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Theory of BEC/BCS crossover with p-wave Feshbach resonances

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p-wave BCS-BEC condensates

Victor Gurarie

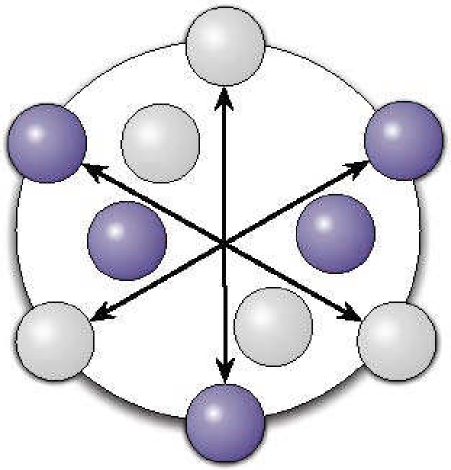
Thanks to:

- L. Radzihovsky (Boulder)
- A. Andreev (UWa, Seattle)
- J. Levinsen (Boulder)
- N. Cooper (Cambridge, UK)

Review: BCS-BEC crossover, *s*-wave

Fermions of two species (spins) with attractive interactions

BCS



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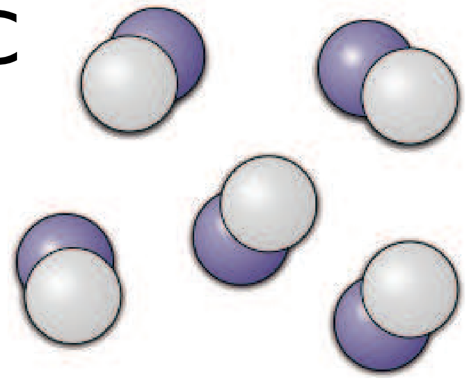
Interactions getting stronger

BEC



Review: BCS-BEC crossover, *s*-wave

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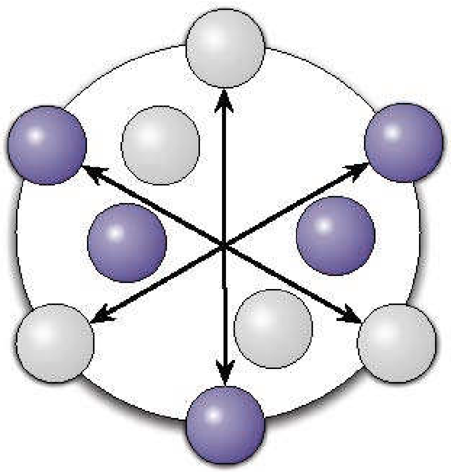

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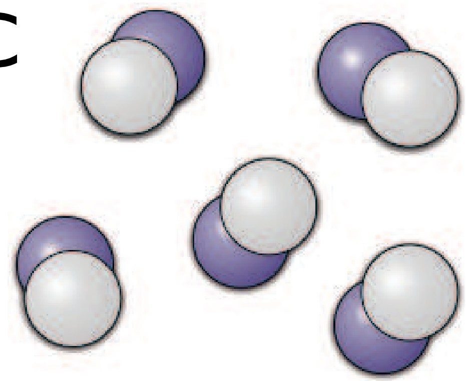
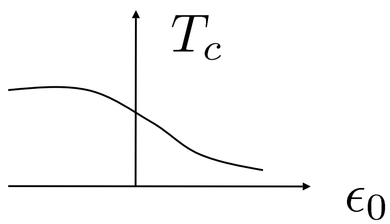
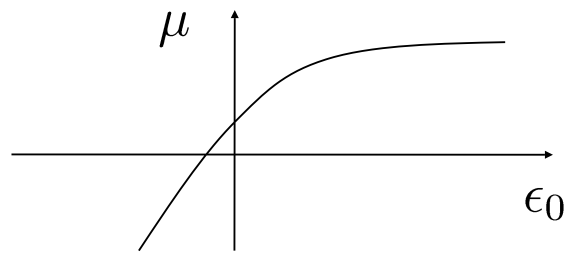


$$H = \sum_{p,\sigma} \frac{p^2}{2m_a}\, a^\dagger_{p\sigma} a_{p\sigma} + \sum_{p} \left(\epsilon_0 + \frac{p^2}{2m_m} \right) b^\dagger_p b_p + \sum_{p,q}\ \frac{g}{\sqrt{V}} \left(b_q\ a^\dagger_{p+q\uparrow} a^\dagger_{-p\downarrow} + b^\dagger_q\ a_{-p\downarrow} a_{p+q\uparrow} \right)$$





p-wave BCS-BEC condensates:

- Using identical fermionic atoms suppresses *s*-wave, leads to *p*-wave.
- Could be in one of two possible regimes: p_x+ip_y (axial) or p_x (polar).
- Identical atoms with *p*-wave resonances prefer to be in the p_x+ip_y regime.
- BCS to BEC is a phase transition. For p_x+ip_y it is a topological phase transition.
- BCS p_x+ip_y phase in 2D is a topological phase which supports non-Abelian excitations and could be used for decoherence-free quantum computing and quantum information storage.
- Not yet clear if the *p*-wave superfluids can be made stable.

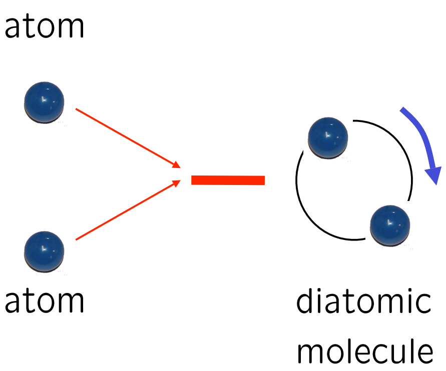
p-wave Feshbach resonance

$$H = \sum_p \frac{p^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \sum_{q,\alpha} \left(\varepsilon + \frac{q^2}{4m} \right) b^\dagger_{\alpha \mathbf{q}} b_{\alpha \mathbf{q}} + \sum_{\mathbf{p},\mathbf{q},\alpha} \frac{g}{\sqrt{V}} \left(b_{\alpha \mathbf{q}}\, p_\alpha\, a^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}} a^\dagger_{-\mathbf{p}+\frac{\mathbf{q}}{2}} + h.c. \right)$$

resonance

molecules

atoms



3D Dimensionless parameters

Narrow vs wide: $\gamma \sim \frac{g^2}{l}$ ← Interparticle spacing

Weak vs strong: $c_2 \sim \frac{g^2}{R_e}$ ← Interaction range

γ small – narrow – mean field theory applies

c_2 large – strong – there are nontrivial few body effects

(trimer formation).

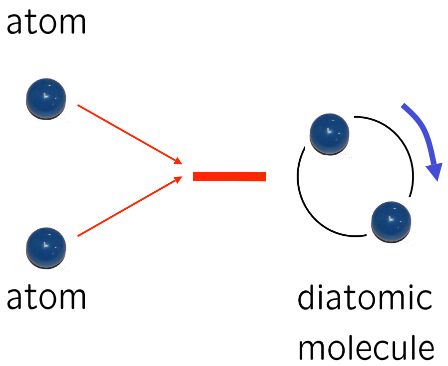
$\gamma \sim 1/20$

$c_2 \sim 10$

2D Dimensionless parameter: g

The physics of the resonance

$$H = \sum_p \frac{p^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \sum_{q,\alpha} \left(\varepsilon + \frac{q^2}{4m} \right) b^\dagger_{\alpha\mathbf{q}} b_{\alpha\mathbf{q}} + \sum_{\mathbf{p},\mathbf{q},\alpha} \frac{g}{\sqrt{V}} \left(b_{\alpha\mathbf{q}}\, p_\alpha\, a^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}} a^\dagger_{-\mathbf{p}+\frac{\mathbf{q}}{2}} + h.c. \right)$$



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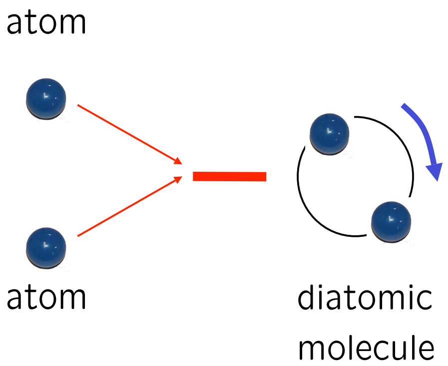

Scattering amplitude

$$f(p) = \frac{p^2}{-\frac{1}{v} + \frac{1}{2} k_0 p^2 - i p^3}$$

Landau & Lifshitz, vol 3

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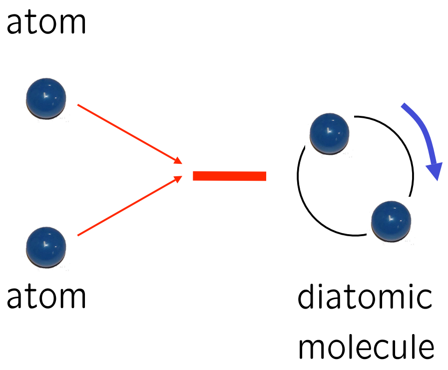

Scattering amplitude

$$f(p) = \frac{p^2}{-\frac{1}{v} + \frac{1}{2} k_0 p^2 - i p^3}$$

Landau & Lifshitz, vol 3

$$\frac{1}{v} = -\frac{6\pi}{mg^2} \left(\varepsilon - \frac{mg^2}{9\pi^2} \Lambda^3 \right)$$

$$k_0 = -\frac{12\pi}{m^2 g^2} \left(1 + \frac{m^2}{3\pi^2} g^2 \Lambda \right)$$

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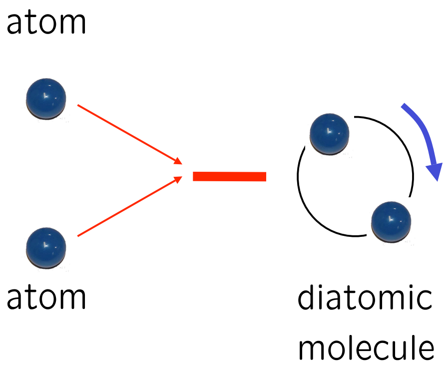
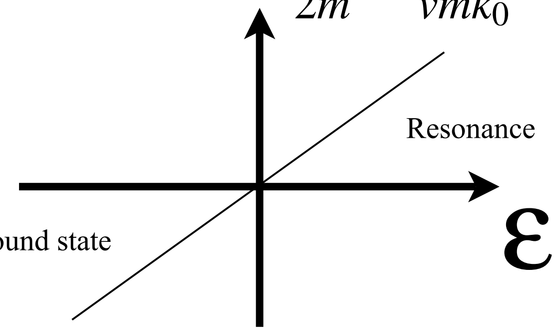


Scattering amplitude

$$f(p) = \frac{p^2}{-\frac{1}{v} + \frac{1}{2} k_0 p^2 - i p^3}$$

Landau & Lifshitz, vol 3

$$f^{-1}(p) = 0 \to \frac{p^2}{2m} \approx \frac{1}{v m k_0}$$



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Mean Field Approximation

$$H = \sum_p \frac{p^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \sum_{q,\alpha} \left(\varepsilon + \frac{q^2}{4m} \right) b^\dagger_{\alpha\mathbf{q}} b_{\alpha\mathbf{q}} + \sum_{\mathbf{p},\mathbf{q},\alpha} \frac{g}{\sqrt{V}} \left(b_{\alpha\mathbf{q}}\ p_\alpha\ a^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}} a^\dagger_{-\mathbf{p}+\frac{\mathbf{q}}{2}} + h.c. \right)$$

Substitute: $\langle b_{\alpha\mathbf{p}} \rangle \rightarrow \delta_{\mathbf{p},0} B_\alpha$

$$B_\alpha = u_\alpha + i v_\alpha$$
$$u_\alpha v_\alpha = 0$$

$\left\langle b_{0,\alpha} \right\rangle = B_\alpha$ is a molecular condensate, which could be

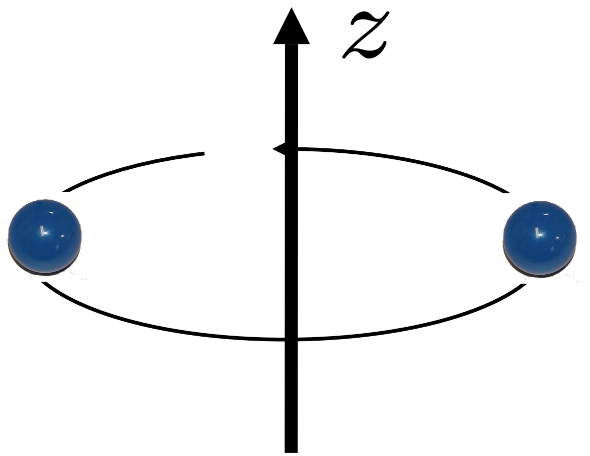
$B_\alpha = B \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$, $p_x + i p_y$ phase

$$u = v$$

$B_\alpha = B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, p_z phase

$$v = 0$$

Two phases of a *p*-wave spinless superfluid



	p_z	p_x+ip_y
L_z	0	± 1
B_α	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\\pm i\\0\end{pmatrix}$

Mean field theory

Mean field: $b_{\alpha\mathbf{q}} = B_\alpha \delta_{\mathbf{q},\mathbf{0}}$

$$B_\alpha = u_\alpha + i v_\alpha$$
$$u_\alpha v_\alpha = 0$$

$$\frac{\delta F[B]}{\delta B_\alpha} = (\epsilon - 2\mu)\, B_\alpha - g^2 \int \frac{d^d p}{(2\pi)^d} \frac{p_\alpha p_\beta B_\beta}{\sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + 4g^2 B_\gamma B^*_\delta p_\gamma p_\delta}} = 0.$$

V. Gurarie, L. Radzihovsky, A. Andreev (2005)

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$$\begin{aligned} F[B_\alpha] \quad &= \quad (\epsilon - 2\mu)\left(u^2 + v^2\right) + \nu(\mu)\left[\left(u^2+v^2\right)\log\left(u+v\right) + \frac{u^3+v^3}{u+v}\right] + \\ &+ \quad \frac{4\pi a}{m}\left[\left(u^2+v^2\right)^2 + \frac{1}{2}\left(u^2-v^2\right)^2\right] \end{aligned}$$

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Free term

$$\begin{aligned} F[B_\alpha] &= (\epsilon - 2\mu)\left(u^2 + v^2\right) + \nu(\mu)\left[\left(u^2 + v^2\right)\log\left(u + v\right) + \frac{u^3 + v^3}{u + v}\right] + \\ &+ \frac{4\pi a}{m}\left[\left(u^2 + v^2\right)^2 + \frac{1}{2}\left(u^2 - v^2\right)^2\right] \end{aligned}$$

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Free term | BCS term

$$F[B_\alpha] = (\epsilon - 2\mu)\left(u^2 + v^2\right) + \nu(\mu)\left[\left(u^2+v^2\right)\log\left(u+v\right) + \frac{u^3+v^3}{u+v}\right] + \\ + \frac{4\pi a}{m}\left[\left(u^2+v^2\right)^2 + \frac{1}{2}\left(u^2-v^2\right)^2\right]$$

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Free term BCS term

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$$+ \frac{4\pi a}{m}\left[\left(u^2+v^2\right)^2 + \frac{1}{2}\left(u^2-v^2\right)^2\right]$$

$(B_\alpha^* B_\alpha)^2$ $B^2 B^{*2}$

BEC term

V. Gurarie, L. Radzihovsky, A. Andreev (2005)

Mean field theory

Mean field: $b_{\alpha\mathbf{q}} = B_\alpha \delta_{\mathbf{q},\mathbf{0}}$

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Free term — BCS term

$$F[B_\alpha] = (\epsilon - 2\mu)\left(u^2 + v^2\right) + \nu(\mu)\left[\left(u^2+v^2\right)\log\left(u+v\right) + \frac{u^3+v^3}{u+v}\right] + $$
$$+ \frac{4\pi a}{m}\left[\left(u^2+v^2\right)^2 + \frac{1}{2}\left(u^2-v^2\right)^2\right]$$

$(B^*_\alpha B_\alpha)^2$ $\quad B^2 B^{*2}$

BEC term

All terms demand $u = v$

Thus we have $p_x + i p_y$

V. Gurarie, L. Radzihovsky, A. Andreev (2005)

$\mu = 0$ phase transitions

Phase	Bogoliubov spectrum	Gapless point $\mu > 0$	$\mu < 0$
p_x	$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + 4B^2 p_x^2}$	$p_x = 0,\ \frac{p^2}{2m} = \mu$	Gapped
$p_x + ip_y$	$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + 4B^2\left(p_x^2 + p_y^2\right)}$	$p_x = 0,\ p_y = 0,\ \frac{p^2}{2m} = \mu$	Gapped

Thus μ=0 represents a quantum phase (BCS to BEC) transition
G. Volovik, (1993). L. Borkowski, C. Sá de Melo, (1999).

In case of $p_x + ip_y$ the transition is topological
(G. Volovik, Universe in a Helium Droplet))

p-wave condensates confined to 2D

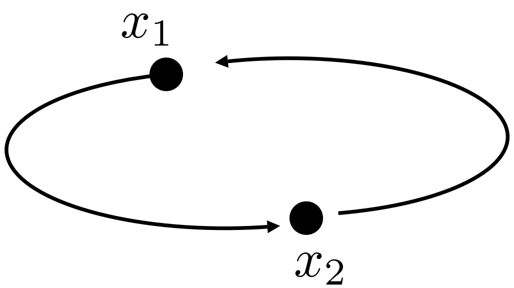
p_x+ip_y condensates, when confined to 2D, in the **BCS** (but not BEC) regime, realize particles with non-Abelian statistics.

N. Read and D. Green, PRB 2000

Non-Abelian statistics

F. Wilczek, 80s

N. Read, G. Moore, 1991



Bosons: $\psi(x_2, x_1) = \psi(x_1, x_2)$

Fermions: $\psi(x_2, x_1) = -\psi(x_1, x_2)$

Anyons: $\psi(x_2, x_1) = e^{i\theta}\psi(x_1, x_2)$ (fractional statistics)

"non-Abelions": $\psi_i(x_2, x_1) = \sum_j U_{ij}\psi_j(x_1, x_2)$ (non-Abelian statistics)

Conditions for the non-Abelian statistics:
2D, gap, fractionalization

Footnote: proposals to realize quantum computing with non-Abelian particles. A. Kitaev (2001)

BdG Equations in a Superconductor

$$\hat{H} = \sum_{ij} \left(\hat{a}_i^\dagger h_{ij} \hat{a}_j - \hat{a}_j h_{ij} \hat{a}_i^\dagger + \hat{a}_i \Delta_{ij} \hat{a}_j + \hat{a}_j^\dagger \Delta_{ij}^* \hat{a}_i^\dagger \right)$$

i, j stand for position, spin, etc.

$$h^\dagger = h$$
$$\Delta^T = -\Delta$$

BdG Equations in a Superconductor

$$\hat{H} = \sum_{ij} \left(\hat{a}_i^\dagger h_{ij} \hat{a}_j - \hat{a}_j h_{ij} \hat{a}_i^\dagger + \hat{a}_i \Delta_{ij} \hat{a}_j + \hat{a}_j^\dagger \Delta_{ij}^* \hat{a}_i^\dagger \right)$$

i, j stand for position, spin, etc.

$$h^\dagger = h$$
$$\Delta^T = -\Delta$$

$$\mathcal{H} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}$$

BdG Equations: $\mathcal{H}\psi_n = E_n \psi_n$

BdG Equations in a Superconductor

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BdG Equations: $\mathcal{H}\psi_n = E_n \psi_n$

Important symmetry:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \mathcal{H}^* = -\sigma_1 \mathcal{H} \sigma_1, \qquad \mathcal{H}\sigma_1\psi_n^* = -\sigma_1\mathcal{H}^*\psi_n^* = -E_n\sigma_1\psi_n^*.$$

Eingevalues come in pairs: $\psi_n, E_n; \;\; \sigma_1\psi_n^*, -E_n$

M. Zirnbauer, 1996: Class D Hamiltonian

Zero modes in a superconductor

$$\mathcal{H}\psi = 0 \ \rightarrow \mathcal{H}\sigma_1\psi^* = 0.$$

$$\psi_0 = \psi + \sigma_1\psi^* \qquad \psi_0 = i\left(\psi - \sigma_1\psi^*\right) \longrightarrow \qquad \sigma_1\psi_0 = \psi_0^*$$

$$\psi_0 = \begin{pmatrix} u \\ u^* \end{pmatrix}$$

Zero mode creation operator

$$\hat{\gamma} = \sum_i u_i^* \hat{a}_i + u_i \hat{a}_i^\dagger$$

Zero mode is a Majorana fermion

$$\hat{\gamma}^\dagger = \hat{\gamma}, \ \hat{\gamma}^2 = 1$$

Stability of the zero modes: VG, L. Radzihovsky, PRB, 2007

Breaking a fermion apart

Two Majorana = one normal (Dirac) fermion

$$\hat{c} = (\hat{\gamma}_1 + i\hat{\gamma}_2)/2$$

$$\hat{c}^\dagger = (\hat{\gamma}_1 - i\hat{\gamma}_2)/2$$

$$\{\hat{c}, \hat{c}\} = 0$$

$$\{\hat{c}^\dagger, \hat{c}\} = 1$$

$$u_1(r) \sim e^{-\frac{|r-r_1|}{l}}$$

$$u_2(r) \sim e^{-\frac{|r-r_2|}{l}}$$

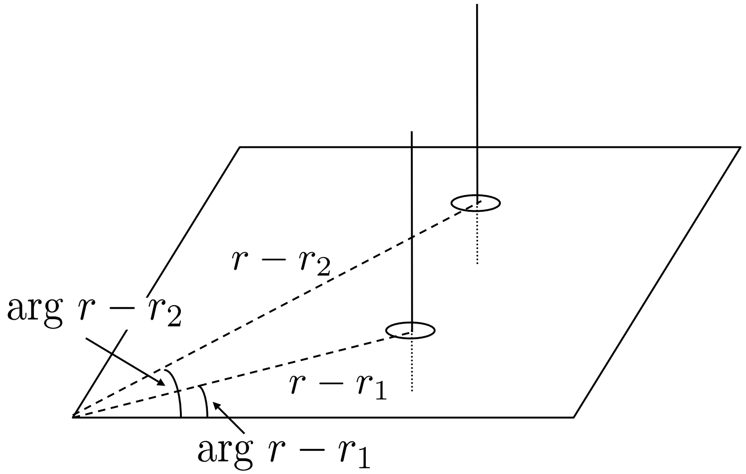
$$\hat{\gamma}_\alpha = \int d^2r \left[u_\alpha^*(r)\hat{a}(r) + u_\alpha(r)\hat{a}^\dagger(r)\right]$$

Thus a superconductor, via its zero modes, breaks the fermions in half: the fractionalization

2D, gap: *p*-wave p_x+ip_y superconductor

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} -\frac{\nabla^2}{2m} - \mu & \sqrt{\Delta(r)}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sqrt{\Delta(r)} \\ \sqrt{\Delta^*(r)}\left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sqrt{\Delta^*(r)} & \frac{\nabla^2}{2m} + \mu \end{pmatrix}$$

$$\Delta(r) = \Delta_0 \;\rightarrow\; E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \Delta_0^2 p^2}.$$



$$\Delta(r) = \prod_\alpha f(r - r_\alpha)\; e^{i \arg(r - r_\alpha)} \xrightarrow[\mu > 0]{} \hat{\gamma}_\alpha$$

N. Read, D. Green, PRB, 2000

Relationship to Quantum Hall Effect

$$|\mathrm{BCS}\rangle = \prod_p \left(u_p + v_p a^\dagger_{-p} a^\dagger_p \right) |0\rangle \qquad \psi(r_1, r_2, \ldots) = \langle 0|\, a(r_1) a(r_2) \ldots |\mathrm{BCS}\rangle$$

Relationship to Quantum Hall Effect

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$$\psi\left(r_1, r_2, \ldots\right) = \mathcal{A}\left[g(r_1 - r_2) g(r_3 - r_4) \ldots\right] \qquad g(r) = \int \frac{d^d p}{(2\pi)^d} \frac{v_p}{u_p} e^{ipr}$$

$$\mu > 0,\ 2D,\ p_x + i p_y \ \rightarrow\ g(r) \sim \frac{1}{z}$$

Relationship to Quantum Hall Effect

$$|\mathrm{BCS}\rangle = \prod_p \left(u_p + v_p a^\dagger_{-p} a^\dagger_p \right) |0\rangle \qquad \psi(r_1, r_2, \ldots) = \langle 0| \, a(r_1) a(r_2) \ldots |\mathrm{BCS}\rangle$$

$$\psi\left(r_1, r_2, \ldots\right) = \mathcal{A}\left[g(r_1 - r_2) g(r_3 - r_4) \ldots\right] \qquad g(r) = \int \frac{d^d p}{(2\pi)^d} \frac{v_p}{u_p} e^{ipr}$$

$$\mu > 0, \; 2D, \; p_x + ip_y \;\; \rightarrow \;\; g(r) \sim \frac{1}{z}$$

$$\psi(z_1, z_2, \ldots, z_N) = \mathcal{A}\left[\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \right]$$

One recognizes the Pfaffian (Moore-Read) state in the quantum Hall effect.

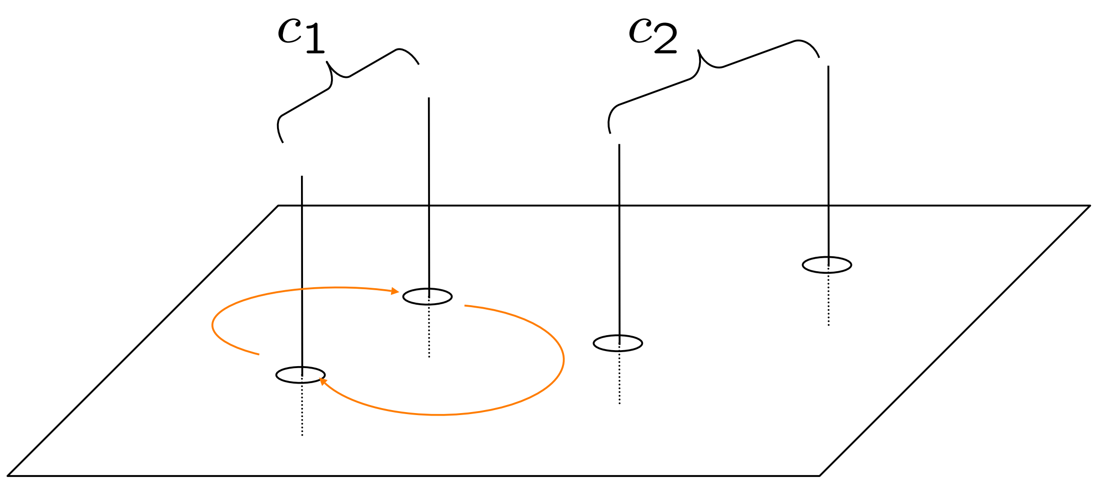
Thus non-Abelian statistics must follow.

C. Nayak, F. Wilczek, 1994

N. Read and D. Green, PRB, 2000

Non-Abelian statistics

D. Ivanov, PRL (2001)



One fermion (two states – either empty or occupied fermion) per two vortices

$2^{\frac{n}{2}}$ states per *n* vortices

In this example, there are two fermions and four states

$$\psi_1 = |0\rangle$$
$$\psi_2 = c_1^\dagger |0\rangle$$
$$\psi_3 = c_2^\dagger |0\rangle$$
$$\psi_4 = c_1^\dagger c_2^\dagger |0\rangle$$

Vortex exchange mixes these states

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \to U \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Universal matrix of vortex exchange independent of anything but the fact that vortices were exchanged.

Candidates to realize the Pfaffian state

QH

- Quantum Hall Effect. At ν=5/2, it is very fragile and experiments are inconclusive.
- Rotating Bosons. Hard to rotate.

SC

- Liquid ^{3}He. Must be in one of the more exotic phases to be p_x+ip_y , it is not clear how to manipulate its zero modes.
- $SrRuO_4$. Fierce debates whether it is a *p*-wave superconductor at all, to say nothing whether it is p_x+ip_y .
- *P*-wave Feshbach resonances. <u>Known</u> to be p_x+ip_y . Stability of the *p*-wave molecules?

BCS-BEC phase transition 2D $p_x + ip_y$ condensate

$$E_p = \sqrt{\left(\frac{p^2}{2m_a} - \mu\right)^2 + 4g^2B^2\left(p_x^2 + p_y^2\right)}$$

In 2D, the spectrum is gapped at both □ > *0* and □ < *0*, and gapless at □ = *0*.

□ > *0* to □ < *0* is a topological phase transition

N. Read, D. Green, Phys. Rev. B **61**, 10261 (2000)

G. Volovik, "Universe in a Helium Droplet"

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$$|\Omega\rangle = \prod_p \left[u_p + v_p\ a_p^\dagger a_{-p}^\dagger\right]|0\rangle$$

Anderson's pseudospin

$$\begin{cases} n_x + in_y = 2v^* u \\ n_z = |v|^2 - |u|^2 \end{cases}$$

$$\vec{n} = \frac{1}{\sqrt{\left(\frac{p^2}{2m_a} - \mu\right)^2 + 4B^2\left(p_x^2 + p_y^2\right)}} \begin{pmatrix} 2gBp_x \\ -2gBp_y \\ \frac{p^2}{2m_a} - \mu \end{pmatrix}$$

BCS-BEC phase transition 2D
p_x+ip_y condensate

$$E_p = \sqrt{\left(\frac{p^2}{2m_a} - \mu\right)^2 + 4g^2B^2\left(p_x^2+p_y^2\right)}$$

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Explicit calculations show that

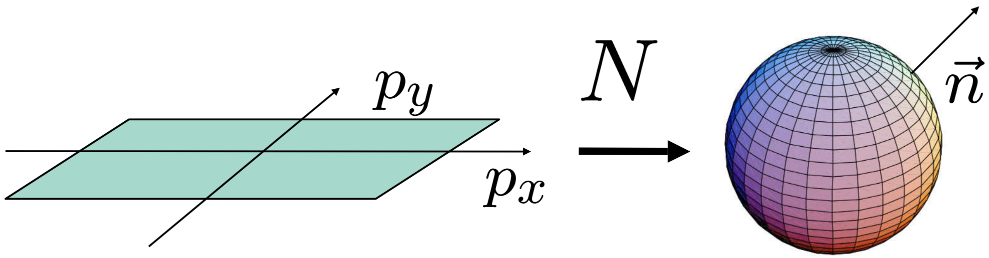
N=*0* if □ < *0*

N=*1* if □ > *0*

$$|\Omega\rangle = \prod_p \left[u_p + v_p\ a_p^\dagger a_{-p}^\dagger\right]|0\rangle$$

Anderson's pseudospin
$$\begin{cases} n_x + i n_y = 2v^* u \\ n_z = |v|^2 - |u|^2 \end{cases}$$

$$\vec{n} = \frac{1}{\sqrt{\left(\frac{p^2}{2m_a}-\mu\right)^2 + 4B^2\left(p_x^2+p_y^2\right)}}\begin{pmatrix} 2gBp_x \\ -2gBp_y \\ \frac{p^2}{2m_a}-\mu \end{pmatrix}$$



$$N = \frac{1}{8\pi}\int d^2p\ \left[\vec{n}\cdot\partial_\alpha\vec{n}\times\partial_\beta\vec{n}\ \epsilon_{\alpha\beta}\right]$$

topological invariant

Experiments

PHYSICAL REVIEW A **70**, 030702(R) (2004)

P-wave Feshbach resonances of ultracold ^{6}Li

J. Zhang,[1,2] E. G. M. van Kempen,[3] T. Bourdel,[1] L. Khaykovich,[1,4] J. Cubizolles,[1] F. Chevy,[1] M. Teichmann,[1] L. Tarruell,[1]
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(Received 18 June 2004; published 30 September 2004)

p-wave Feshbach molecules

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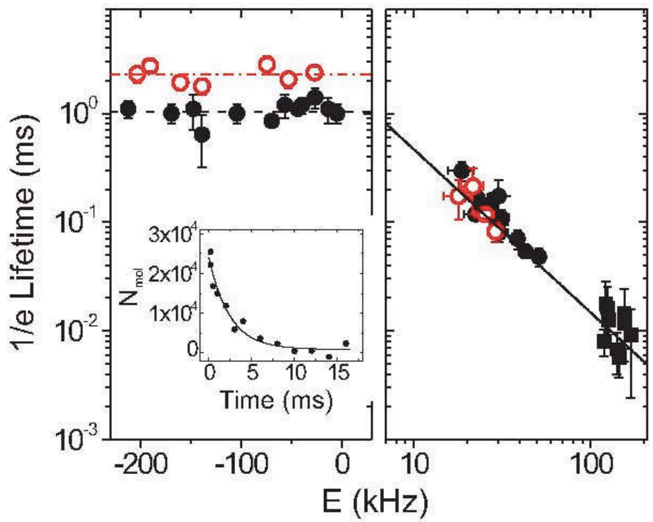
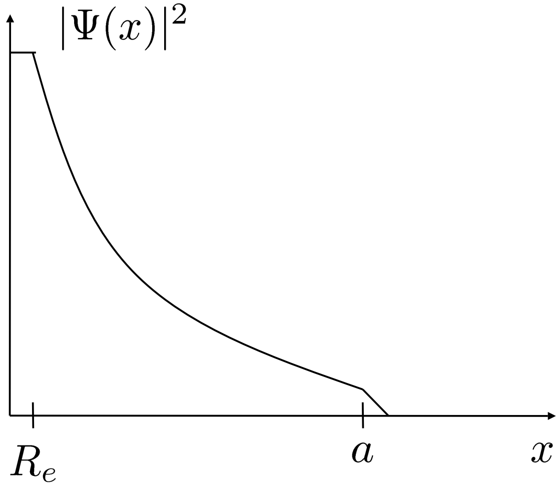
(Dated: March 3, 2007)

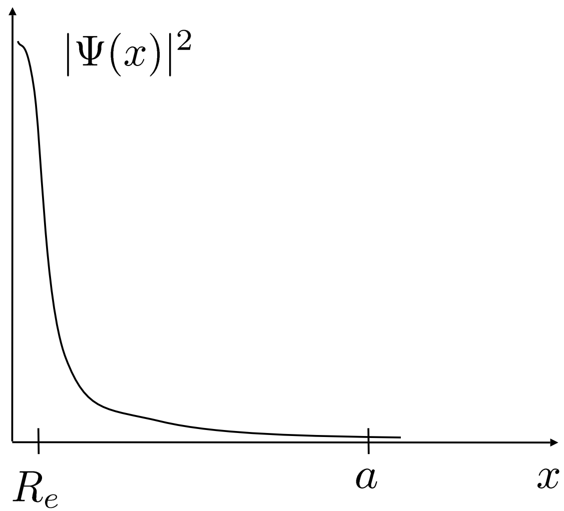


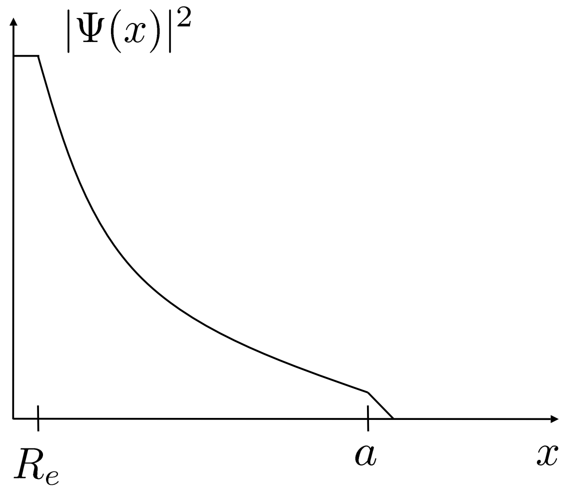
Bottom line:
the molecules are unstable,
with $\tau \sim 2ms$

Origin of stability: Molecular size

$3D$ s-wave	$3D$ p-wave	$2D$ p-wave
$\Psi(x) \sim \frac{1}{x}$	$\Psi(x) \sim \frac{1}{x^2}$	$\Psi(x) \sim \frac{1}{x}$







$$\int_{R_e}^{a} d^3x \, |\Psi(x)|^2 \sim a$$

Large molecule, hard to collapse

$$\int_{R_e}^{a} d^3x \, |\Psi(x)|^2 \sim \frac{1}{R_e}$$

Small molecule, easy to collapse

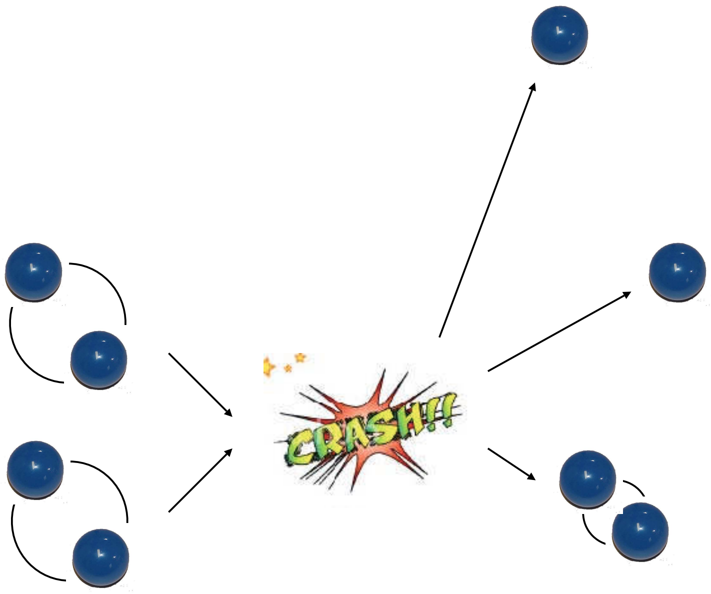
$$\int_{R_e}^{a} d^2x \, |\Psi(x)|^2 \sim \log \frac{a}{R_e}$$

Intermediate case, maybe not so easy to collapse??

R_e force range

$$E_{\text{binding}} = -\frac{\hbar^2}{ma^2}$$

3-body recombination (large molecules)



$$\Gamma = n v \sigma \sim$$

$$\sim \frac{1}{a^d}\, v\, a^{d-1}\, \frac{a}{v}\, \frac{\hbar}{m R_e^2} \left(\frac{R_e}{a}\right)^{2d+2\gamma}$$

- $\frac{1}{a^d}$ ← n
- v ← v
- a^{d-1}: Elastic cross-section
- $\frac{a}{v}$: Time the molecules spent in the vicinity of each other
- $\frac{\hbar}{m R_e^2}$: Probability of collapse
- $\left(\frac{R_e}{a}\right)^{2d+2\gamma}$: Probability that 3 atoms are within distance R_e to each other

$$\Gamma \sim \frac{\hbar}{m a^2} \left(\frac{R_e}{a}\right)^{2d+2\gamma-2}$$

$$\frac{a}{R_e} \sim 200 - 1000$$

3D s-wave: $\gamma \sim -0.2$

$$\Gamma \sim \frac{\hbar}{m a^2} \left(\frac{R_e}{a}\right)^{3.55}$$

Petrov, Salomon, Shlyapnikov (2005)

$$\psi_{3B} \sim r^{\gamma}$$

3D bosons:
$\Gamma \sim \frac{\hbar}{m a^2}$

3-body recombination, 2D *p*-wave (small molecules)

$$\Gamma = n v \sigma \sim \frac{1}{a^3} v R_e^2 \frac{R_e}{v} \frac{\hbar}{m R_e^2} = \frac{\hbar}{m a^2} \frac{R_e}{a}$$

$\frac{\hbar}{ma^2}$ → 10KHz

$\frac{R_e}{a}$ → $\sim 1/500$

Alternative picture:

$$\Gamma = n v \sigma = \frac{1}{a^3} v \frac{v_f}{v} R_e^2 \sim \frac{\hbar}{m a^2} \frac{R_e}{a}$$

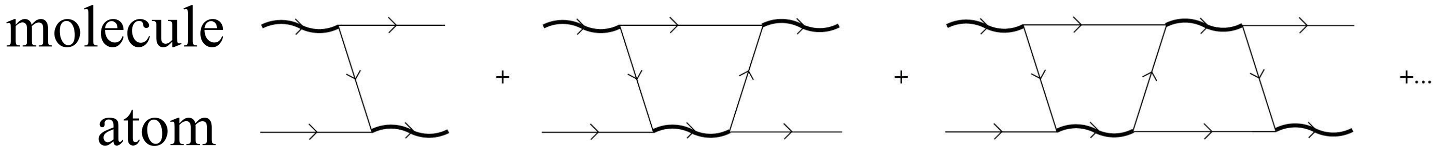
R_e^2 → Inelastic amplitude

The atom-molecule hybridization rate is also $\frac{\hbar}{ma^2} \frac{R_e}{a}$

J. Levinsen, N. Cooper, VG, PRL (2007)

L. Jona-Lasinio, L. Pricoupenko, Y. Castin, (2007)

3-body p-wave bound states

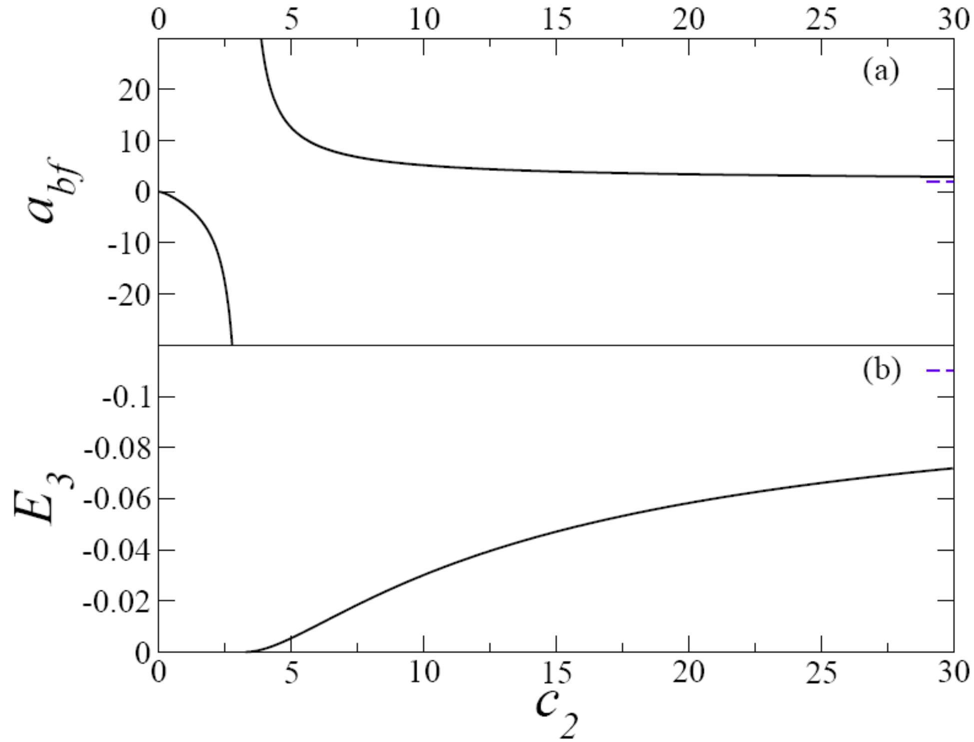


At large $c_2 \sim m^2 g^2 / R_e$
there are bound states of three atoms

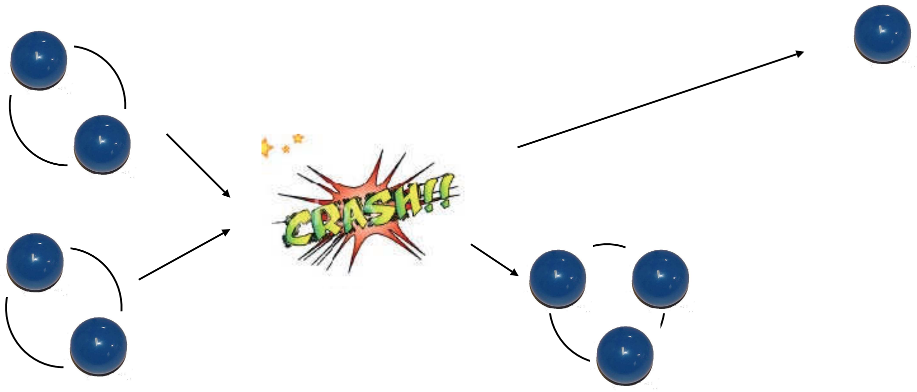
Experimentally $c_2 \sim 10$

M. Jona-Lasinio,
L. Pricoupenko, Y. Castin (2007)

J. Levinsen, N. Cooper, VG,
PRL (2007)



Decay into trimers



Molecule-molecule scattering length is no longer real – another source of instability

$$\Gamma = nv\sigma = \frac{1}{a^3} v \frac{v_f}{v} R_e^2 \sim \frac{\hbar}{ma^2} \frac{R_e}{a}$$

Same lifetime as the collapse into deeply bound molecular state

Stability of 2D p-wave condensate

$$\Gamma \sim \frac{\hbar}{ma^2}\left(\frac{R_e}{a}\right)^{2d+2\gamma-2} = \frac{\hbar}{ma^2}\left(\frac{R_e}{a}\right)^{2+2\gamma}$$

What is γ in 2D p-wave?

3D s-wave bosons	3D p-wave fermions	3D s-wave fermions
$\Gamma \sim \frac{\hbar}{ma^2}$	$\Gamma \sim \frac{\hbar}{ma^2}\frac{R_e}{a}$	$\Gamma \sim \frac{\hbar}{ma^2}\left(\frac{R_e}{a}\right)^{3.55}$
Unstable	Not too stable	Stable

Conclusions

- *p*-wave in 2D: a path to new exotic physics
- p_x+ip_y is formed robustly by single species fermions and, at the same time, is needed for this new physics
- The lifetime of *p*-wave molecules in 3D is short, but perhaps is not hopelessly short
- In 2D, the lifetime is not yet known
- Once the topological state is created, its non-Abelian excitations can be directly probed.