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Synchronization in Complex Networks

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Synchronization in Complex Networks

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Toolbox TOCSY

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Outline

- Introduction
- Complete synchronization in complex networks
- Hierarchical (clustered) transitions to complex synchronization in complex networks
- Structure vs. functionality in complex brain networks – network of networks
- Conclusions

Synchronization in ensembles of active subsystems: Biology

- Ensemble of doves (wings in synchrony)
- Menstruation (e.g. female students living in one room in a dormitory)
- Collective firing of neurons – cause of any action of animals having neurons
- Fireflies, crickets, frogs in Asia (India, Vietnam...)

Basic Model in Statistical Physics and Nonlinear Sciences for ensembles

- Traditional Approach:

  Regular chain or lattice of coupled oscillators; global or nearest neighbour coupling

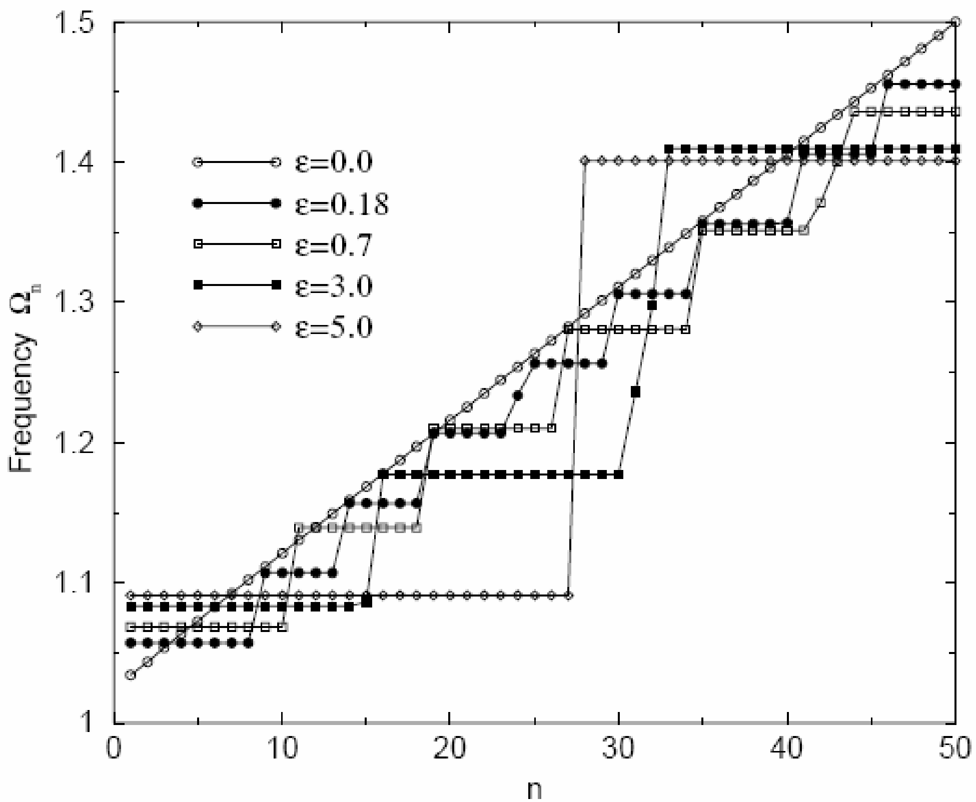
Synchronization in spatially extended systems

$$\dot{x}_n = -\omega_n y_n - z_n \ ,$$

$$\dot{y}_n = \omega_n x_n + a y_n + \epsilon(y_{n+1} - 2y_n + y_{n-1}) \ ,$$

$$\dot{z}_n = 0.4 + (x_n - 8.5) z_n \ .$$

Chain of diffusively coupled Roessler oscillators



Soft transition to phase synchronization

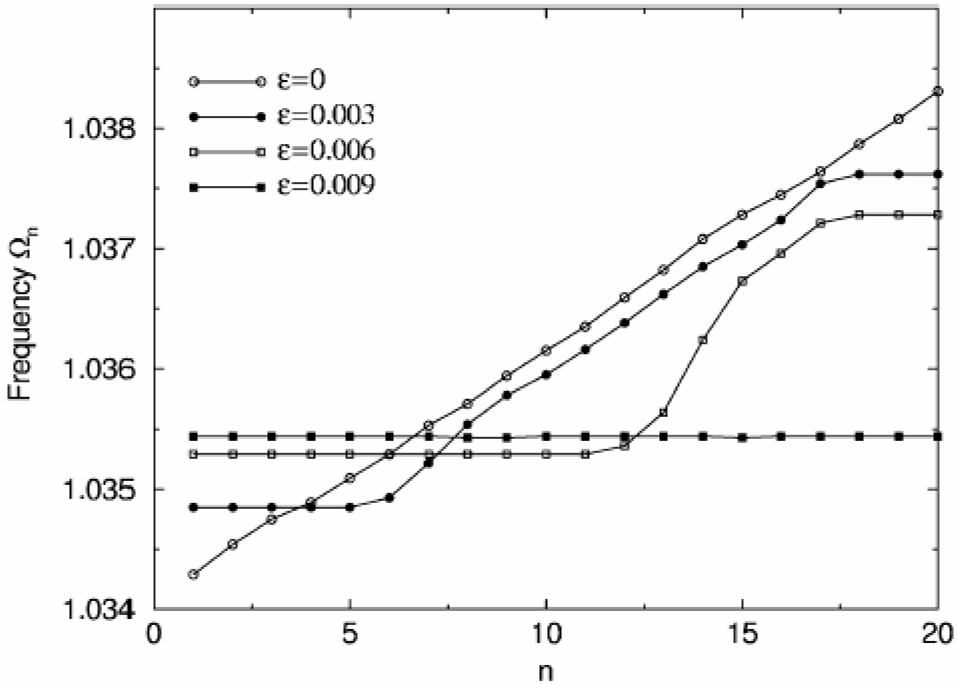


Fig. 6.3. Soft transition to global synchronization in a chain of Rössler oscillators (6.24). Mean frequencies Ω_n for different values of coupling ϵ. The parameters are: $N = 20$, the frequency mismatch $\delta = 2 \times 10^{-4}$ and $\omega_1 = 1$.

Soft transition to PS

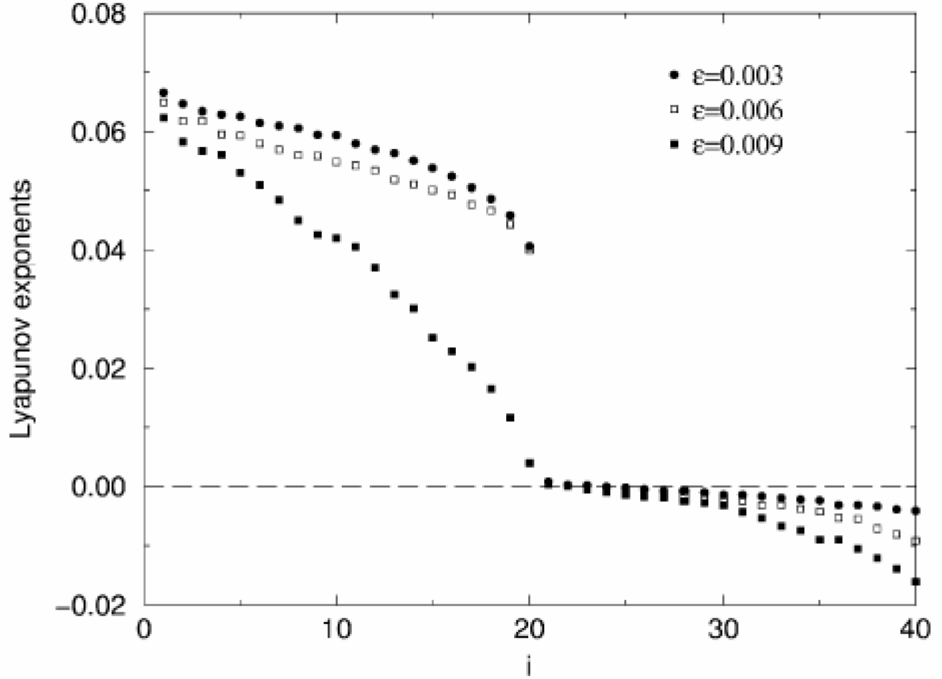


Fig. 6.4. 40 largest Lyapunov exponents λ_i for the regimes reported in Fig. 6.3.

Hard transition to PS

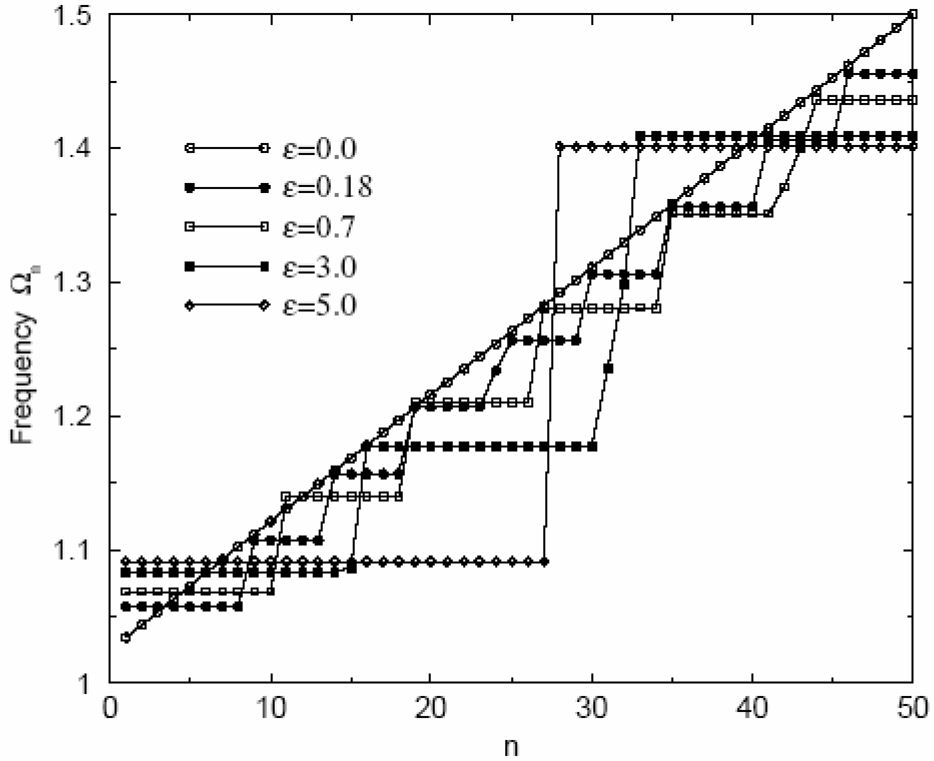


Fig. 6.5. Hard transition to global synchronization in a chain of Rössler oscillators (Eq. (6.24)). Mean frequencies Ω_n for different values of coupling ϵ. The parameters are: $N = 50$, the frequency mismatch $\delta = 9 \times 10^{-3}$ and $\omega_1 = 1$.

Hard transition to PS

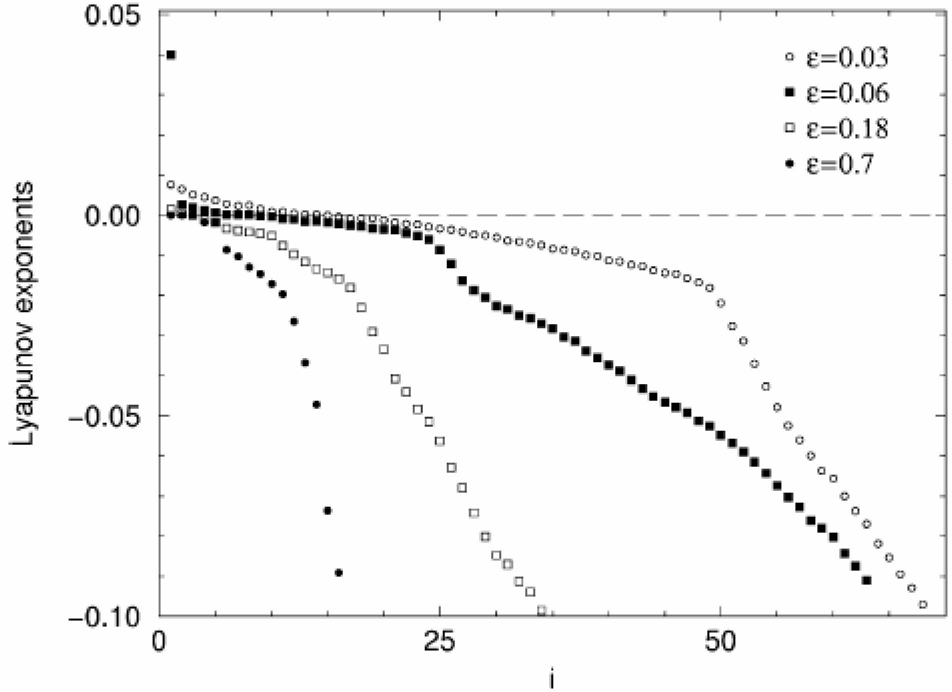


Fig. 6.6. 70 largest Lyapunov exponents λ_i for the regimes reported in Fig. 6.5.

Hard transition to PS

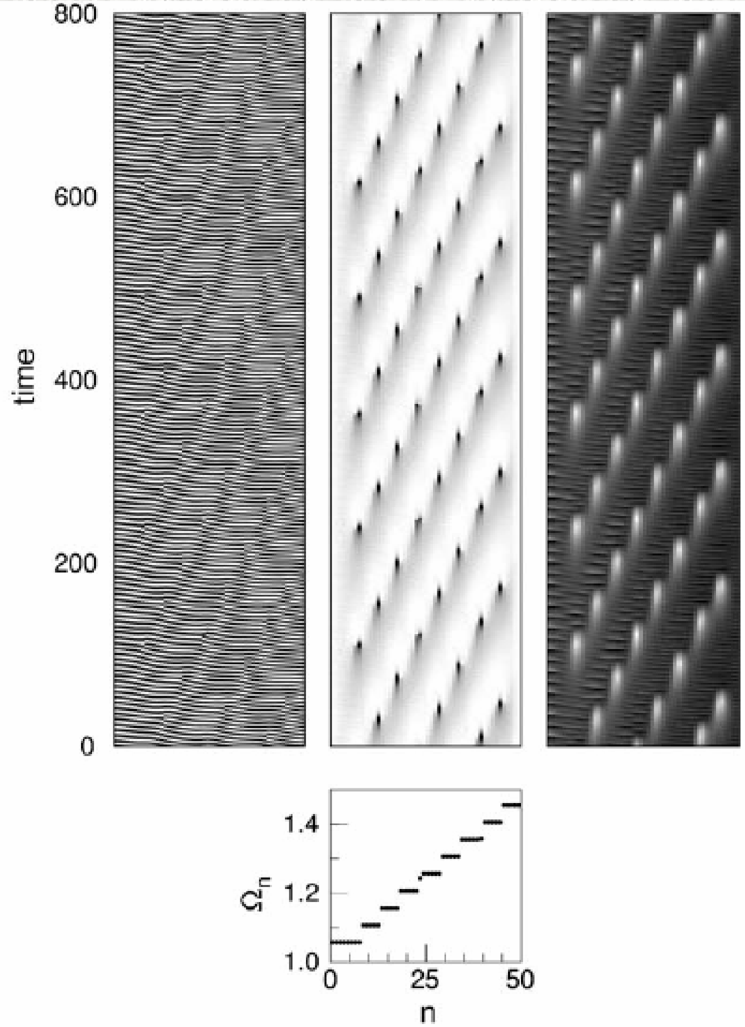


Fig. 6.7. Mean frequencies Ω_n and space–time plots in a chain of 50 coupled Rössler oscillators with a frequency mismatch $\delta = 9 \times 10^{-3}$ and coupling $\epsilon = 0.18$. All plots show a gray-scale representation of corresponding quantities. Minimal values are represented by white and maximal by black.

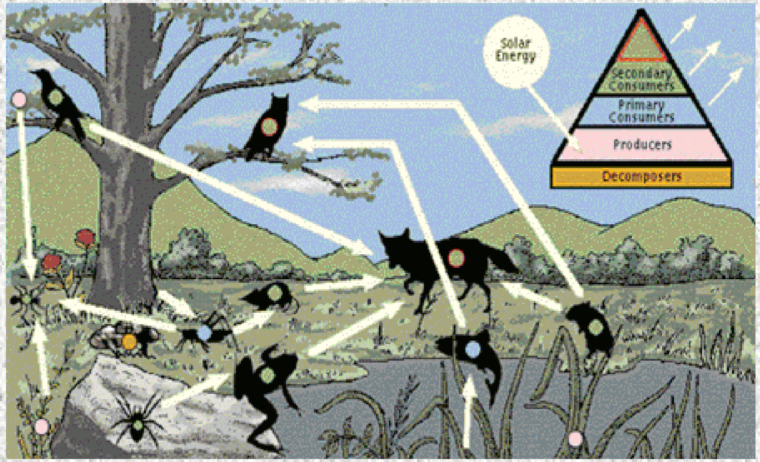
Networks with Complex Topology

- Random graphs/networks (Erdös, Renyi, 1959)
- Small-world networks (Watts, Strogatz, 1998)
- Scale-free networks (Barabasi, Albert, 1999)

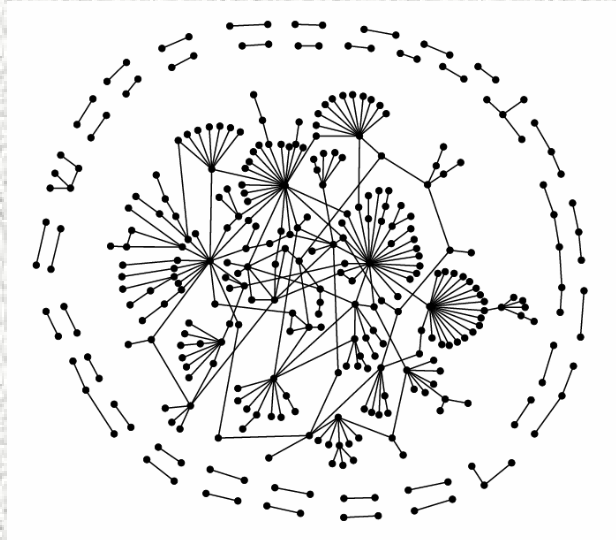
- Applications: neuroscience, cell biology, epidemic spreading, internet, traffic, systems biology
- Many participants (nodes) with complex interactions and **complex dynamics at the nodes**

Biological Networks

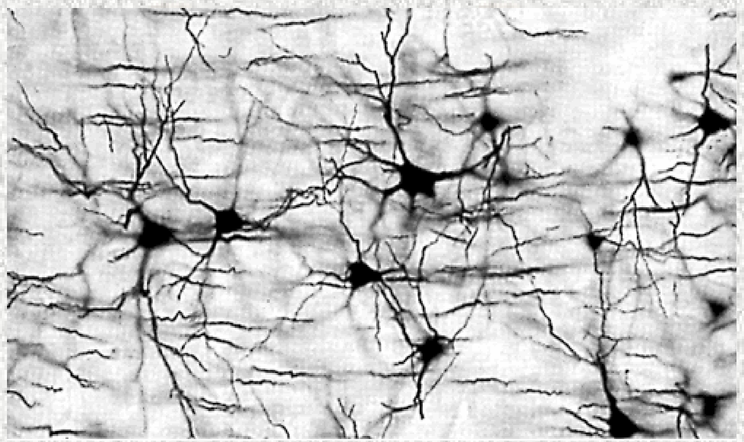
Ecological Webs



Protein interaction



Neural Networks

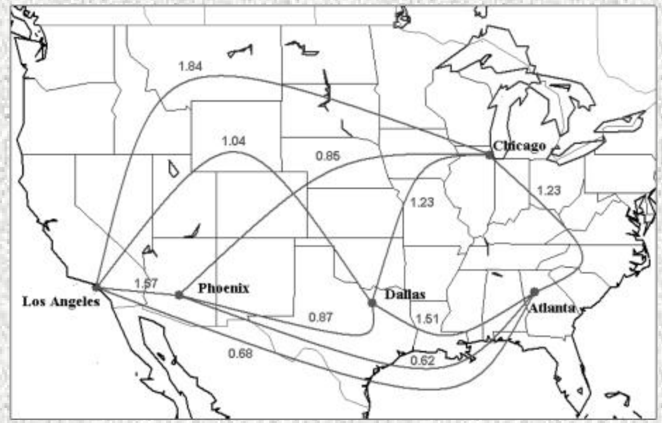


Genetic Networks

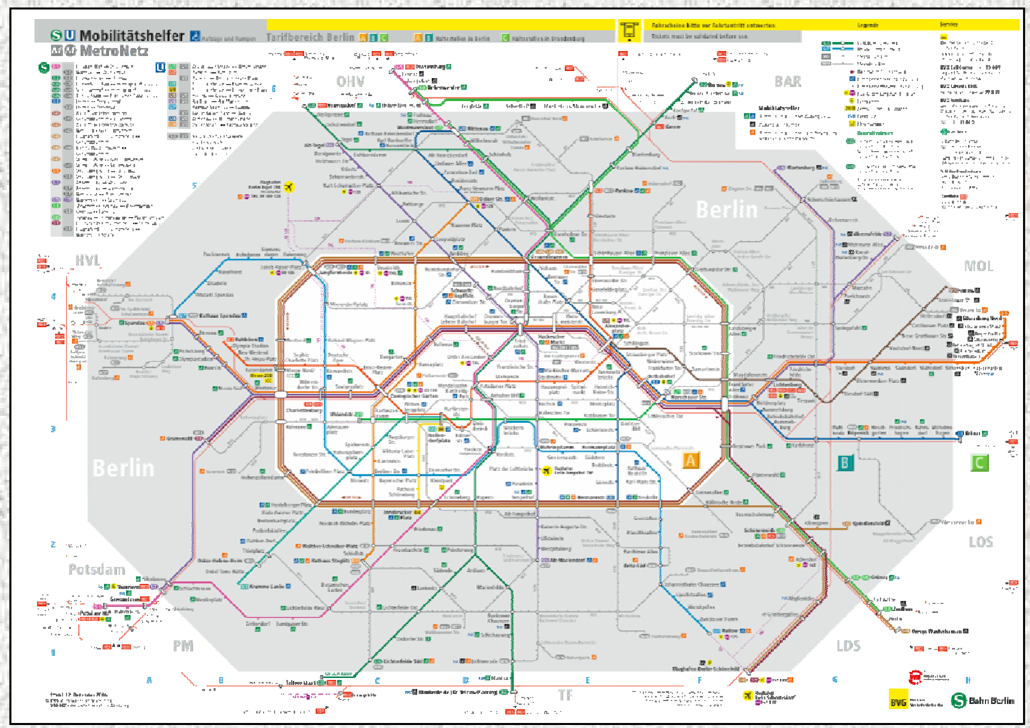
Metabolic Networks

Transportation Networks

Airport Networks



Local Transportation



Road Maps

Scale-freee Networks

Network resiliance

- Highly robust against random failure of a node
- Highly vulnerable to deliberate attacks on **hubs**

Applications

- **Immunization** in networks of computers, humans, ...

Synchronization in such networks

- Synchronization properties strongly influenced by the network´s structure (Jost/Joy, Barahona/Pecora, Nishikawa/Lai, Timme et al., Hasler/Belykh(s) etc.)
- Self-organized synchronized clusters can be formed (Jalan/Amritkar)
- Previous works mainly focused on the influence of the connection´s topology (assuming coupling strength uniform)

Universality in the synchronization of weighted random networks

Our intention:

Include the influence of weighted coupling for complete synchronization

(Motter, Zhou, Kurths: Phys. Rev. Lett., 96, 034101, 2006)

Weighted Network of N Identical Oscillators

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} W_{ij} A_{ij} [\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \\ &= \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots, N,\end{aligned}$$

F – dynamics of each oscillator

H – output function

G – coupling matrix combining adjacency A and weight W

$G_{ij} = -W_{ij}$ for $i \neq j$ $\qquad$ $G_{ii} = \sum_j W_{ij} A_{ij} = S_i$

S_i - intensity of node i (includes topology and weights)

General Condition for Synchronizability

Stability of synchronized state

$$\{\mathbf{x}_i = \mathbf{s}, \forall\, i \,|\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})\}$$

N eigenmodes of

$$\dot{\xi}_i = [D\mathbf{F}(\mathbf{s}) - \sigma\lambda_i D\mathbf{H}(\mathbf{s})]\xi_i,$$

λ_i *i*th eigenvalue of G

Synchronizability

G has real eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N, \text{ with } \lambda_1 = 0$$

Network synchronizable for some σ if

$$\epsilon_1 < \sigma\lambda_i < \epsilon_2 \text{ is satisfied for all } i \geq 2$$

or for the **eigenratio R**

$$R \equiv \lambda_N / \lambda_2 < \epsilon_2 / \epsilon_1$$

R small ➔ more synchronizable

Cost involved in the network‘s coupling

Cost = total input strength of the connections of all nodes at the synchronization threshold $\sigma_{\min} \equiv \epsilon_1/\lambda_2$

$$C = \sigma_{\min} \sum_{i,j} W_{ij} A_{ij} = \sigma_{\min} \sum_{i=1}^{N} S_i$$

Normalized cost

$$C_0 \equiv C/(N\epsilon_1) = \Omega/\lambda_2$$

$$\Omega = \sum_{i=1}^{N} S_i/N$$ Mean Intensity

Main result for general weighted random networks

$$R = A_R \frac{S_{\max}}{S_{\min}} R_H(K), \quad C_0 = A_C \frac{\Omega}{S_{\min}} C_H(K)$$

Distribution of eigenvalues in large random networks leads then to:

$$R \approx R_H(K) \equiv \frac{1 + 2/\sqrt{K}}{1 - 2/\sqrt{K}}$$

$$C_0 \approx C_H(K) \equiv \frac{1}{1 - 2/\sqrt{K}}$$

Main results

Synchronizability universally determined by:

- mean degree K and

- heterogeneity of the intensities

$$\frac{S_{\max}}{S_{\min}} \quad \text{or} \quad \frac{\Omega}{S_{\min}}$$

$S_{\min}, S_{\max}$ - minimum/ maximum intensities

Hierarchical Organization of Synchronization in Complex Networks

Homogeneous (constant number of connections in each node)

vs.

Scale-free networks

$$\dot{\mathbf{x}}_j = \tau_j \mathbf{F}(\mathbf{x}_j) + \frac{g}{K} \sum_{i=1}^{N} A_{ji} (\mathbf{x}_i - \mathbf{x}_j)$$

Zhou, Kurths: CHAOS (focus issue: 16, 015104 (2006))

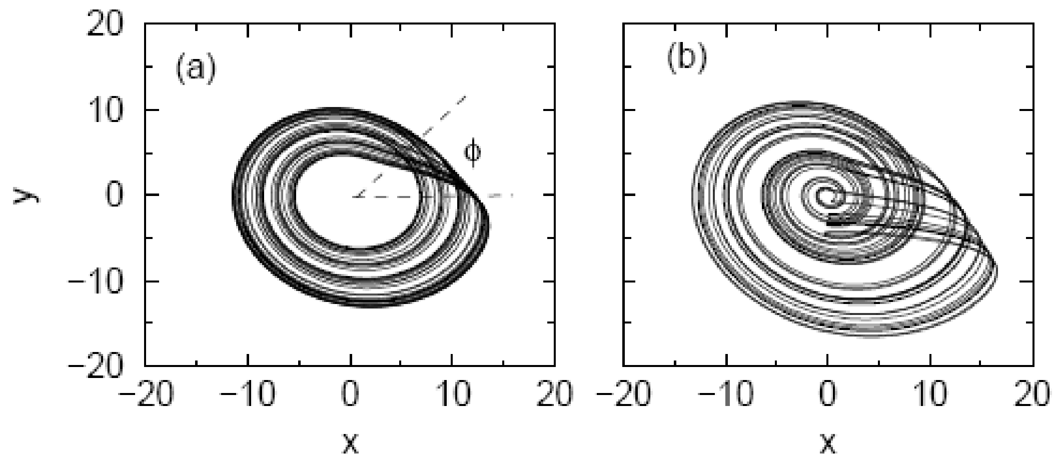


FIG. 1: Chaotic attractors of the Rössler oscillator in the phase coherent regime (a) and phase non-coherent regime (b).

$$\begin{aligned}
\dot{x} &= -0.97x - z, \\
\dot{y} &= 0.97x + ay, \\
\dot{z} &= x(z - 8.5) + 0.4,
\end{aligned}$$

Identical oscillators

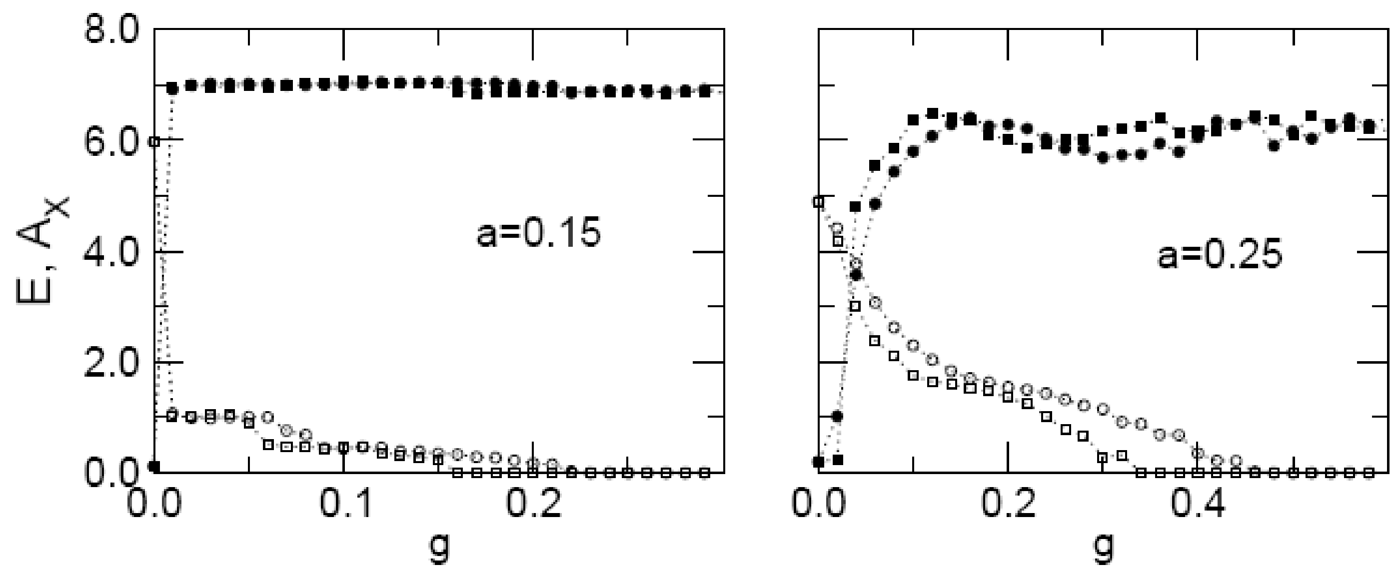


FIG. 3: Transition to CS in the SFN and the HN, indicated by the synchronization error E (squares) and the amplitude A_X of the mean field X (circles). The filled symbols are for the SFNs and the open symbols for the HNs. (a) Phase coherent oscillations at $a = 0.15$. (b) Phase non-coherent oscillations at $a = 0.25$. In both networks, $N = 1000$ and $K = 10$.

Transition to synchronization

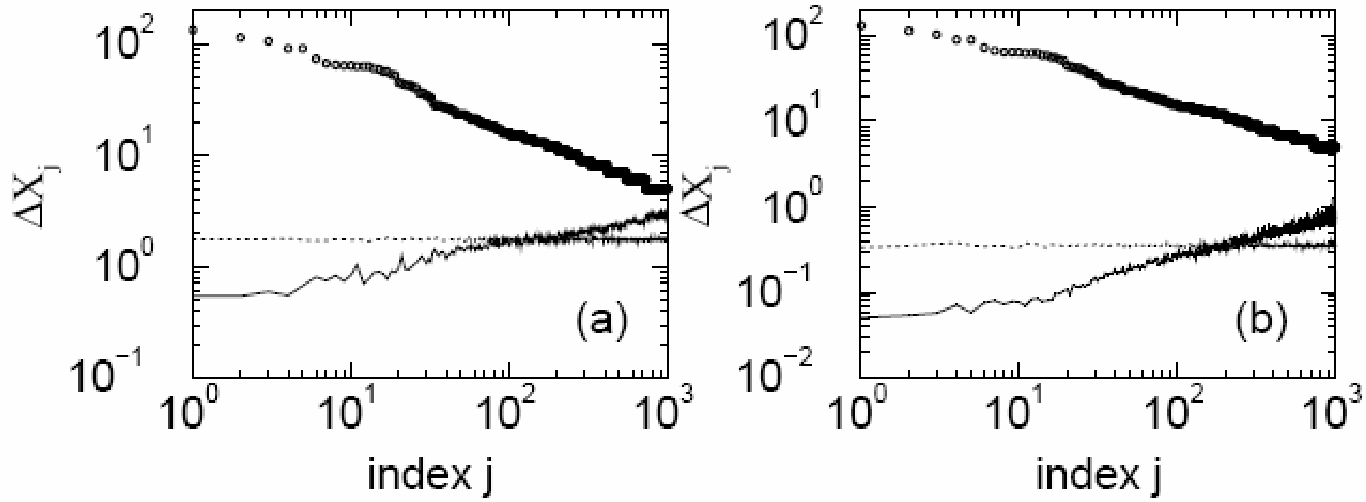


FIG. 4: Synchronization difference ΔX_j of the oscillators with respect to the global mean field X in the SFN (solid line) and HN (dotted line). The symbol ($\circ$) denotes the degree k_j of the nodes. Note the log-log scales of the plots. (a) When the coupling strength is weak ($g = 0.1$) and (b) when the synchronized state ($g = 0.5$) is perturbed by noise ($D = 0.5$). Here the oscillations are phase non-coherent at $a = 0.25$ and the behavior is very similar for phase coherent oscillations at $a = 0.15$.

$$\Delta X_j = \langle |x_j - X| \rangle_t$$

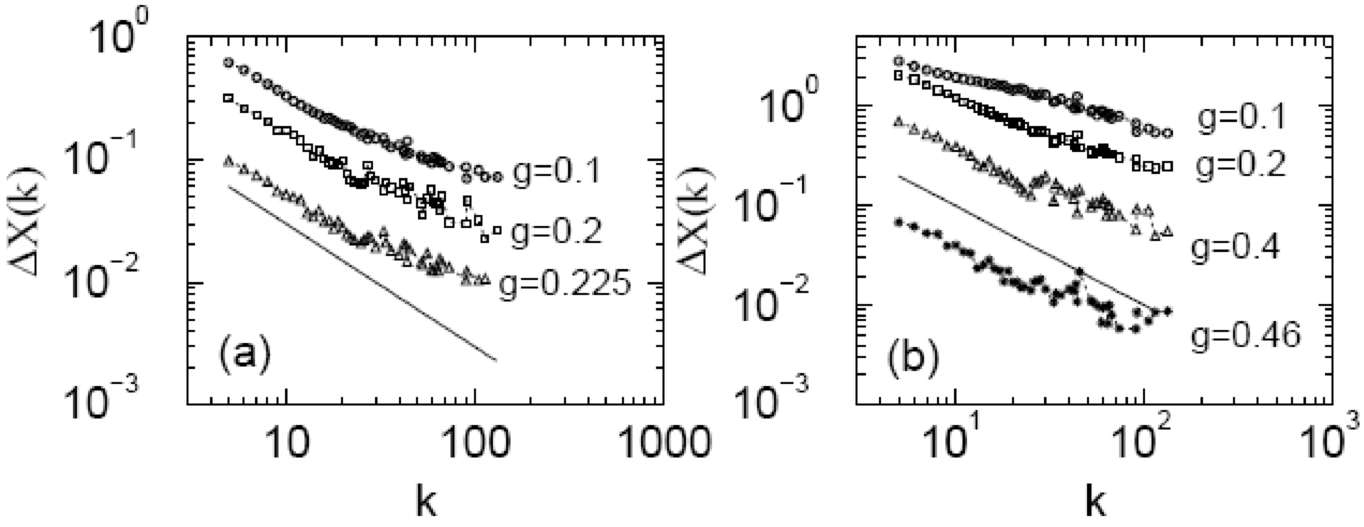


FIG. 5: The average values $\Delta X(k)$ as a function of k at various coupling strength g in the SFN. (a) Phase coherent regime $a = 0.15$. (b) phase non-coherent regime $a = 0.25$. The solid line with slop -1 is plotted for reference.

$$\Delta X(k) \sim k^{-\alpha}$$

$$\Delta X(k) = \frac{1}{N_k} \sum_{k_j = k} \Delta X_j$$

$$\alpha \approx 1$$

Mean-field approximation

$$\dot{\mathbf{x}}_j = \tau_j \mathbf{F}(\mathbf{x}_j) + \frac{g k_j}{K}(\mathbf{X} - \mathbf{x}_j), \quad k_j \gg 1.$$

Each oscillator forced by a common signal

Coupling strength ~ degree

$$\frac{d}{dt}\Delta X(k) = \Lambda(k)\Delta X(k) + c.$$

For nodes with rather large degree

➔ Scaling: $\Delta X(k) \sim k^{-1},$

Clusters of synchronization

$$\Delta X_{ij} = \langle |x_i - x_j| \rangle_t$$

$$\Delta X_{ij} \leq \Delta_{th}$$

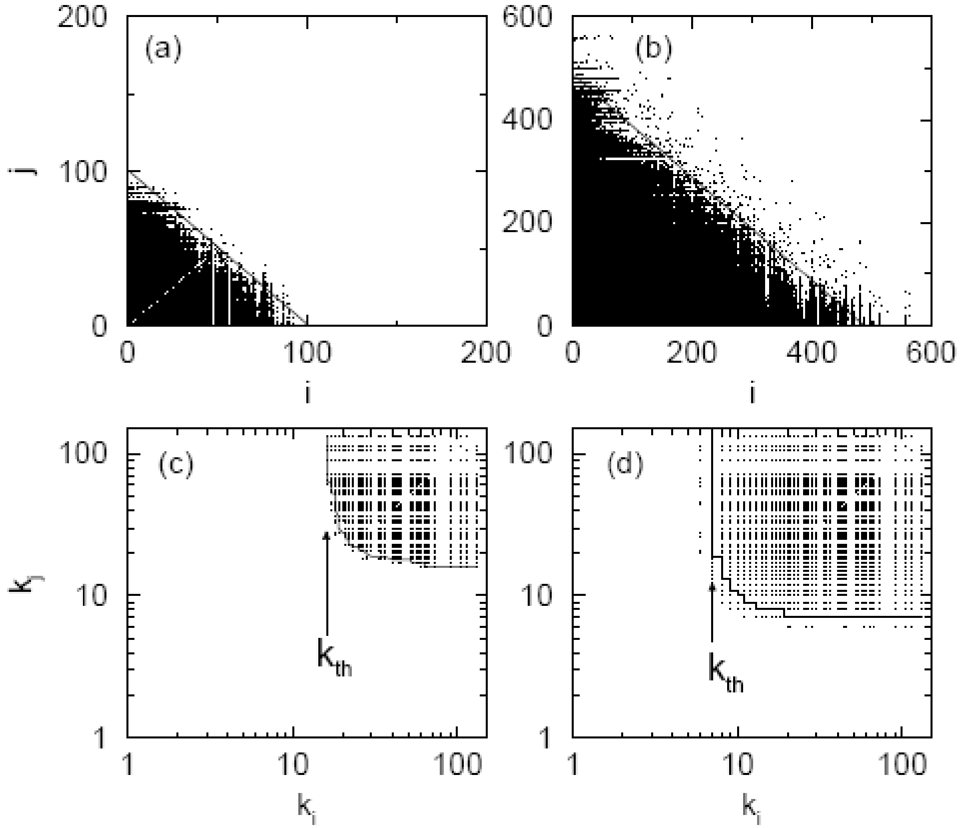


FIG. 7: The effective synchronization clusters in the synchronized network in the presence of noise ($a = 0.25$, $g = 0.5$, and $D = 0.5$), represented simultaneously in the index space (i, j) (a, b) and in the degree space (k_i, k_j) (c, d). A dot is plotted when $\Delta X_{ij} \leq \Delta_{th}$. (a) and (c) for the threshold value $\Delta_{th} = 0.25$, and (b) and (d) for $\Delta_{th} = 0.50$. The solid lines in (a) and (b) denote $i + j = J_{th}$ and are also plotted in (c) and (d) correspondingly. Note the different scales in (a) and (b) and the log-log scales in (c) and (d).

Non-identical oscillators

➔ phase synchronization

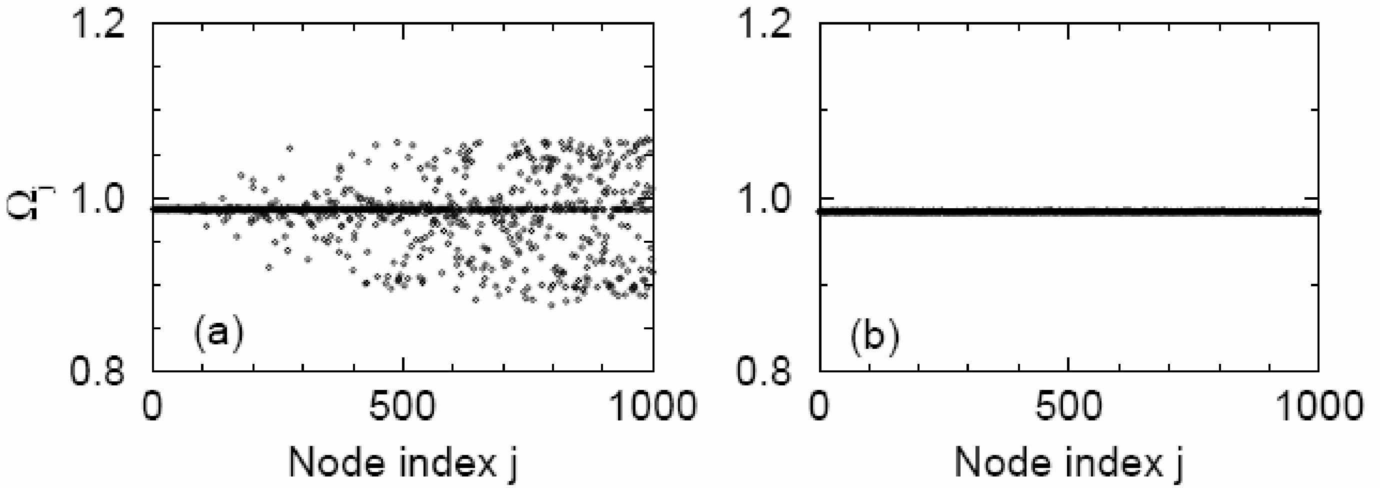


FIG. 11: The mean oscillation frequencies Ω_j of the oscillators in the SFN (a) and the HN (b) at the coupling strength $g = 0.13$.

Transition to synchronization in complex networks

- Hierarchical transition to synchronization via clustering
- Hubs are the „engines“ in cluster formation AND they become synchronized first among themselves

Cat Cerebal Cortex

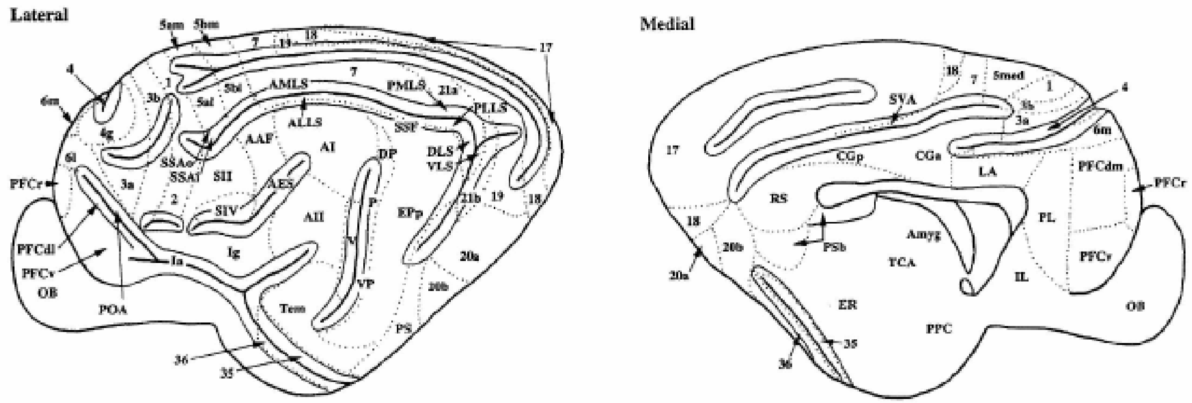


Fig. 1. Topographical map of cat cerebral cortex (from [20]).

Anatomical Connectivity

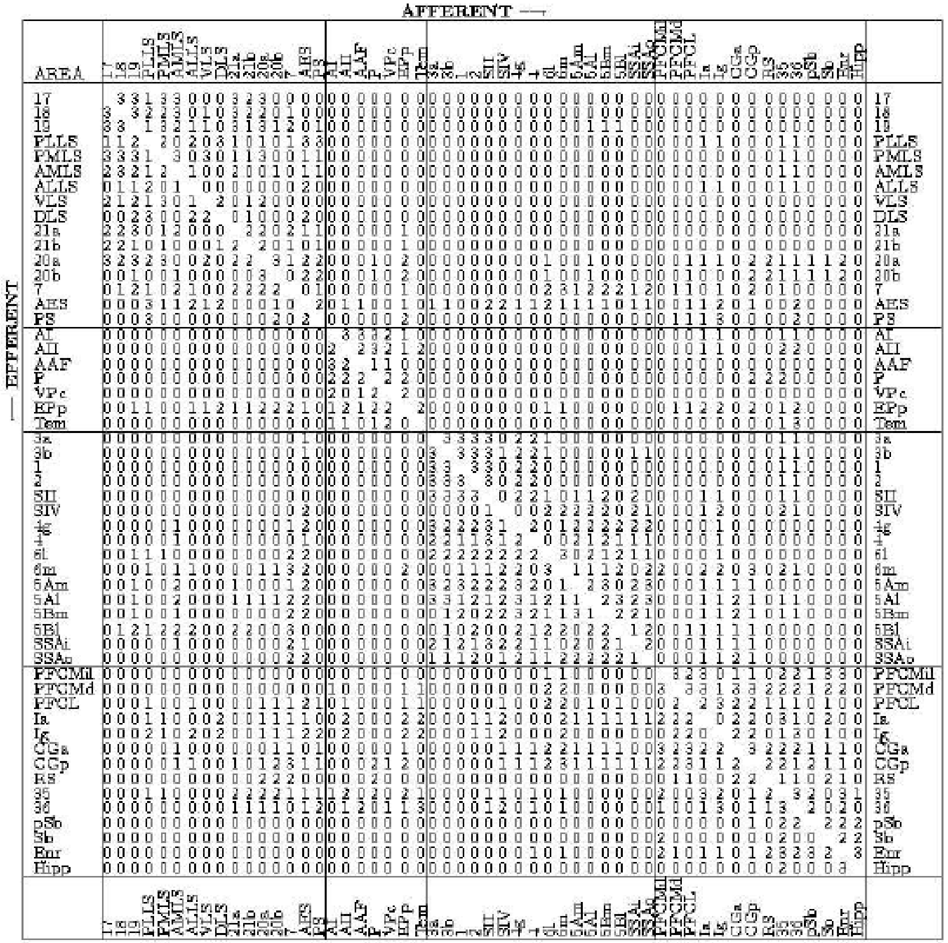


Fig. 2. Connectivity matrix representing connections between 53 cortical areas of cat brain.

Scannell et al.,

Cereb. Cort., 1999

Modelling ➔Functional Organization in Networks

- Intention:

Macroscopic ➔ Mesoscopic Modelling

Network of Networks

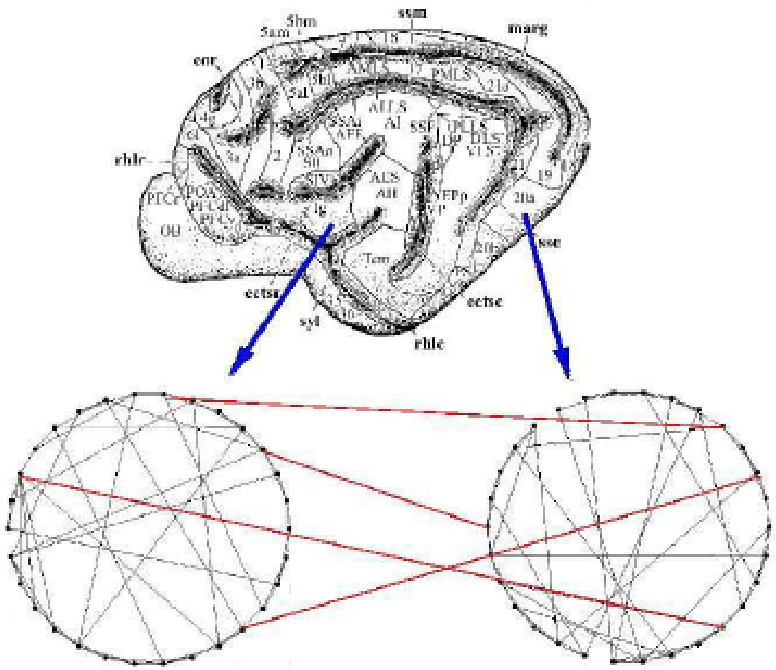
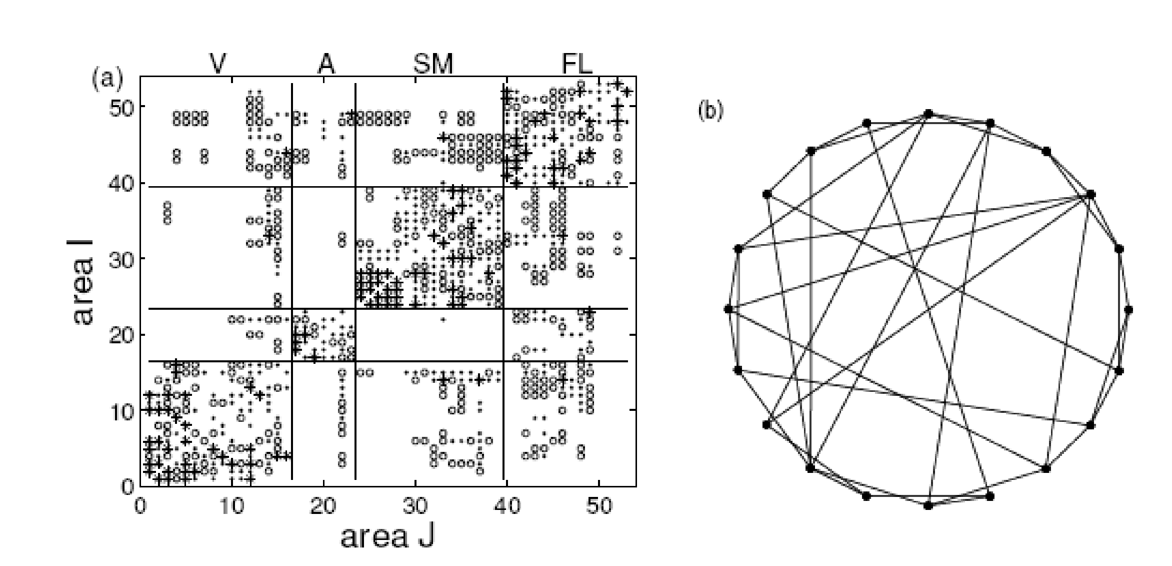


Fig. 3. The modeled system — a network of networks. Note that local subnetworks have small-world structure.

Hierarchical organization in complex brain networks

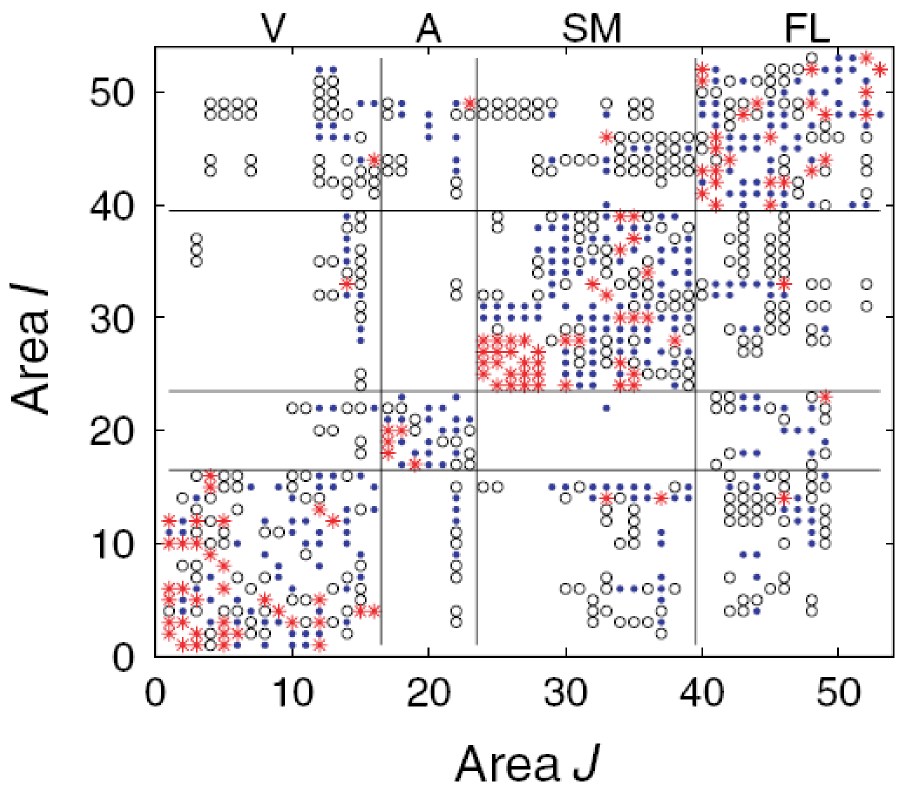


a) Connection matrix of the cortical network of the cat brain (anatomical)

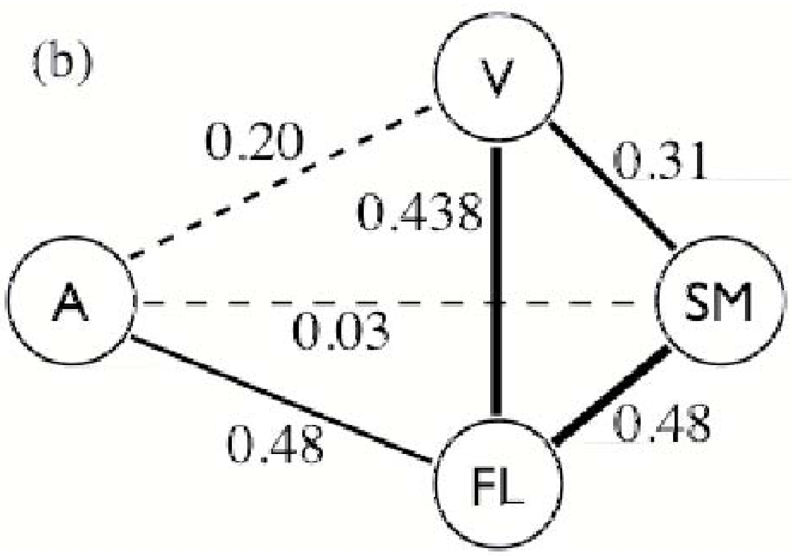
b) Small world sub-network to model each node in the network (200 nodes each, FitzHugh Nagumo neuron models - excitable)

➔ **Network of networks**

Phys Rev Lett 97 (2006), Physica D 224 (2006), New J. Phys. 9, 178 (2007)



Density of connections between the four communities



- Connections among the nodes: 2-3 … 35
- 830 connections
- Mean degree: 15

Model for neuron i in area I

$$\epsilon \dot{x}_{I,i} = f(x_{I,i}) + \frac{g_1}{k_a} \sum_{j}^{N_a} M_I^L(i,j)(x_{I,j} - x_{I,i})$$

$$+ \frac{g_2}{\langle w \rangle} \sum_{J}^{N} M^C(I,J) L_{I,J}(i)(\bar{x}_J - x_{I,i}),$$

$$\dot{y}_{I,i} = x_{I,i} + a_{I,i} + D\xi_{I,i}(t),$$

where

$$f(x_{I,i}) = x_{I,i} - \frac{x_{I,i}^3}{3} - y_{I,i}.$$

FHN model

Transition to synchronized firing

g – coupling strength – control parameter

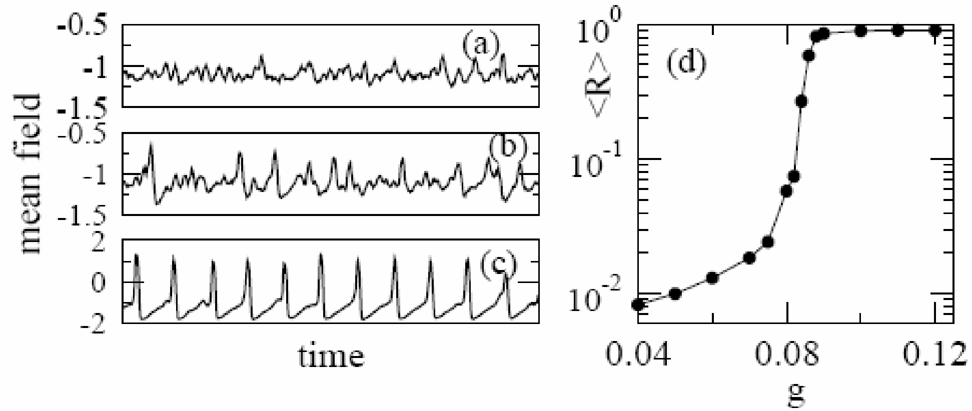


FIG. 2: Mean activity X of one area at various couplings (a) $g = 0.06$, (b) $g = 0.082$ (c) $g = 0.09$. The average correlation coefficient $\langle R \rangle = \frac{1}{N(N-1)} \sum_{I \neq J}^{N} R(I, J)$ $(N = 53)$ vs. g.

Network topology vs. Functional organization in networks

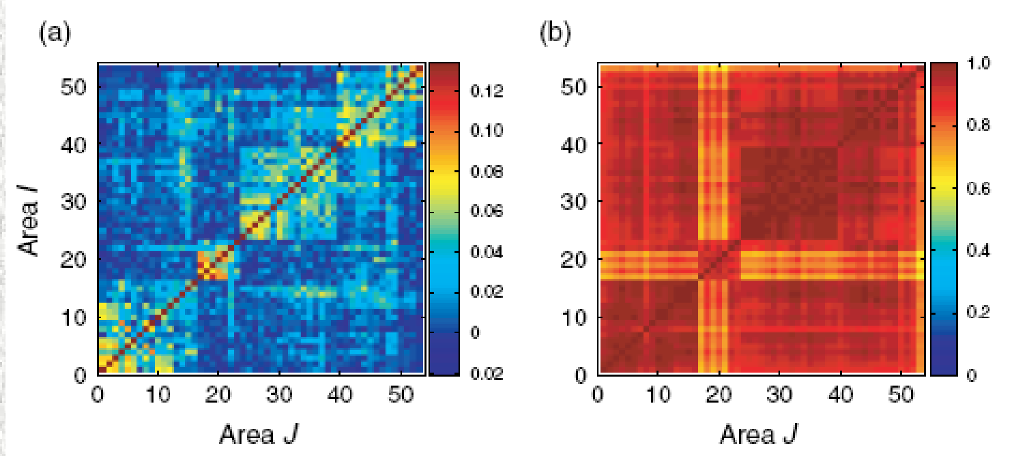


Figure 7. Correlation matrices R_{IJ} at weak coupling $g = 0.07$ (a) and strong coupling $g = 0.12$ (b).

Weak-coupling dynamics ➔ non-trivial organization

➔ relationship to underlying network topology

Dendrograms

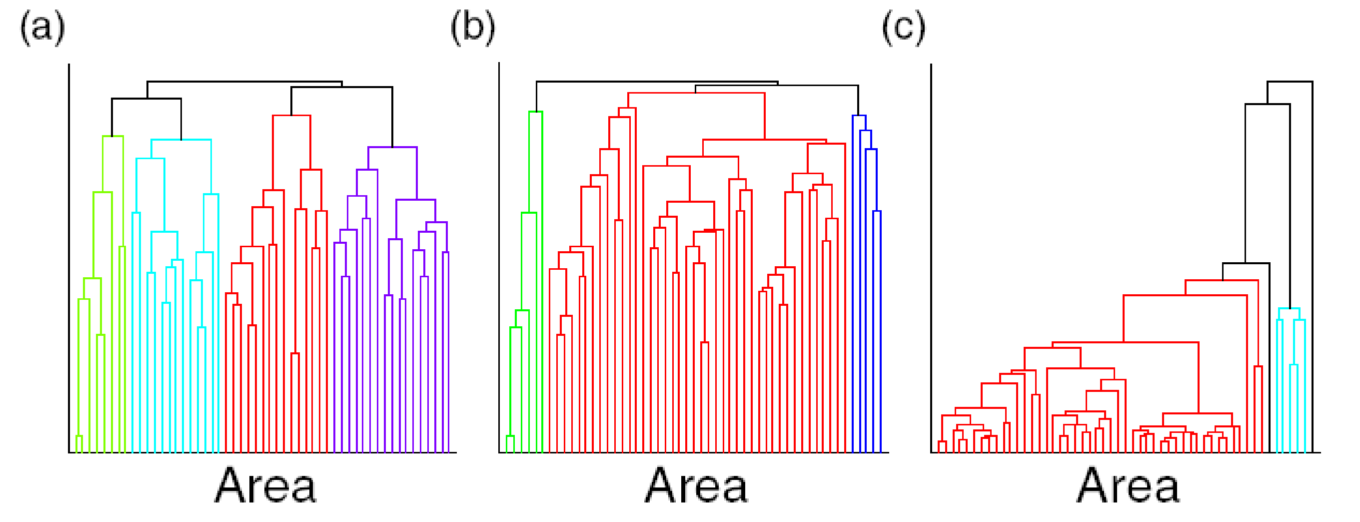


Figure 13. Typical hierarchical tree of the dynamical clusters in the weak coupling regime (a) $g = 0.07$, transient regime (b) $g = 0.082$ and strong coupling regime (c) $g = 0.12$.

Functional vs. Structural Coupling

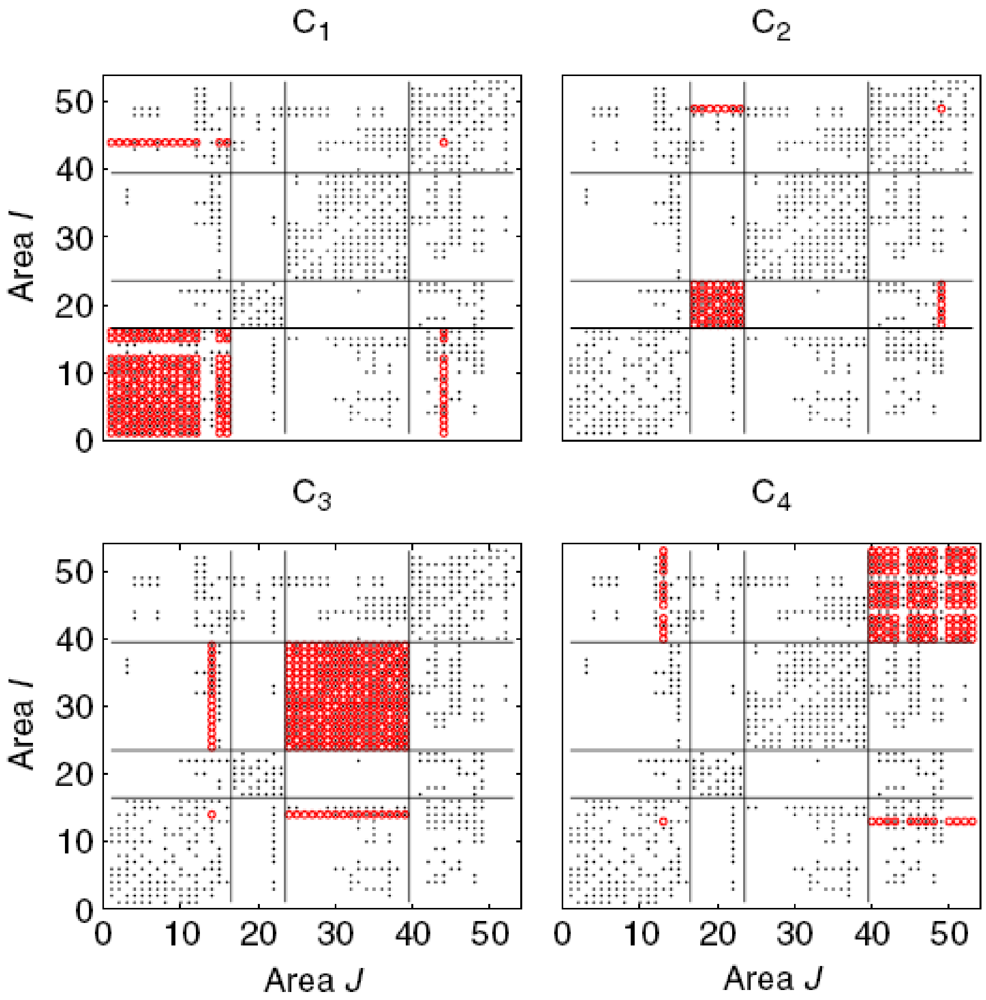
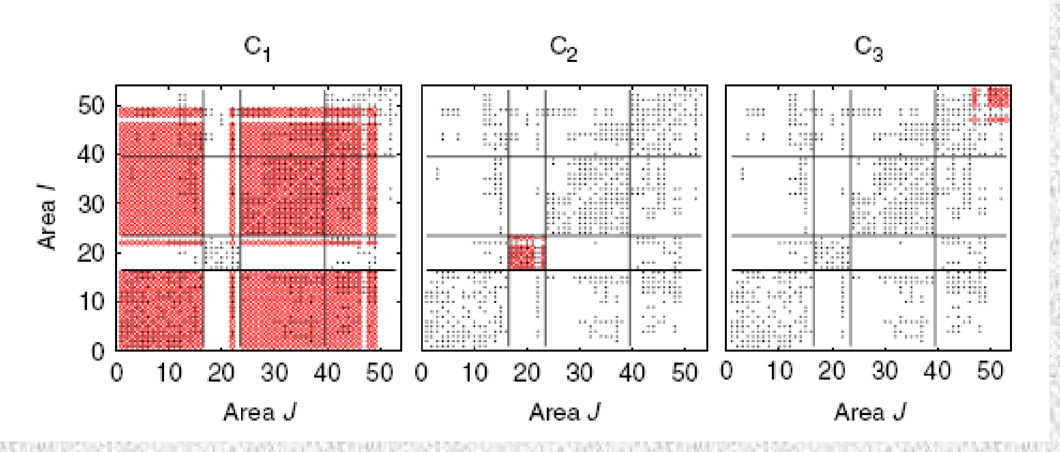


Figure 14. Four major dynamical clusters ($\circ$) with weak coupling strength $g = 0.07$, compared to the underlying anatomical connections ($\cdot$).

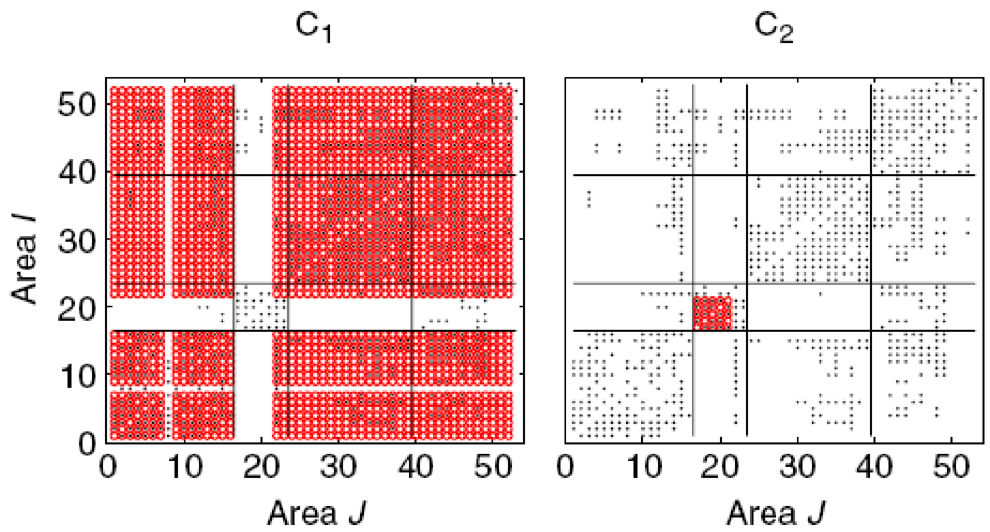
Intermediate Coupling



Intermediate Coupling:

3 main dynamical clusters

Strong Coupling

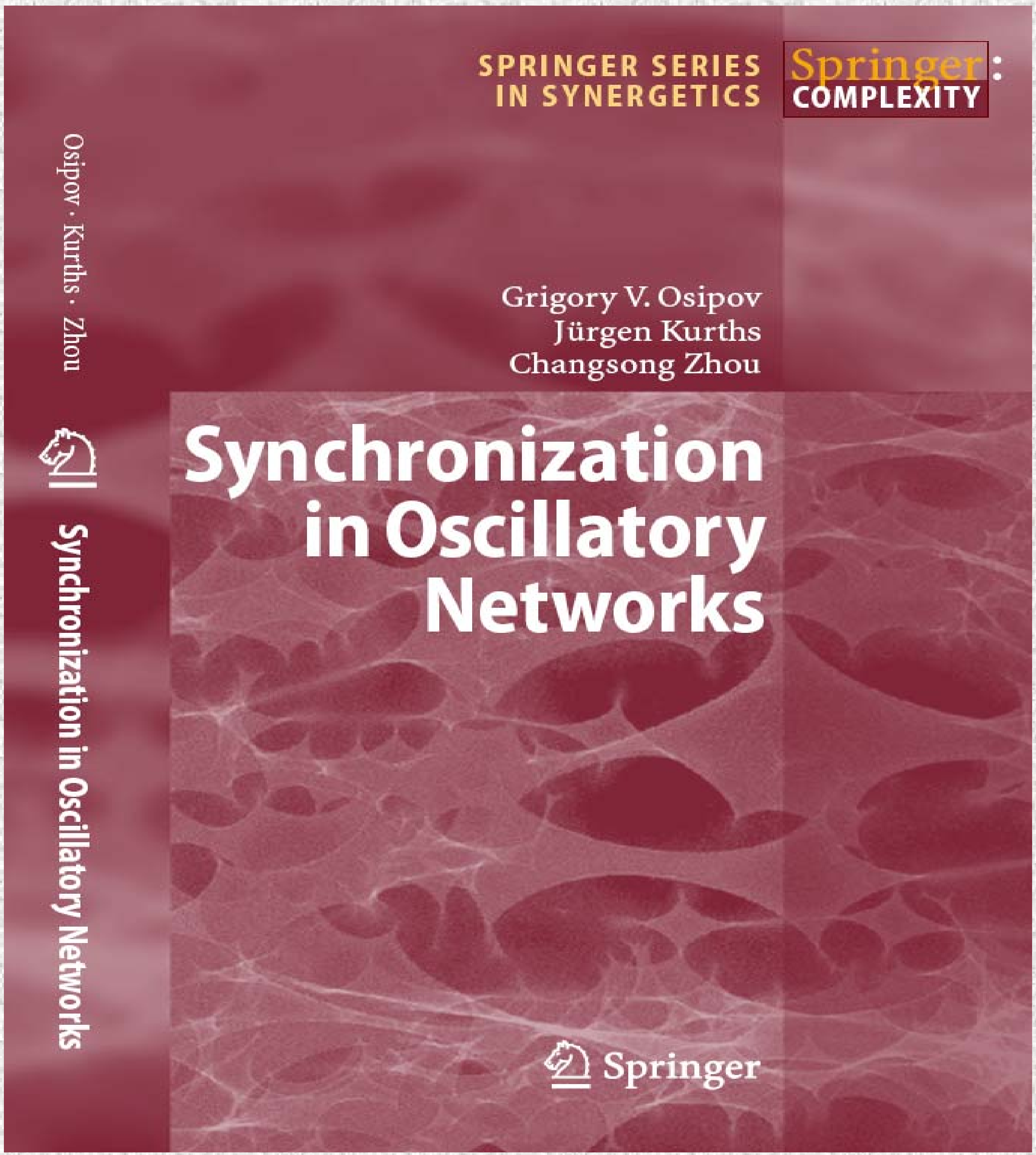


Summary - Take Home Messages

- There are rich synchronization phenomena in complex networks (**self-organized structure formation**) – hierarchical transitions
- This approach seems to be promising for modelling and understanding of some aspects of **brain dynamics** (e.g. cognition) but also **metabolic networks** etc.

Our papers on complex networks

Europhys. Lett. 69, 334 (2005)
Phys. Rev. E 71, 016116 (2005)
CHAOS 16, 015104 (2006)
Physica D 224, 202 (2006)
Phys: Rev. Lett. 96, 034101 (2006)
Phys. Rev. Lett. 96, 164102 (2006)
Phys. Rev. Lett. 96, 208103 (2006)
Phys. Rev. Lett. 97, 238103 (2006)
New J. Physics 9, 178 (2007)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