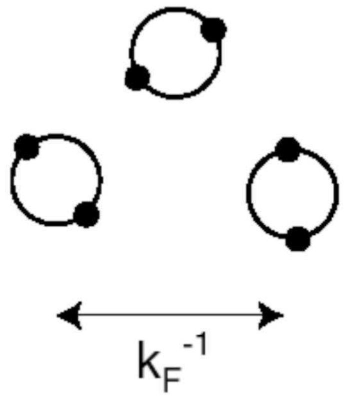
$N \ll N^*$: pairs formed within a single shell $\Delta_0 \ll \hbar\omega$

as in atomic nuclei (Bruun, Heiselberg, Mottelson 02)

BEC limit: strong coupling $|\epsilon_b| \gg \varepsilon_F \rightarrow k_F a_s \ll 1$

point bosons $\xi_b \approx a_s \rightarrow 0$ form far above T_c

pair size $\boxed{k_F \xi_b \ll 1}$



$\xi_b \neq$ coherence length ξ_0

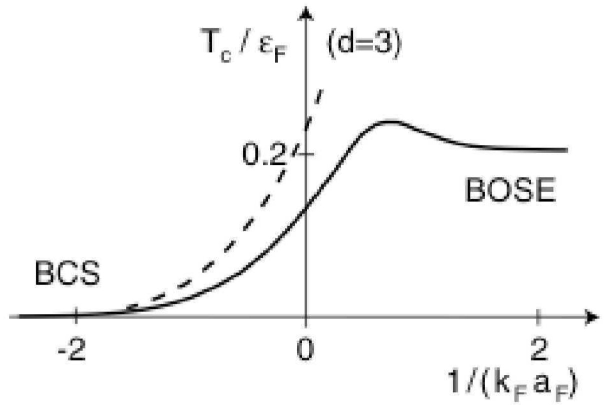
condensation temperature $T_c = 0.218\, T_F$ from

ideal Bose gas with $n_B = n/2$ and $m_B = 2m$

critical temperature

Nozieres/SR '85

Drechsler/Zw. '92



attractive fermions turn into repulsive bosons

$a_{dd} = 0.60\, a > 0$ Petrov/Shlyapnikov/Salomon '03

Universality for broad resonances $k_F r^\star \ll 1$ Ho '04

$\Delta_0 \approx 0.5\, \epsilon_F, \;\; \mu \approx 0.4\, \epsilon_F$ at $T = 0$

$T_c \approx 0.16\, T_F, \;\; \mu(T_c) \approx 0.42\, \epsilon_F, \;\; s(T_c) \approx 0.7\, k_B$

Outline:

I) BCS-BEC crossover in 3D

1) Thermodynamics   Haussmann, Rantner, Cerrito

2) Universal properties at resonance

II) Exact solution of the 1D problem

1) Bethe-Ansatz   Fuchs, Recati

2) Imbalanced case, FFLO

III) Imbalanced gases, RF-spectroscopy   Punk

BCS groundstate (variational Ansatz for arbitrary g)

$$\psi_{\text{BCS},N} = \hat{A}\,\{\underbrace{\phi(1\,2)\,\phi(3\,4)\cdots\phi(N-1\,N)}\}$$

ideal Bose gas of identical pairs

pair-state $\quad \phi(1,2) = \psi(\vec{x}) \cdot \chi(\vec{r}) \cdot \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

internal wavefunction $\chi(\vec{r}) \sim \exp -r/\xi_b \rightarrow$ pair radius ξ_b

COM-coord. $\vec{x}$ $\rightarrow$ inhomogeneous pair-amplitude $\psi(\vec{x})$

increase g: extended $\rightarrow$ local pairs $\rightarrow$ smooth crossover

(Keldysh 1965, Eagles 1969, Leggett 1980)

BCS Hamiltonian $\hat{H}_{\mathrm{BCS}} = -\frac{\tilde{g}}{V}\, b_0^\dagger b_0$ (only $q = 0$ pairs)

exact ground state for fixed μ is a coherent state

$$|\psi\rangle_{\mathrm{BCS}} = \prod_k (u_k|00\rangle_k + v_k|11\rangle_k) \sim \exp \sum_k \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |0\rangle$$

at fixed N (even): $|\psi_N\rangle_{\mathrm{BCS}} \sim \left(\sum_k \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right)^{N/2} |0\rangle$

$\chi_k = v_k/u_k$ = Fouriertransf. of internal wavefunction gap parameter Δ_k follows from gap-equation

$$\frac{1}{\tilde{g}} = \frac{1}{V} \sum_k \frac{1}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}}$$

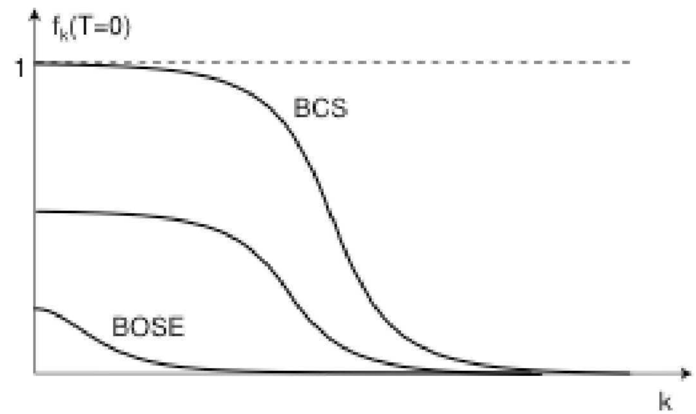
chemical potential μ from number equation ($\mu_{\text{BCS}} = \epsilon_F$)

$$\langle N \rangle = 2 \sum_k v_k^2 = \sum_k \left(1 - \frac{\epsilon_k - \mu}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}} \right)$$

momentum distrib.

$$f_k = v_k^2 \rightarrow \frac{4\pi n \xi_b^3}{(1 + (k\xi_b)^2)^2}$$

since $\mu_{\text{BEC}} \rightarrow -\epsilon_b/2$



exact solution of $\hat{H}_{\text{BCS}}$ at $T = 0$ and fixed N

Richardson +Gaudin 1963, Ortiz/Dukelsky PRA 2005

exact dynamics Barankov/Levitov ...

are Cooper pairs Bosons ?

antisymmetrization $\hat{A}$ reduces the condensate fraction

N_0/N to $\approx \Delta_0/\epsilon_F \ll 1$ in weak coupling, nevertheless

BCS-superfluidity is BEC of pairs !

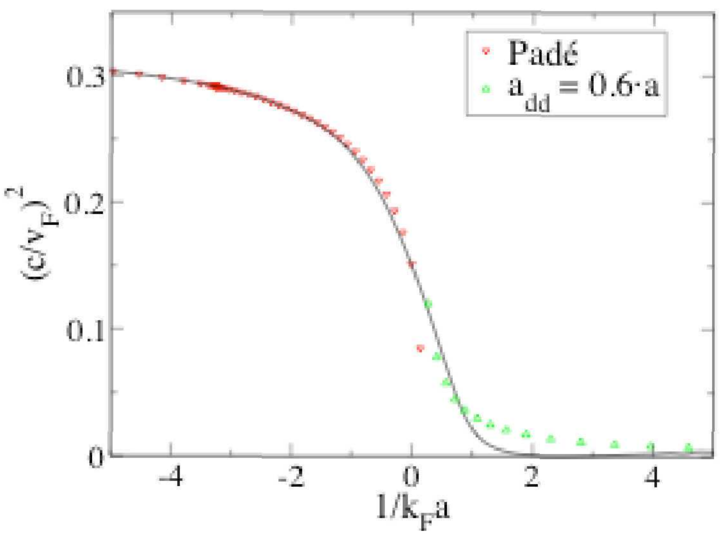
Low energy description ground state is a

neutral s-wave superfluid for arbitrary coupling

excitations $\omega(q) = cq$ Bogoliubov/Anderson '58

universal thermodynamics at low temperatures

$$S(T)/V = \frac{2\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3$$



sound velocity c

Many-body theory single channel model with

pseudopotential $V_{\uparrow\downarrow}(\mathbf{x}) = \bar{g}\,\delta(\mathbf{x})$ ren. $g = 4\pi\hbar^2 a/m$

Luttinger/Ward '60 $\Omega = -T \ln Z = \Omega(\widehat{G})$

$$\Omega[\widehat{G}] = \beta^{-1}\Big(-\tfrac{1}{2}\mathrm{Tr}\{-\ln\widehat{G} + [\widehat{G}_0^{-1}\widehat{G} - 1]\} - \Phi[\widehat{G}]\Big)$$

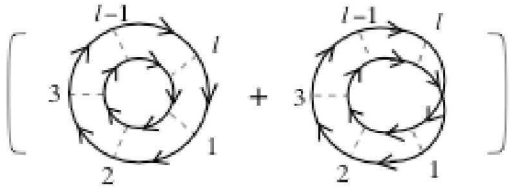
DeDominicis/Martin '64 entropy $S[\widehat{G},\widehat{\Gamma}]$ as a functional

of one- and two-particle functions $\widehat{G}(\mathbf{k},\omega)$ and $\widehat{\Gamma}(\mathbf{K},\Omega)$

$$\delta\Omega[\widehat{G},\widehat{\Gamma}]/\delta\widehat{G} = 0 \quad \text{and} \quad \delta\Omega[\widehat{G},\widehat{\Gamma}]/\delta\widehat{\Gamma} = 0$$

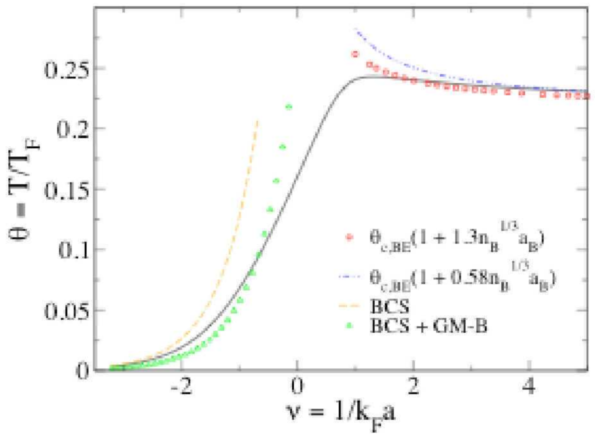
variational principle for functions !

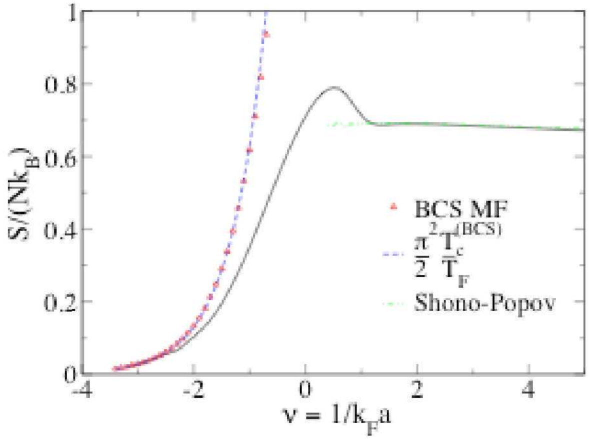
Ladder-approx. $\Phi[G] = \sum_{l=1}^{\infty}$



enforce gapless nature by a modified coupling constant

critical temperature and **entropy** at T_c

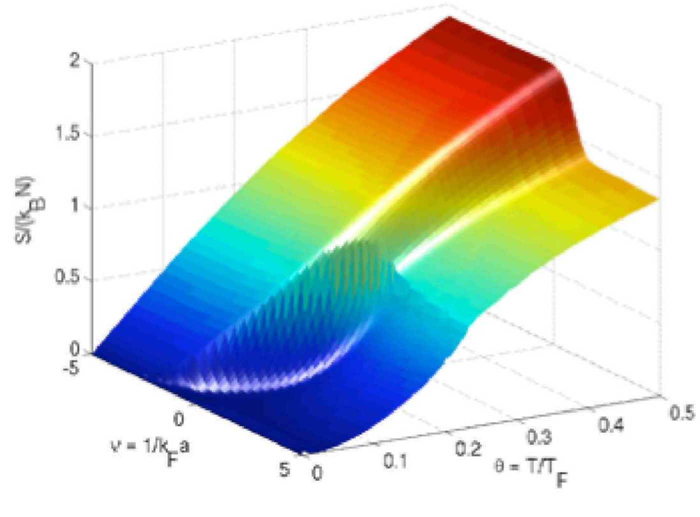




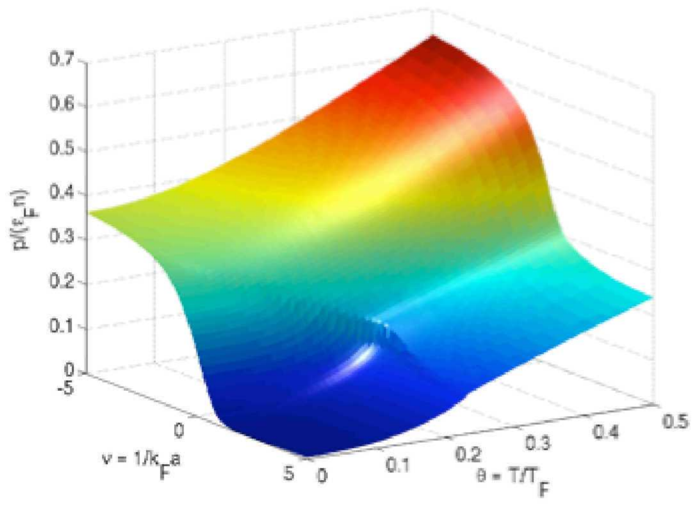
at unitarity $T_c/\varepsilon_F = 0.160$ $S/Nk_B = 0.71$

Burovski et al '06 $T_c/\varepsilon_F = 0.152(7)$ $S/Nk_B = 0.16$?

entropy



pressure



Universality at $a = \infty$ $F/N = \varepsilon_F \cdot g(T/T_F)$ Ho '04

implies $p = 2U/3V$ as if interactions were $1/r^2$

chemical potential $\mu/\varepsilon_F = \xi(T/T_F) = \begin{cases} 0.36 \text{ at } T = 0 \\ 0.39 \text{ at } T = T_c \end{cases}$

determines cloud size in a trap $R_{TF} = R_{TF}^{(0)} \cdot \xi^{1/4}$

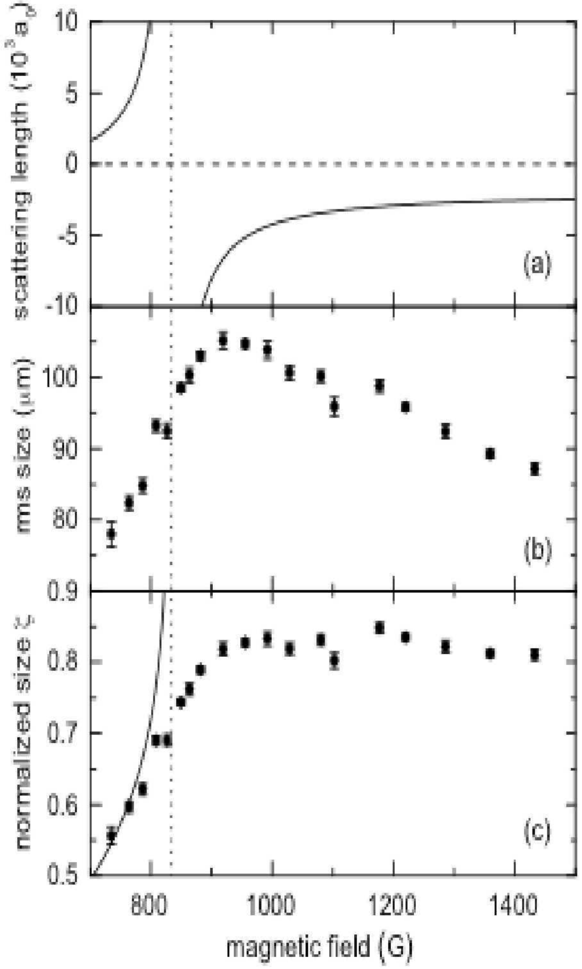
and critical temp. $T_c/\varepsilon_F(N) = 0.16 \cdot \xi^{-1/2}(T_c) \approx 0.25$

field theory $\epsilon = 4 - d$ - expansion Nishida/Son '06

$\xi(T = 0) = \frac{1}{2}\epsilon^{3/2} + \frac{\epsilon^{5/2}}{16} \ln \epsilon + \ldots = 0.47$ at $\epsilon = 1$

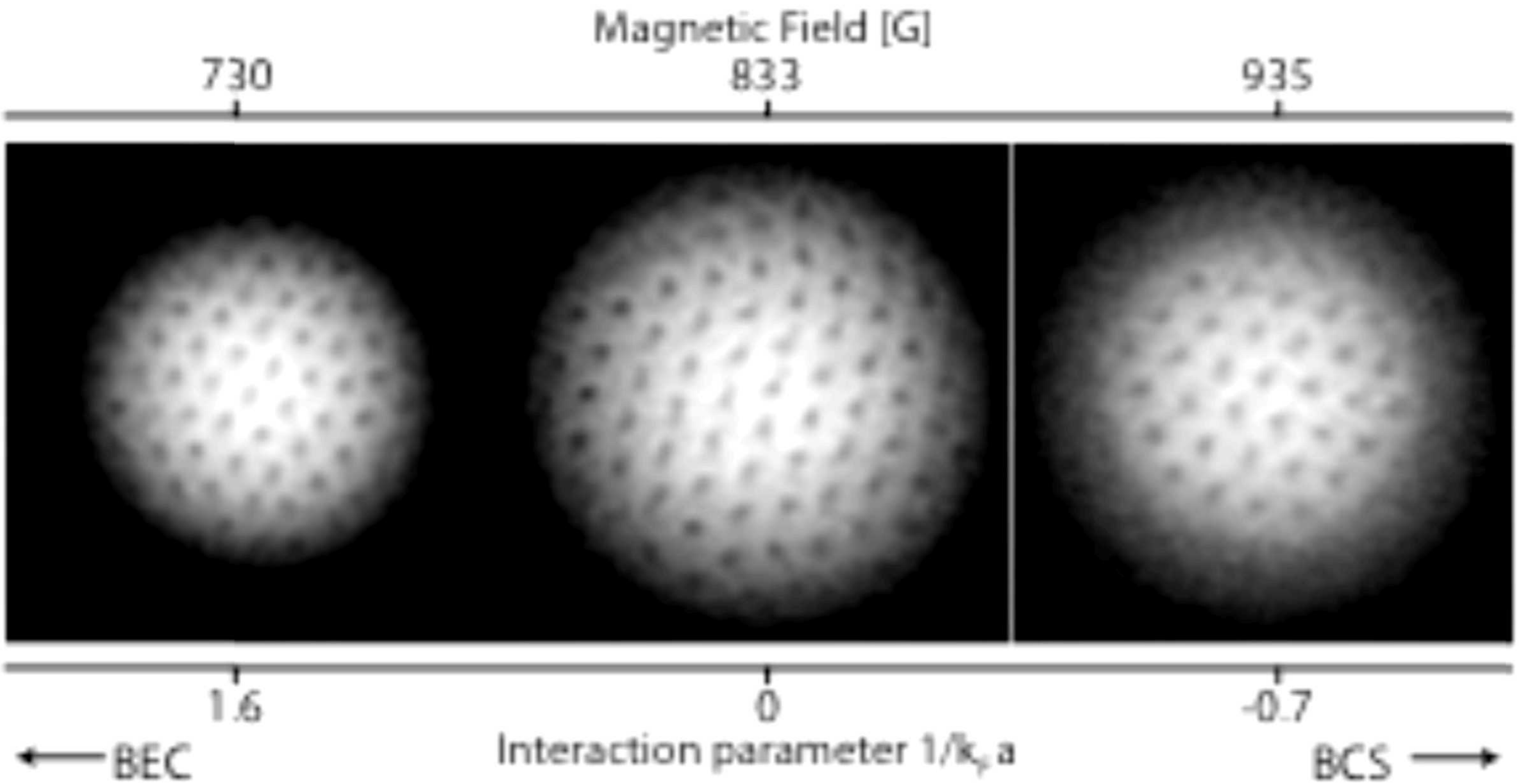
$1/N$ - expansion $\xi(T_c) = 0.59$ at $N = 1$ Nikolic/S. '06

cloud size (in situ)



		β
Experimental results	Bartenstein *et al.* [61]	$-0.68^{+0.13}_{-0.10}$
	Bourdel *et al.* [62]	−0.64(15)
	Duke [63]	−0.49(4)
	Partridge *et al.* [64]	−0.54(5)
Calculated values	Astrakharchik *et al.* [17]	−0.58(1)
	Carlson *et al.* [16]	−0.56(1)
	Hu, Liu, and Drummond [67]	−0.599
	Perali *et al.* [65]	−0.545
	Padé approximation[8,9]	−0.67
	Present work	−0.64

vortices as a signature of superfluidity MIT

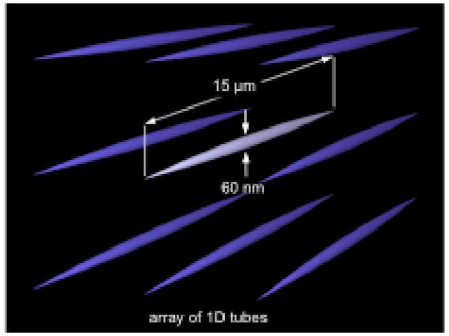


Cold Atoms in 1D (Bloch, Weiss, Schmiedmayer, ..)

strong optical lattice

in two directions

typ. $\approx 10^3$ wires



pseudopotential $U(x) = g_1 \delta(x)$ with $g_1 = 2\hbar\omega_\perp \cdot a_s$

repulsive Bosons $\gamma = \frac{\epsilon_{int}}{\epsilon_{kin}} = \frac{g_1 n}{\frac{\hbar^2}{m} n^2} = \frac{2a_s}{n l_\perp^2}$

strong coupling at low 1D densities $n \ll a_s / l_\perp^2$

Tonks-limit $\gamma = \infty$: 1D Bosons with hard-core int.

$$\Psi_B(x_1 \dots x_N) = \prod_{i<j} |\sin\left[\pi(x_j - x_i)/L\right]| =$$

$$= |\Psi_F^{(0)}(x_1 \dots x_N)| \leftrightarrow$$ free Fermions (Girardeau)

Strongly interacting fermions in 1D

arbitr. small attraction gives two-particle binding !

Gaudin/Yang '67: Bethe-Ansatz solution of

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_\sigma \int \hat{\Psi}_\sigma^\dagger \, \partial_x^2 \hat{\Psi}_\sigma + \frac{g_1}{2}\sum_\sigma \int \hat{\Psi}_\sigma^\dagger \hat{\Psi}_{-\sigma}^\dagger \hat{\Psi}_{-\sigma} \hat{\Psi}_\sigma$$

dim.less interaction constant $\gamma = mg_1/\hbar^2 n < 0$

H describes a Luther-Emery liquid : extended

pairs if $|\gamma| \ll 1$ transform to local pairs if $|\gamma| \gg 1$

BEC-limit: pairs are hard–core bosons (TG-gas)

Olshanii '98: scattering theory in a single channel wire

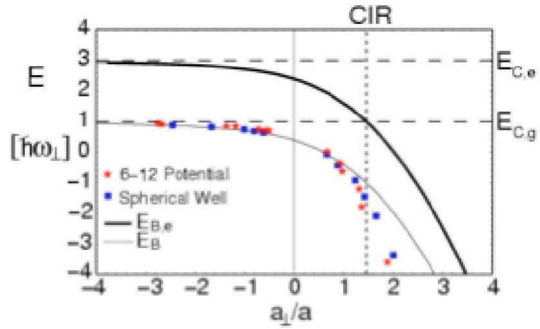
eff. pot. between fermions with opposite spin $g_1\,\delta(x)$

$$g_1(a) = 2\hbar\omega_\perp \cdot \frac{a}{1 - A\,a/a_\perp}$$

confinement induced resonance (CIR) at $a = a_\perp/A \approx a_\perp$

plus a bound state

for arbitrary sign of a



binding energy at CIR: $|\epsilon_b| = 2\hbar\omega_\perp \gg \epsilon_F \;\; \rightarrow$

BEC–side has unbreakable dimers with repulsive

interaction $g_1 \rightarrow 2\hbar\omega_\perp \cdot 0.6\,a$ (Mora et al '05)

$\gamma < 0$: increasing $|\gamma|$ transforms Cooper-pairs

to tightly bound molecules; size at CIR is $\approx a_\perp$ $\rightarrow$

hard core Bosons form Tonks gas

$\gamma > 0$: Boson size shrinks $\rightarrow$ weakly interacting BEC

Bethe-Ansatz (Gaudin/Yang $\rightarrow$ Lieb/Liniger)

$$\psi_\sigma(x_1, \ldots x_N) = \sum_P A_\sigma\{P\} \exp\, i \sum_j k_{j_P} x_j$$

$N \rightarrow \infty$: quasimomenta distribution function $\rho(k)$

$$\pi\rho(k) = 1 + \int_{-K}^{K} \frac{dq}{n} \frac{\gamma\rho(k)}{\gamma^2 + [(k-q)/n]^2}$$

for both $\gamma < 0$ and $\gamma > 0$

excitations: a) **fermionic** for $\gamma < 0$

triplet excit. (2 spinons): $\omega^s(q) = \sqrt{(\Delta/2\hbar)^2 + (v_s q)^2}$

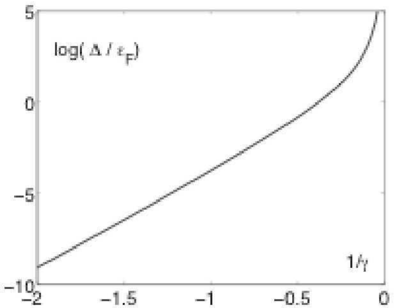
gap $\quad \Delta \to 2\Delta_{BCS} = \epsilon_F\, 16\pi^{-3/2}|\gamma|^{1/2} \exp -\pi^2/2|\gamma|$

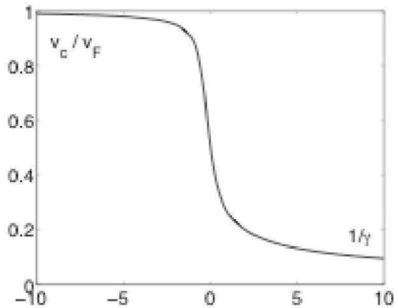
pair size reaches n^{-1} at $\quad \gamma \approx -2$

spin gap at strong coupling $\Delta \to |\epsilon_b| = 2\hbar\omega_\perp$

b) **bosonic** collective excit. for arbitrary γ

Bogoliubov-Anderson mode: $\omega^c(q) = v_c \cdot q$





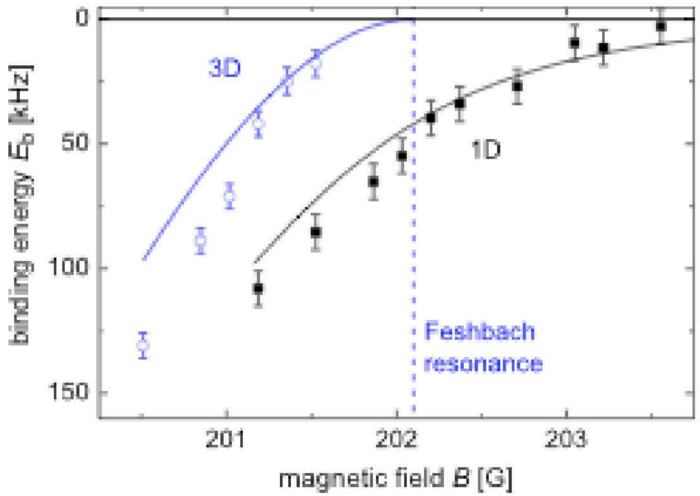
Experiments: Feshbach in **1D** (^{40}K, Moritz '05)

$\omega_z \approx 2\pi \cdot 250\,\mathrm{Hz}$, particle number $N \sim \epsilon_F/\hbar\omega_z \approx 100$

$\omega_\perp \approx 2\pi \cdot 69\,\mathrm{kHz}$, temperature $T \approx 0.2\,T_F$

RF-spectroscopy of

two-particle binding



Imbalanced 1D gases Hu/Liu/Drummond, Orso '07

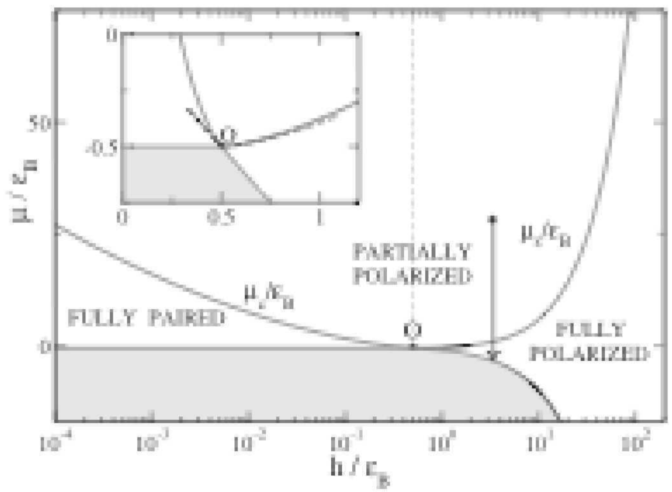
Clogston-Chandrasekhar $h_c \equiv \Delta/2$ where $n_\uparrow \neq n_\downarrow$

saturation field h_s beyond which $n_\downarrow \equiv 0$

$T = 0$ phase diagram

partially polarized phase

exhibits FFLO - order



$$\langle \hat{\Psi}^\dagger_\uparrow(x)\hat{\Psi}^\dagger_\downarrow(x)\hat{\Psi}_\downarrow(x')\hat{\Psi}_\uparrow(x')\rangle \sim \exp\left[iQ(x-x')\right]/|x-x'|^{1/K}$$

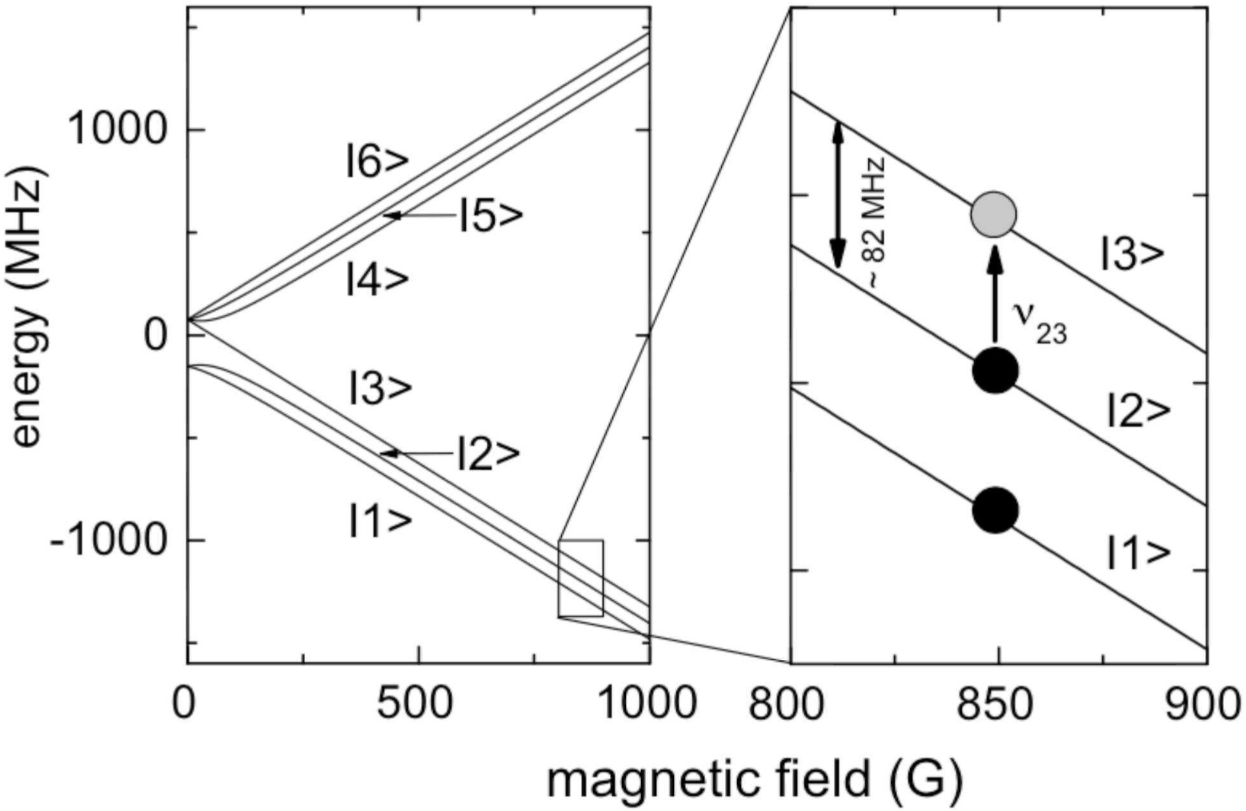
Novel phases in imbalanced Fermi gases

fermionic superfluids possess a pairing gap

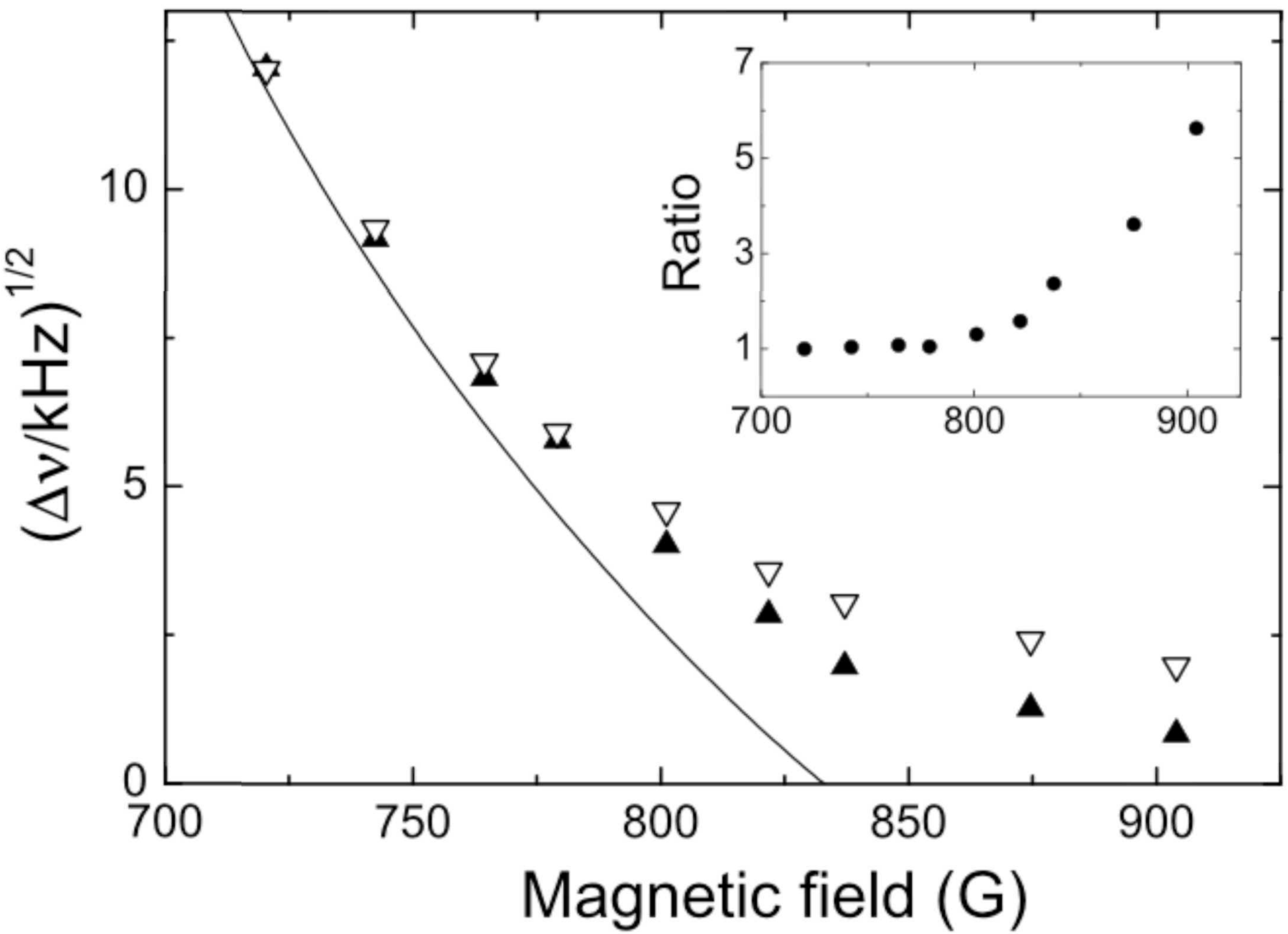
measurable by **RF-spectroscopy** Chin et al '04

a) what is actually measured near unitarity ?

b) is the appearance of a 'gap' proof of **superfluidity** ?



energy (MHz)
1000
0
-1000
|6>
|5>
|4>
|3>
|2>
|1>
0
500
1000
800
850
900
~ 82 MHz
ν23
|3>
|2>
|1>
magnetic field (G)



$(\Delta\nu/\mathrm{kHz})^{1/2}$
0
5
10
700
750
800
850
900
Magnetic field (G)
Ratio
1
3
5
7
700
800
900

RF-spectrum as a probe of superfluidity

$$I(\omega) \sim \int dt\, e^{i\omega t} \langle \left[\hat{\Psi}_3^\dagger(t) \hat{\Psi}_\downarrow(t), \hat{\Psi}_\downarrow^\dagger(0) \hat{\Psi}_3(0) \right] \rangle$$

peak shift $\hbar\bar{\omega} = \frac{\langle H'_{12} \rangle}{N_2} \left(\frac{\bar{g}_{13}}{\bar{g}_{12}} - 1 \right) \to \frac{\langle H'_{12} \rangle}{N_2 \Lambda} \frac{\pi}{2} \left(\frac{1}{a_{13}} - \frac{1}{a_{12}} \right)$

energy $E = 2 \sum_k \varepsilon_{\mathbf{k}} (n_{\mathbf{k}} - C/k^4)$ $\quad C = s k_F^4$

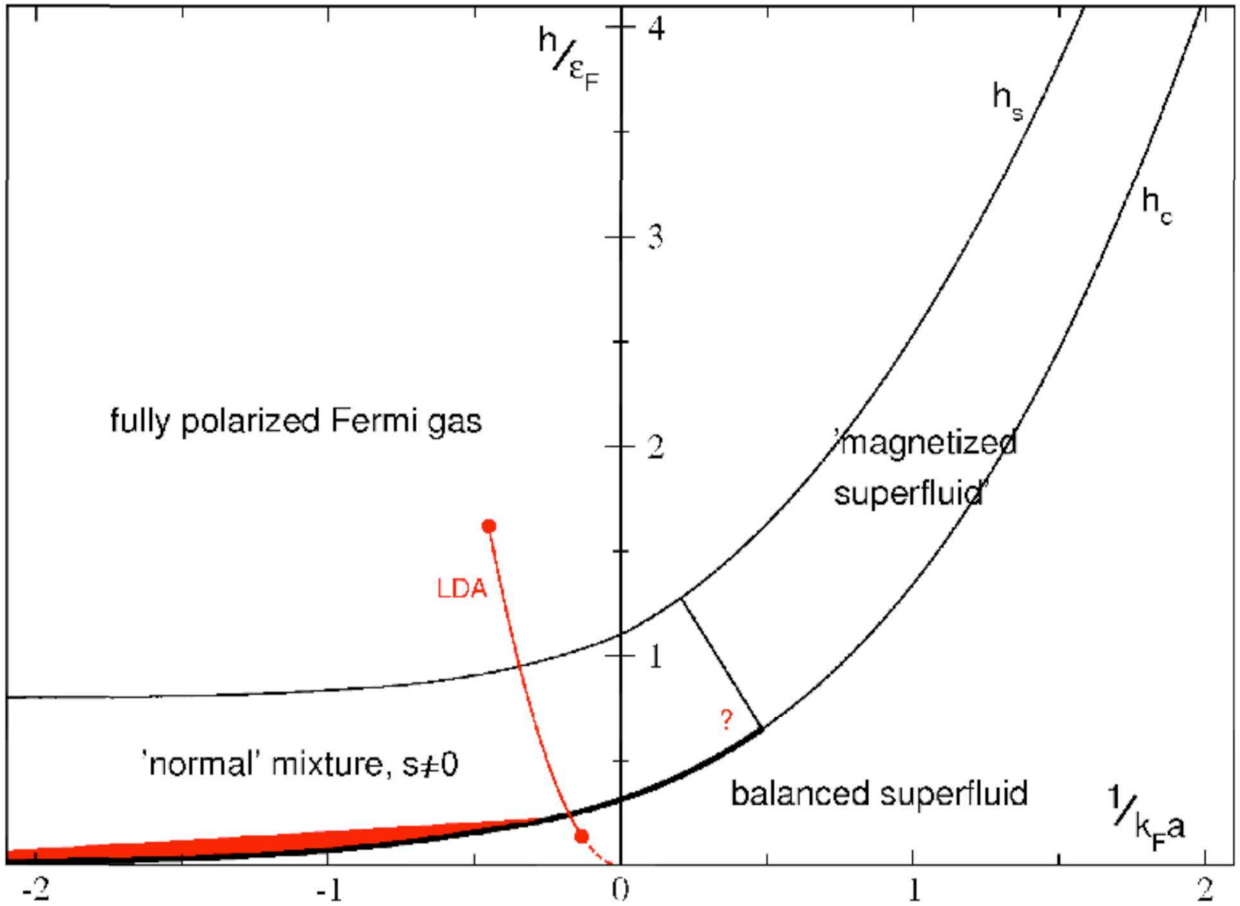
$$\hbar\bar{\omega} = s \cdot \frac{4\varepsilon_F^2}{n_2} \left(\frac{1}{g_{12}} - \frac{1}{g_{13}} \right)$$

Punk/Zw. '07

gives $\hbar\bar{\omega} = \begin{cases} 2\varepsilon_b (1 - a_{12}/a_{13}) & \text{BEC-limit} \\ -0.19\, \hbar^2 k_F / m a_{13} & \text{at unitarity} \end{cases}$

Imbalanced Fermi gases Partridge, Zwierlein '06

$N_\uparrow - N_\downarrow \neq 0$ implies $\mu_\uparrow - \mu_\downarrow \equiv 2h \neq 0$



Clogston-Chandrasekhar field near unitarity

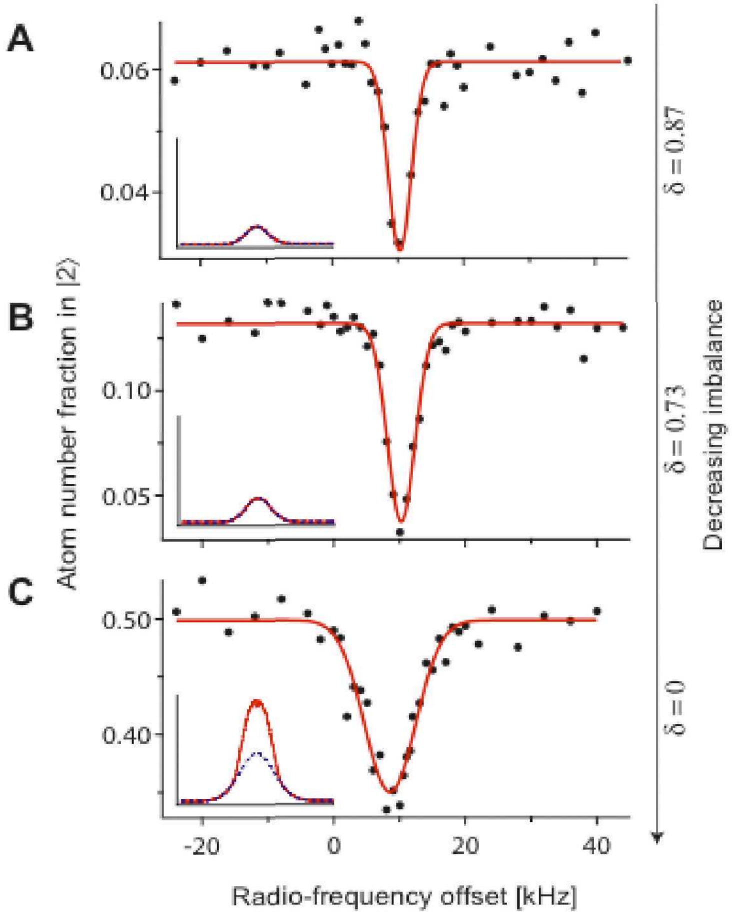
Zwierlein et al '06 disappearance of vortex lattice

beyond a critical imbalance $\delta_c \approx 0.77$

Lobo et al '06 $h_c \approx 0.96\,\mu \quad \rightarrow h_c \approx 0.4\,\epsilon_F$

saturation field h_s to a fully polarized Fermi gas

Chevy '06: $h_s \geq 0.6\,\mu_\uparrow$ implies $h_s \geq 1.27\,\epsilon_F$



RF in imbalanced gases

Schunck et al '07

Conclusions

1) The BCS-BEC crossover in the balanced case is qualitatively well understood, precise results for universal ratio's and dynamical prop. are still open.

2) Imbalanced gases exhibit novel phases, like a 'magnetized superfluid' or (possibly) FFLO order. FFLO appears accessible in 1D imbalanced gases.

Collective mode damping and viscosity

shear viscosity η from sound damping (hydrodynamic)

$$\omega(q) = cq - i\,\frac{2\eta}{3\rho}\cdot q^2 + \dots$$

$\alpha_\eta = \eta/\hbar n \sim m\langle v\rangle\ell/\hbar \to k_F\ell \geq (6\pi)^{-1}$ Shuryak '04

is the unitary Fermi gas a perfect liquid ? Son '07

quantum hydrodynamics $\hat{H}_0 = \frac{c}{2}\int\left[\frac{\rho}{c}\,(\nabla\varphi)^2 + \frac{c}{\rho}\,\Pi^2\right]$

phase $\hat{\varphi}(\mathbf{x})$ and density fluctuations $\hat{\Pi}(\mathbf{x})$

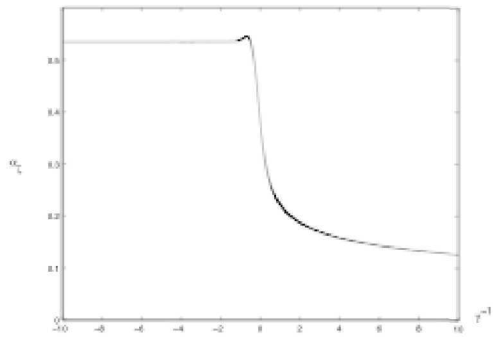
$$\hat{H}' = \frac{1}{6}\int\left[\nabla\varphi\Pi\nabla\varphi + (\nabla\varphi)^2\Pi + \Pi(\nabla\varphi)^2 + \frac{d}{d\rho}\left(\frac{c^2}{\rho}\right)\Pi^3\right]$$

self-energy $\Sigma(q, \omega = cq) = -i\,\frac{\zeta}{2\rho}\cdot q^2$ at $T = 0$

$\zeta = \alpha_\zeta \hbar n$ with $\alpha_\zeta = \frac{1}{4}\sqrt{\left(\frac{\pi}{4} - \frac{1}{2}\right)\frac{c}{v_F}}\left\{3 + \frac{v_F^2}{c^2}\frac{d}{dv_F}\left(\frac{c^2}{v_F}\right)\right\}$

at unitarity (CIR)

$\alpha_\zeta = \frac{1}{2}\sqrt{\pi/2 - 1} = 0.38$



$T \neq 0$ Im $\Sigma(q, \omega = cq) \sim \sqrt{T} \cdot q^{3/2}$ Andreev '80

sound damping is not hydrodynamic at finite T

dynamical scaling $\Gamma_q = \frac{\hbar q^2}{2m}\Phi(q\xi_T)$ with

$\xi_T = \hbar c / T$ and $\Phi(x) = \begin{cases} \alpha_\zeta & \text{for } q\xi_T \gg 1 \\ 3.7\alpha_\zeta/\sqrt{x} & \text{for } q\xi_T \ll 1 \end{cases}$

partially polarized, normal phase $h_c < h < h_s$

poles of pair scattering amplitude $\Gamma_{\uparrow,\downarrow}(\mathbf{q}=0,\omega)$

excitations at $2h - \Omega_+$ and $-2h + \Omega_-$

binding energy of $\uparrow,\downarrow$-pairs

approaches $0.6\,\mu_\uparrow$ for

$h \gg h_c$ (Chevy '06)