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**Synchronization in Complex
Networks**

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Synchronization in Complex Networks

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[Toolbox TOCSY](#)

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Outline

- Introduction
- Complete synchronization in complex networks
- Hierarchical (clustered) transitions to complex synchronization in complex networks
- Structure vs. functionality in complex brain networks – network of networks
- Conclusions

Synchronization in ensembles of active subsystems: Biology

- Ensemble of doves (wings in synchrony)
- Menstruation (e.g. female students living in one room in a dormitory)
- Collective firing of neurons – cause of any action of animals having neurons
- Fireflies, crickets, frogs in Asia (India, Vietnam...)

Basic Model in Statistical Physics and Nonlinear Sciences for ensembles

- Traditional Approach:
Regular chain or lattice of coupled oscillators;
global or nearest neighbour coupling

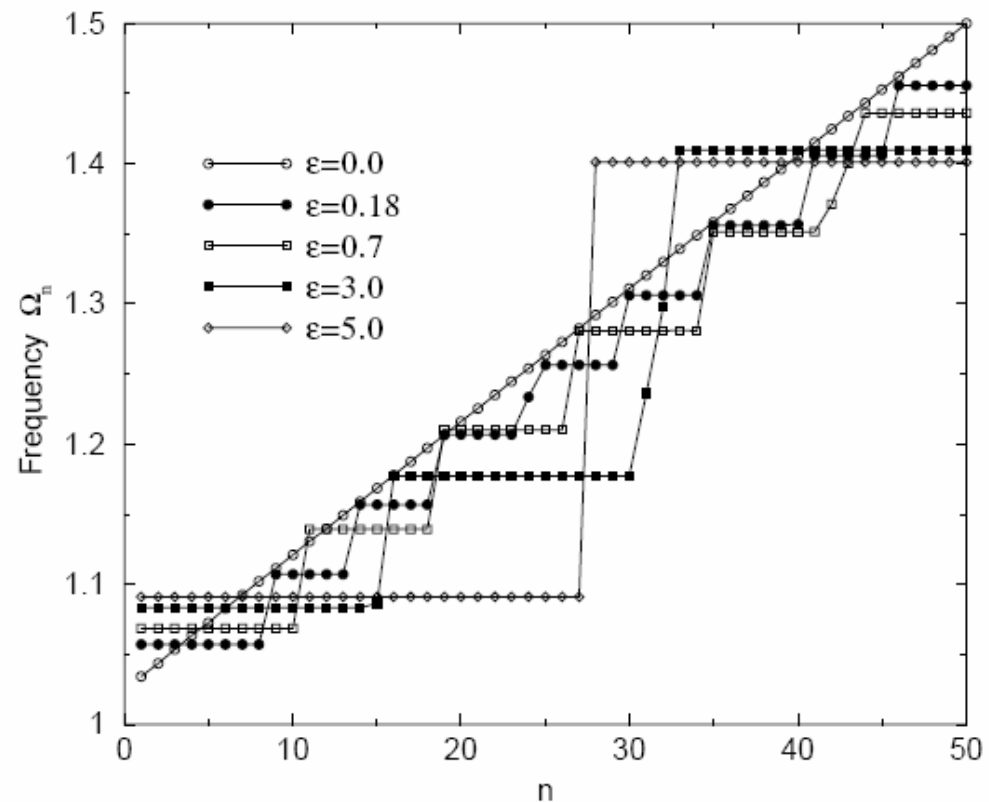
Synchronization in spatially extended systems

$$\dot{x}_n = -\omega_n y_n - z_n ,$$

$$\dot{y}_n = \omega_n x_n + a y_n + \epsilon(y_{n+1} - 2y_n + y_{n-1}) ,$$

$$\dot{z}_n = 0.4 + (x_n - 8.5)z_n .$$

Chain of diffusively
coupled Roessler
oscillators



Soft transition to phase synchronization

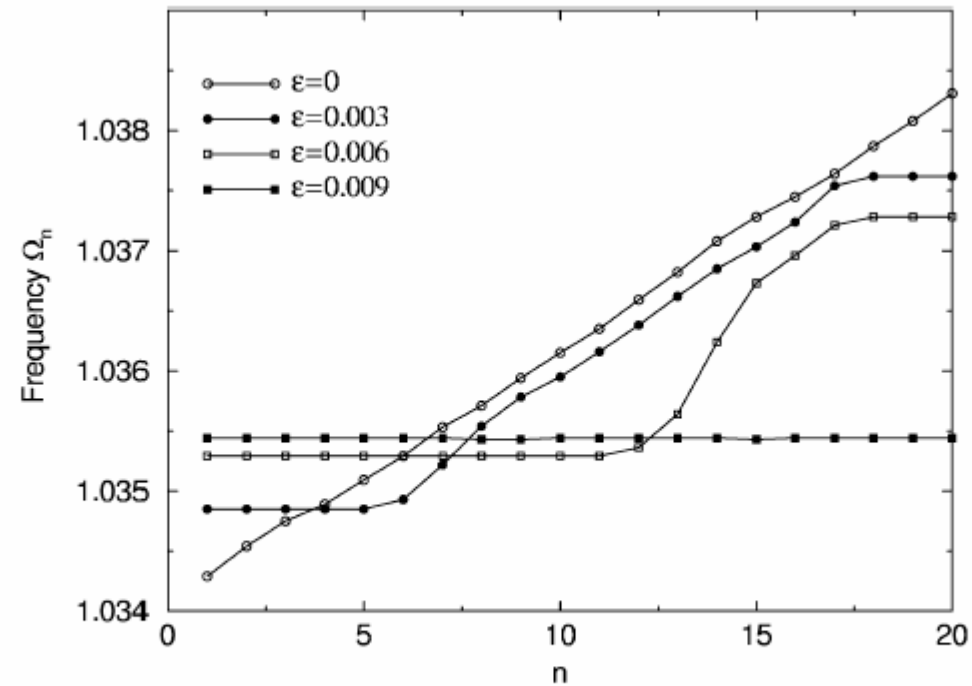


Fig. 6.3. Soft transition to global synchronization in a chain of Rössler oscillators (6.24). Mean frequencies Ω_n for different values of coupling ϵ . The parameters are: $N = 20$, the frequency mismatch $\delta = 2 \times 10^{-4}$ and $\omega_1 = 1$.

Soft transition to PS

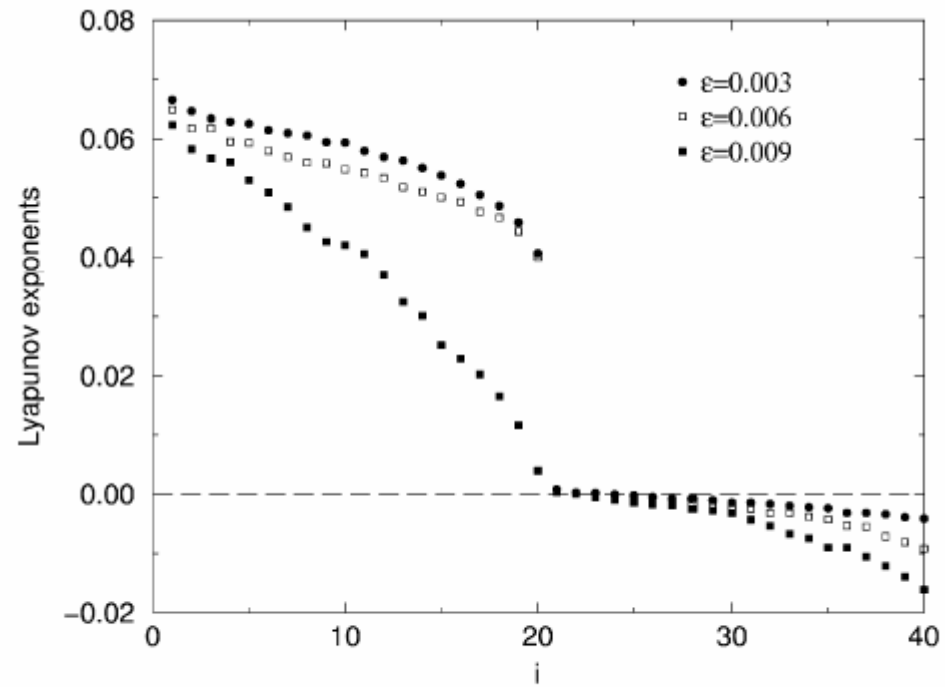


Fig. 6.4. 40 largest Lyapunov exponents λ_i for the regimes reported in Fig. 6.3.

Hard transition to PS

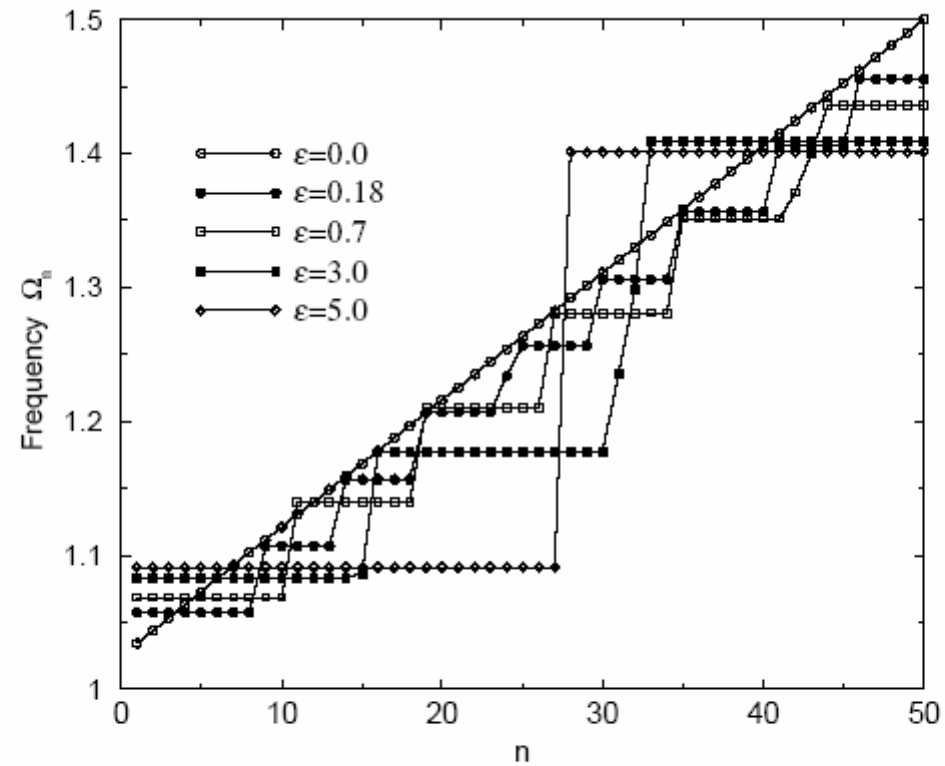


Fig. 6.5. Hard transition to global synchronization in a chain of Rössler oscillators (Eq. (6.24)). Mean frequencies Ω_n for different values of coupling ϵ . The parameters are: $N = 50$, the frequency mismatch $\delta = 9 \times 10^{-3}$ and $\omega_1 = 1$.

Hard transition to PS

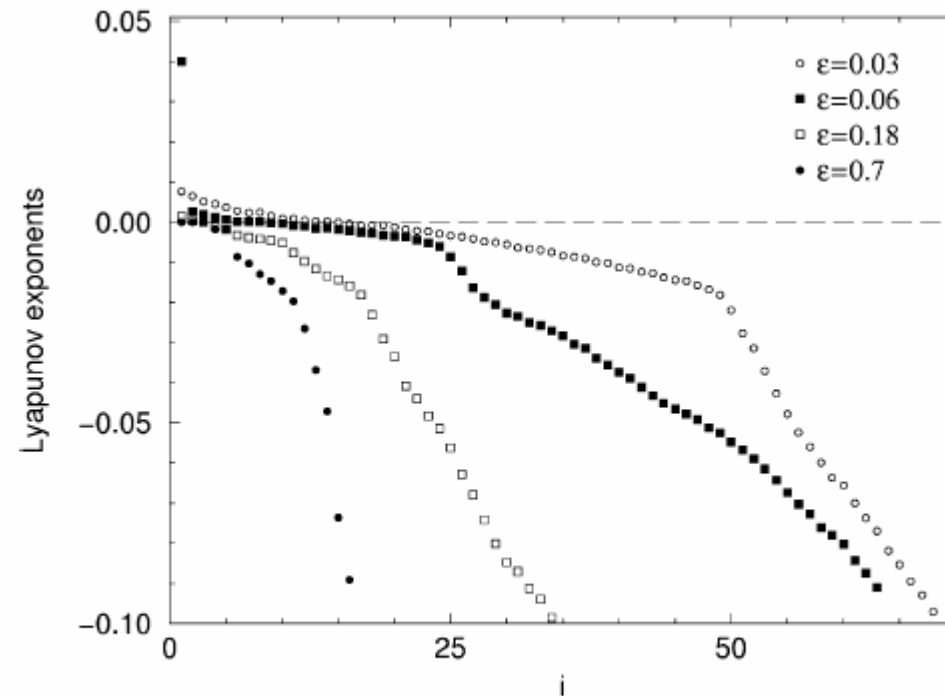


Fig. 6.6. 70 largest Lyapunov exponents λ_i for the regimes reported in Fig. 6.5.

Hard transition to PS

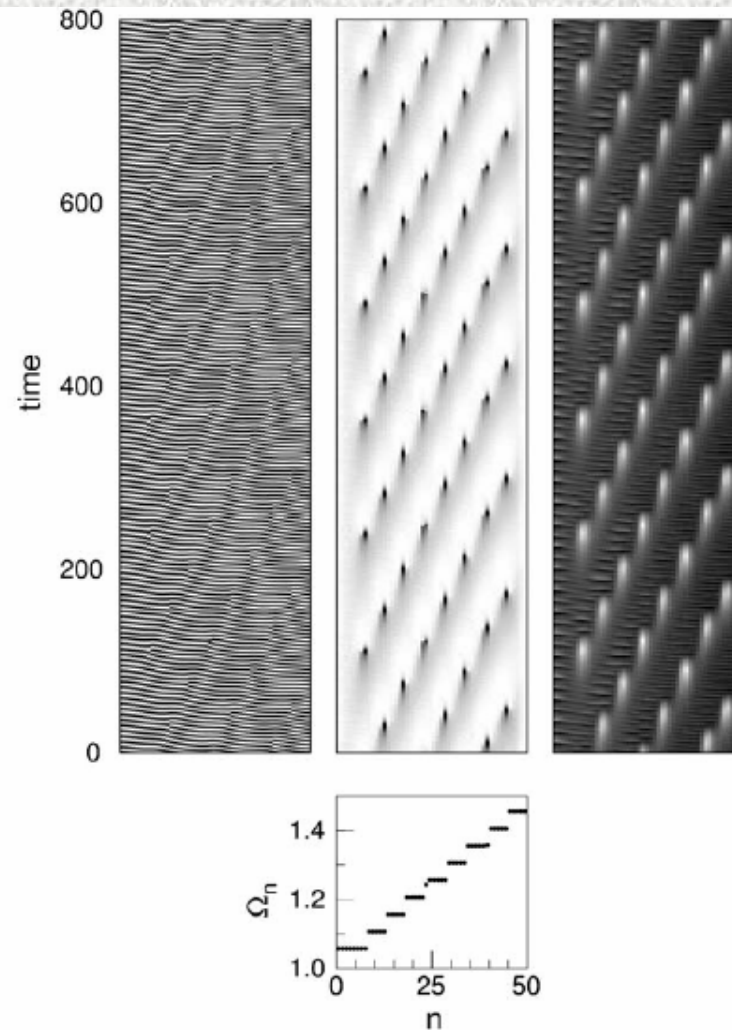


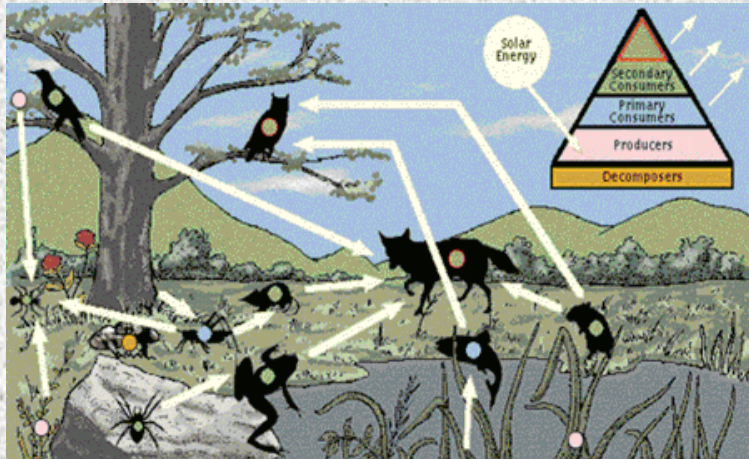
Fig. 6.7. Mean frequencies Ω_n and space-time plots in a chain of 50 coupled Rössler oscillators with a frequency mismatch $\delta = 9 \times 10^{-3}$ and coupling $\epsilon = 0.18$. All plots show a gray-scale representation of corresponding quantities. Minimal values are represented by white and maximal by black.

Networks with Complex Topology

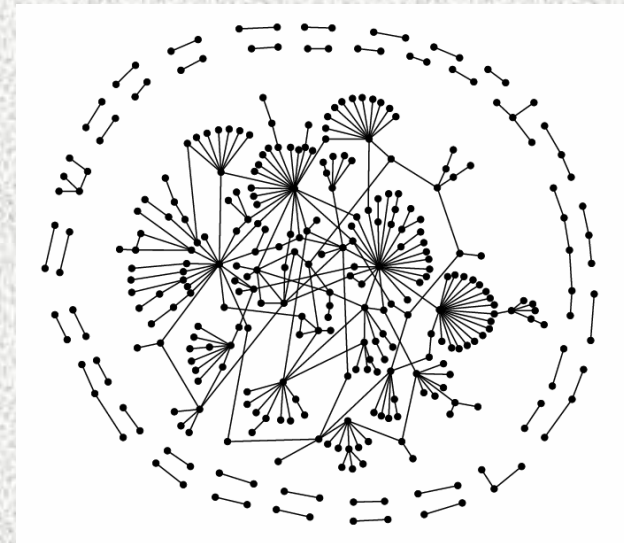
- Random graphs/networks (Erdős, Renyi, 1959)
- Small-world networks (Watts, Strogatz, 1998)
- Scale-free networks (Barabasi, Albert, 1999)
- Applications: neuroscience, cell biology, epidemic spreading, internet, traffic, systems biology
- Many participants (nodes) with complex interactions and **complex dynamics at the nodes**

Biological Networks

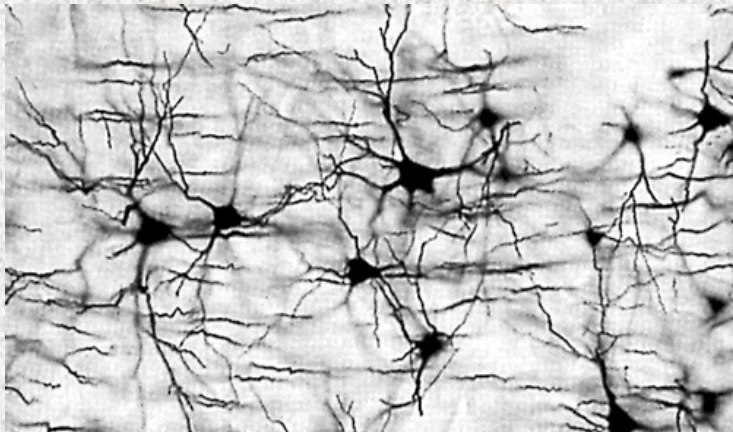
Ecological Webs



Protein interaction



Neural Networks

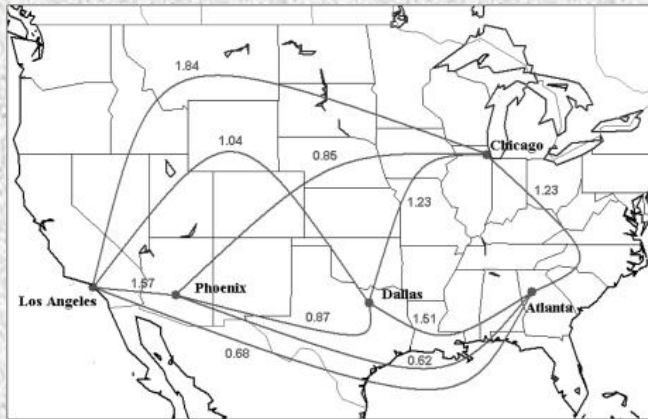


Genetic Networks

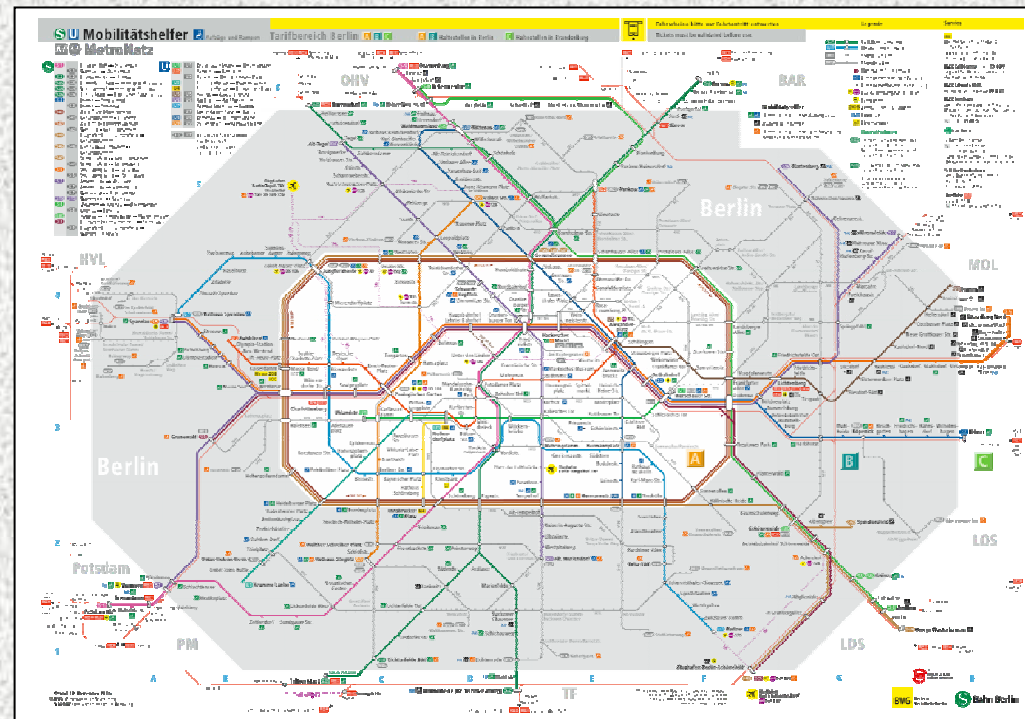
Metabolic Networks

Transportation Networks

Airport Networks



Local Transportation



Road Maps



Scale-free Networks

Network resilience

- Highly robust against random failure of a node
- Highly vulnerable to deliberate attacks on **hubs**

Applications

- **Immunization** in networks of computers, humans, ...

Synchronization in such networks

- Synchronization properties strongly influenced by the network's structure (Jost/Joy, Barahona/Pecora, Nishikawa/Lai, Timme et al., Hasler/Belykh(s) etc.)
- Self-organized synchronized clusters can be formed (Jalan/Amritkar)
- Previous works mainly focused on the influence of the connection's topology (assuming coupling strength uniform)

Universality in the synchronization of weighted random networks

Our intention:

Include the influence of weighted coupling for complete synchronization

(Motter, Zhou, Kurths: Phys. Rev. Lett., 96, 034101, 2006)

Weighted Network of N Identical Oscillators

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^N W_{ij} A_{ij} [\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \\ &= \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^N G_{ij} \mathbf{H}(\mathbf{x}_j), \quad i = 1, \dots, N,\end{aligned}$$

F – dynamics of each oscillator

H – output function

G – coupling matrix combining adjacency A and weight W

$$G_{ij} = -W_{ij} \text{ for } i \neq j$$

$$G_{ii} = \sum_j W_{ij} A_{ij} = S_i$$

S_i - intensity of node i (includes topology and weights)

General Condition for Synchronizability

Stability of synchronized state

$$\{\mathbf{x}_i = \mathbf{s}, \forall i \mid \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})\}$$

N eigenmodes of

$$\dot{\xi}_i = [D\mathbf{F}(\mathbf{s}) - \sigma \lambda_i D\mathbf{H}(\mathbf{s})]\xi_i,$$

λ_i i th eigenvalue of G

Synchronizability

G has real eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N, \text{ with } \lambda_1 = 0$$

Network synchronizable for some σ if

$$\epsilon_1 < \sigma \lambda_i < \epsilon_2 \text{ is satisfied for all } i \geq 2$$

or for the **eigenratio R**

$$R \equiv \lambda_N / \lambda_2 < \epsilon_2 / \epsilon_1$$

R small \rightarrow more synchronizable

Cost involved in the network's coupling

Cost = total input strength of the connections of all nodes at the synchronization threshold $\sigma_{\min} \equiv \epsilon_1 / \lambda_2$

$$C = \sigma_{\min} \sum_{i,j} W_{ij} A_{ij} = \sigma_{\min} \sum_{i=1}^N S_i$$

Normalized cost

$$C_0 \equiv C / (N \epsilon_1) = \Omega / \lambda_2$$

$$\Omega = \sum_{i=1}^N S_i / N \quad \text{Mean Intensity}$$

Main result for general weighted random networks

$$R = A_R \frac{S_{\max}}{S_{\min}} R_H(K), \quad C_0 = A_C \frac{\Omega}{S_{\min}} C_H(K)$$

Distribution of eigenvalues in large random networks leads then to:

$$R \approx R_H(K) \equiv \frac{1 + 2/\sqrt{K}}{1 - 2/\sqrt{K}}$$

$$C_0 \approx C_H(K) \equiv \frac{1}{1 - 2/\sqrt{K}}$$

Main results

Synchronizability universally determined by:

- mean degree K and
- heterogeneity of the intensities

$$\frac{S_{\max}}{S_{\min}}$$

or

$$\frac{\Omega}{S_{\min}}$$

S_{\min}, S_{\max} - minimum/ maximum intensities

Hierarchical Organization of Synchronization in Complex Networks

Homogeneous (constant number of connections in each node)

vs.

Scale-free networks

$$\dot{\mathbf{x}}_j = \tau_j \mathbf{F}(\mathbf{x}_j) + \frac{g}{K} \sum_{i=1}^N A_{ji} (\mathbf{x}_i - \mathbf{x}_j)$$

Zhou, Kurths: CHAOS (focus issue: 16, 015104 (2006))

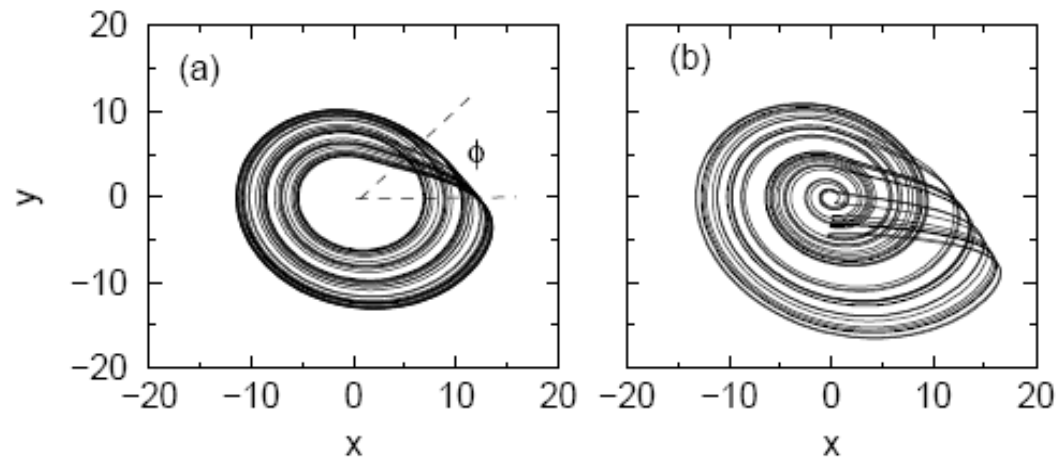


FIG. 1: Chaotic attractors of the Rössler oscillator in the phase coherent regime (a) and phase non-coherent regime (b).

$$\begin{aligned}\dot{x} &= -0.97x - z, \\ \dot{y} &= 0.97x + ay, \\ \dot{z} &= x(z - 8.5) + 0.4,\end{aligned}$$

Identical oscillators

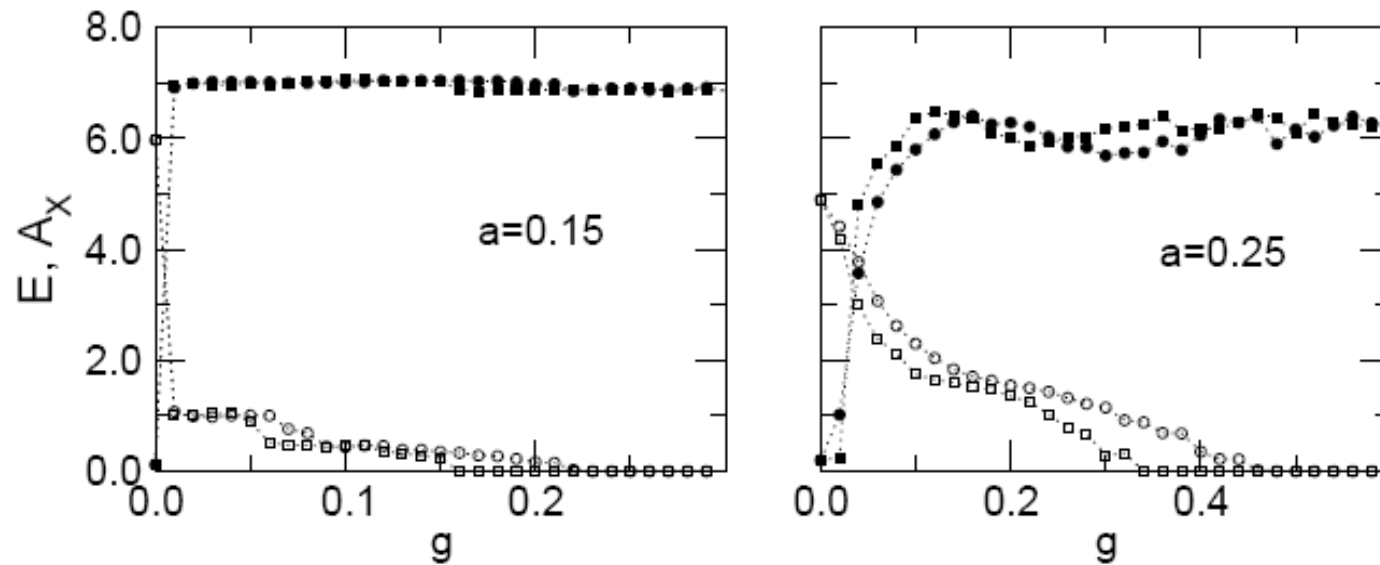


FIG. 3: Transition to CS in the SFN and the HN, indicated by the synchronization error E (squares) and the amplitude A_X of the mean field X (circles). The filled symbols are for the SFNs and the open symbols for the HNs. (a) Phase coherent oscillations at $a = 0.15$. (b) Phase non-coherent oscillations at $a = 0.25$. In both networks, $N = 1000$ and $K = 10$.

Transition to synchronization

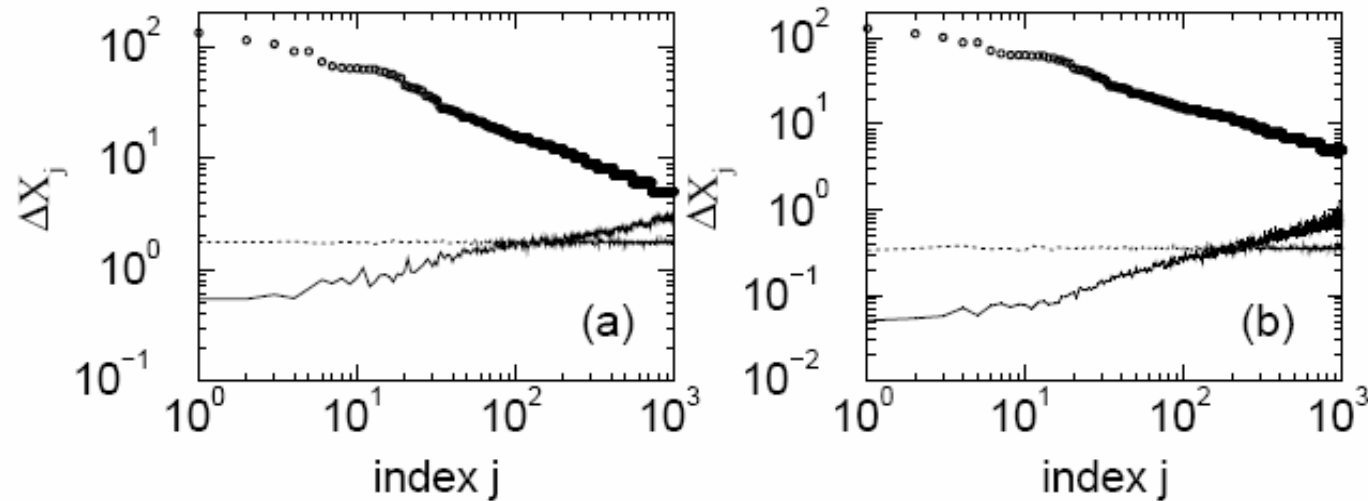


FIG. 4: Synchronization difference ΔX_j of the oscillators with respect to the global mean field X in the SFN (solid line) and HN (dotted line). The symbol (\circ) denotes the degree k_j of the nodes. Note the log-log scales of the plots. (a) When the coupling strength is weak ($g = 0.1$) and (b) when the synchronized state ($g = 0.5$) is perturbed by noise ($D = 0.5$). Here the oscillations are phase non-coherent at $a = 0.25$ and the behavior is very similar for phase coherent oscillations at $a = 0.15$.

$$\Delta X_j = \langle |x_j - X| \rangle_t$$

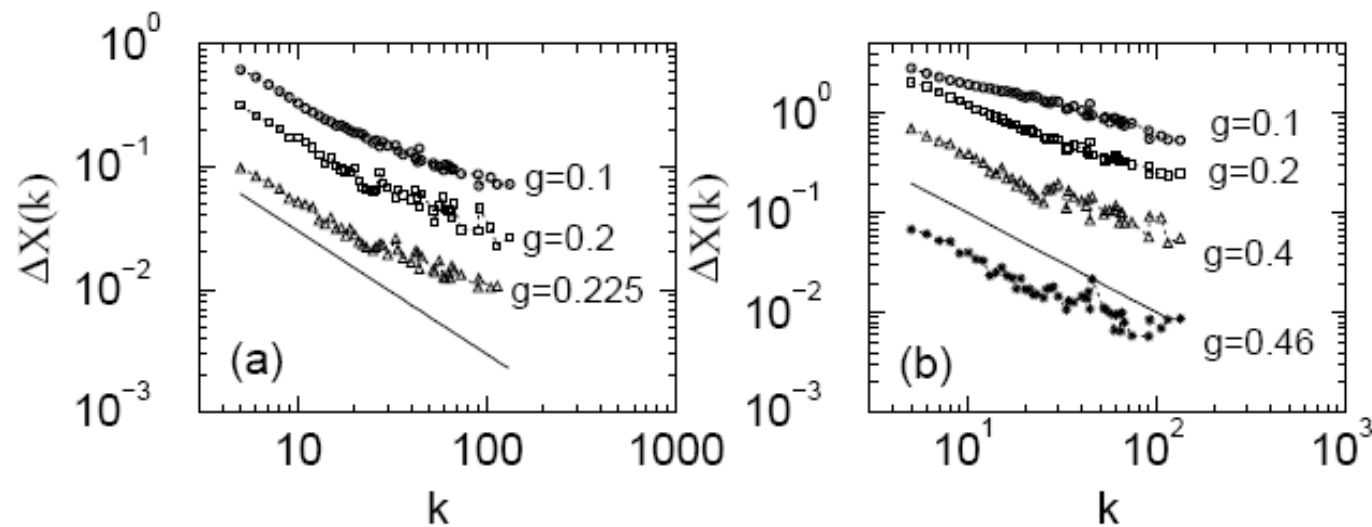


FIG. 5: The average values $\Delta X(k)$ as a function of k at various coupling strength g in the SFN. (a) Phase coherent regime $a = 0.15$. (b) phase non-coherent regime $a = 0.25$. The solid line with slop -1 is plotted for reference.

$$\Delta X(k) \sim k^{-\alpha}$$

$$\alpha \approx 1$$

$$\Delta X(k) = \frac{1}{N_k} \sum_{k_j=k} \Delta X_j$$

Mean-field approximation

$$\dot{\mathbf{x}}_j = \tau_j \mathbf{F}(\mathbf{x}_j) + \frac{gk_j}{K} (\mathbf{X} - \mathbf{x}_j), \quad k_j \gg 1.$$

Each oscillator forced by a common signal

Coupling strength \sim degree

$$\frac{d}{dt} \Delta X(k) = \Lambda(k) \Delta X(k) + c.$$

For nodes with rather large degree

→ Scaling: $\Delta X(k) \sim k^{-1},$

Clusters of synchronization

$$\Delta X_{ij} = \langle |x_i - x_j| \rangle_t$$

$$\Delta X_{ij} \leq \Delta_{th}$$

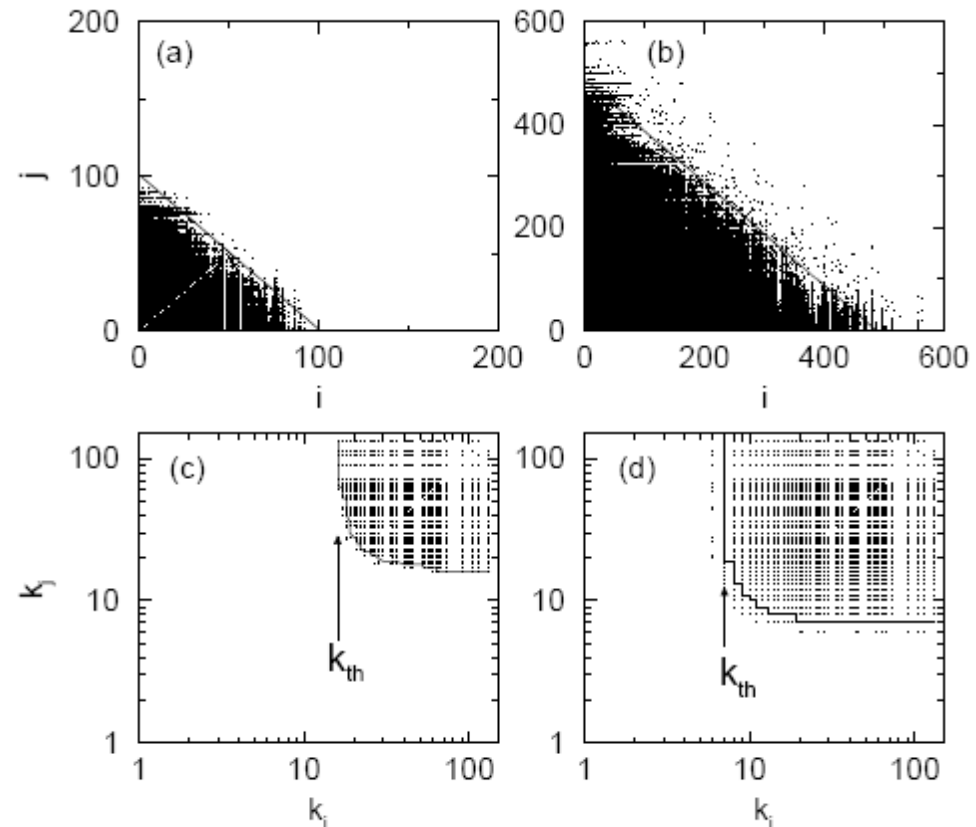
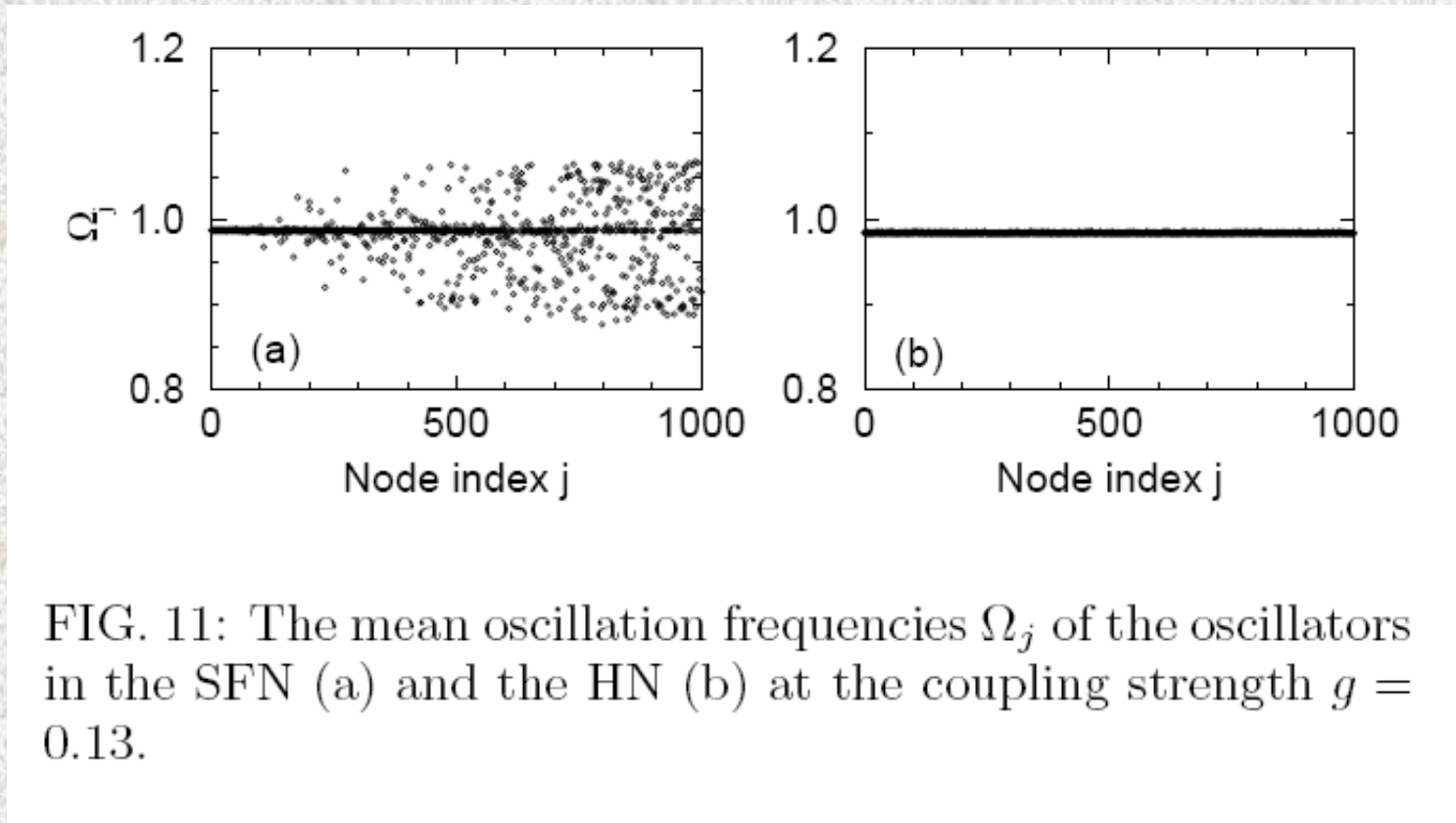


FIG. 7: The effective synchronization clusters in the synchronized network in the presence of noise ($a = 0.25$, $g = 0.5$, and $D = 0.5$), represented simultaneously in the index space (i, j) (a, b) and in the degree space (k_i, k_j) (c, d). A dot is plotted when $\Delta X_{ij} \leq \Delta_{th}$. (a) and (c) for the threshold value $\Delta_{th} = 0.25$, and (b) and (d) for $\Delta_{th} = 0.50$. The solid lines in (a) and (b) denote $i + j = J_{th}$ and are also plotted in (c) and (d) correspondingly. Note the different scales in (a) and (b) and the log-log scales in (c) and (d).

Non-identical oscillators

→ phase synchronization



Transition to synchronization in complex networks

- Hierarchical transition to synchronization via clustering
- Hubs are the „engines“ in cluster formation AND they become synchronized first among themselves

Cat Cerebral Cortex

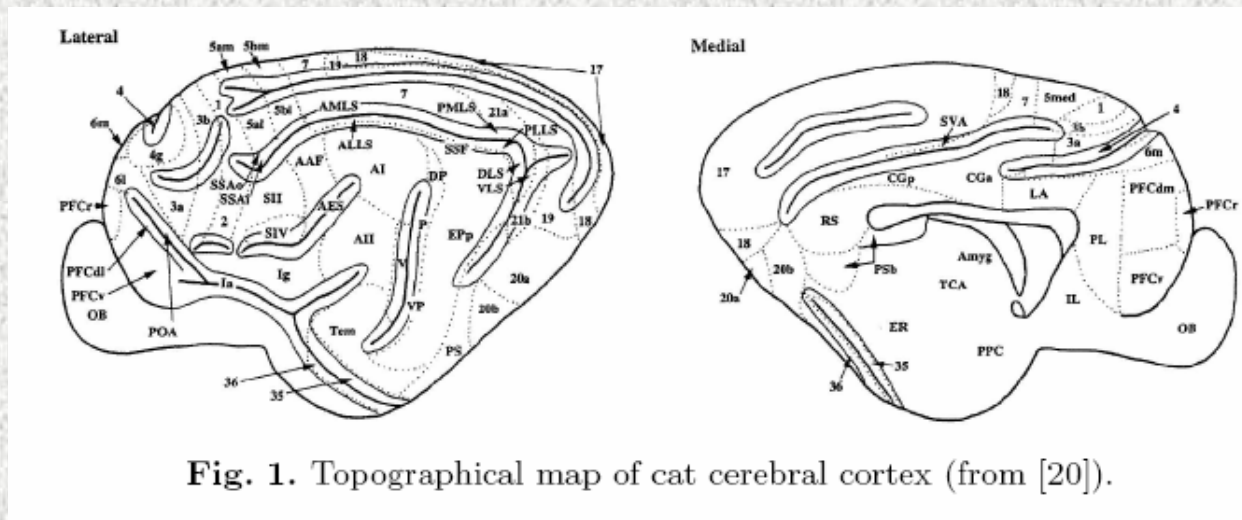


Fig. 1. Topographical map of cat cerebral cortex (from [20]).

Anatomical Connectivity

		AFFERENT																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																								
		17	18	19	FLLS	FMLLS	AMLLS	ALLS	VLLS	DLS	21a	21b	20a	20b	7	AES	PS	Al	AlAF	VPc	EPp	Tem	3a	3b	1a	1b	SII	SIV	4a	4b	5l	5m	5Am	5Al	5Bm	5Bl	5SAi	5SAo	PFCMI	PFCMd	PFCL	Ia	Ib	CGa	CGp	RS	35	35b	35c	35d	35e	35f	35g	35h	35i	35j	35k	35l	35m	35n	35o	35p	35q	35r	35s	35t	35u	35v	35w	35x	35y	35z	35aa	35ab	35ac	35ad	35ae	35af	35ag	35ah	35ai	35aj	35ak	35al	35am	35an	35ao	35ap	35aq	35ar	35as	35at	35au	35av	35aw	35ax	35ay	35az	35ba	35bb	35bc	35bd	35be	35bf	35bg	35bh	35bi	35bj	35bk	35bl	35bm	35bn	35bo	35bp	35bq	35br	35bs	35bt	35bu	35bv	35bw	35bx	35by	35bz	35ca	35cb	35cc	35cd	35ce	35cf	35cg	35ch	35ci	35cj	35ck	35cl	35cm	35cn	35co	35cp	35cq	35cr	35cs	35ct	35cu	35cv	35cw	35cx	35cy	35cz	35da	35db	35dc	35dd	35de	35df	35dg	35dh	35di	35dj	35dk	35dl	35dm	35dn	35do	35dp	35dq	35dr	35ds	35dt	35du	35dv	35dw	35dx	35dy	35dz	35ea	35eb	35ec	35ed	35ee	35ef	35eg	35eh	35ei	35ej	35ek	35el	35em	35en	35eo	35ep	35eq	35er	35es	35et	35eu	35ev	35ew	35ex	35ey	35ez	35fa	35fb	35fc	35fd	35fe	35ff	35fg	35fh	35fi	35fj	35fk	35fl	35fm	35fn	35fo	35fp	35fq	35fr	35fs	35ft	35fu	35fv	35fw	35fx	35fy	35fz	35ga	35gb	35gc	35gd	35ge	35gf	35gg	35gh	35gi	35gj	35gk	35gl	35gm	35gn	35go	35gp	35gq	35gr	35gs	35gt	35gu	35gv	35gw	35gx	35gy	35gz	35ha	35hb	35hc	35hd	35he	35hf	35hg	35hh	35hi	35hj	35hk	35hl	35hm	35hn	35ho	35hp	35hq	35hr	35hs	35ht	35hu	35hv	35hw	35hx	35hy	35hz	35ia	35ib	35ic	35id	35ie	35if	35ig	35ih	35ii	35ij	35ik	35il	35im	35in	35io	35ip	35iq	35ir	35is	35it	35iu	35iv	35iw	35ix	35iy	35iz	35ja	35jb	35jc	35jd	35je	35jf	35jg	35jh	35ji	35jj	35jk	35jl	35jm	35jn	35jo	35jp	35jq	35jr	35js	35jt	35ju	35jv	35jw	35jx	35jy	35jz	35ka	35kb	35kc	35kd	35ke	35kf	35kg	35kh	35ki	35kj	35kl	35km	35kn	35ko	35kp	35kq	35kr	35ks	35kt	35ku	35kv	35kw	35kx	35ky	35kz	35la	35lb	35lc	35ld	35le	35lf	35lg	35lh	35li	35lj	35lk	35ll	35lm	35ln	35lo	35lp	35lq	35lr	35ls	35lt	35lu	35lv	35lw	35lx	35ly	35lz	35ma	35mb	35mc	35md	35me	35mf	35mg	35mh	35mi	35mj	35mk	35ml	35mn	35mo	35mp	35mq	35mr	35ms	35mt	35mu	35mv	35mw	35mx	35my	35mz	35na	35nb	35nc	35nd	35ne	35nf	35ng	35nh	35ni	35nj	35nk	35nl	35nm	35nn	35no	35np	35nq	35nr	35ns	35nt	35nu	35nv	35nw	35nx	35ny	35nz	35oa	35ob	35oc	35od	35oe	35of	35og	35oh	35oi	35oj	35ok	35ol	35om	35on	35oo	35op	35oq	35or	35os	35ot	35ou	35ov	35ow	35ox	35oy	35oz	35pa	35pb	35pc	35pd	35pe	35pf	35pg	35ph	35pi	35pj	35pk	35pl	35pm	35pn	35po	35pp	35pq	35pr	35ps	35pt	35pu	35pv	35pw	35px	35py	35pz	35qa	35qb	35qc	35qd	35qe	35qf	35qg	35qh	35qi	35qj	35qk	35ql	35qm	35qn	35qo	35qp	35qq	35qr	35qs	35qt	35qu	35qv	35qw	35qx	35qy	35qz	35ra	35rb	35rc	35rd	35re	35rf	35rg	35rh	35ri	35rj	35rk	35rl	35rm	35rn	35ro	35rp	35rq	35rr	35rs	35rt	35ru	35rv	35rw	35rx	35ry	35rz	35sa	35sb	35sc	35sd	35se	35sf	35sg	35sh	35si	35sj	35sk	35sl	35sm	35sn	35so	35sp	35sq	35sr	35ss	35st	35su	35sv	35sw	35sx	35sy	35sz	35ta	35tb	35tc	35td	35te	35tf	35tg	35th	35ti	35tj	35tk	35tl	35tm	35tn	35to	35tp	35tq	35tr	35ts	35tt	35tu	35tv	35tw	35tx	35ty	35tz	35ua	35ub	35uc	35ud	35ue	35uf	35ug	35uh	35ui	35uj	35uk	35ul	35um	35un	35uo	35up	35uq	35ur	35us	35ut	35uu	35uv	35uw	35ux	35uy	35uz	35va	35vb	35vc	35vd	35ve	35vf	35vg	35vh	35vi	35vj	35vk	35vl	35vm	35vn	35vo	35vp	35vq	35vr	35vs	35vt	35vu	35vv	35vw	35vx	35vy	35vz	35wa	35wb	35wc	35wd	35we	35wf	35wg	35wh	35wi	35wj	35wk	35wl	35wm	35wn	35wo	35wp	35wq	35wr	35ws	35wt	35wu	35wv	35ww	35wx	35wy	35wz	35xa	35xb	35xc	35xd	35xe	35xf	35xg	35xh	35xi	35xj	35xk	35xl	35xm	35xn	35xo	35xp	35xq	35xr	35xs	35xt	35xu	35xv	35xw	35xx	35xy	35xz	35ya	35yb	35yc	35yd	35ye	35yf	35yg	35yh	35yi	35yj	35yk	35yl	35ym	35yn	35yo	35yp	35yq	35yr	35ys	35yt	35yu	35yv	35yw	35yx	35yy	35yz	35za	35zb	35zc	35zd	35ze	35zf	35zg	35zh	35zi	35zj	35zk	35zl	35zm	35zn	35zo	35zp	35zq	35zr	35zs	35zt	35zu	35zv	35zw	35zx	35zy	35zz

Fig. 2. Connectivity matrix representing connections between 53 cortical areas of cat brain.

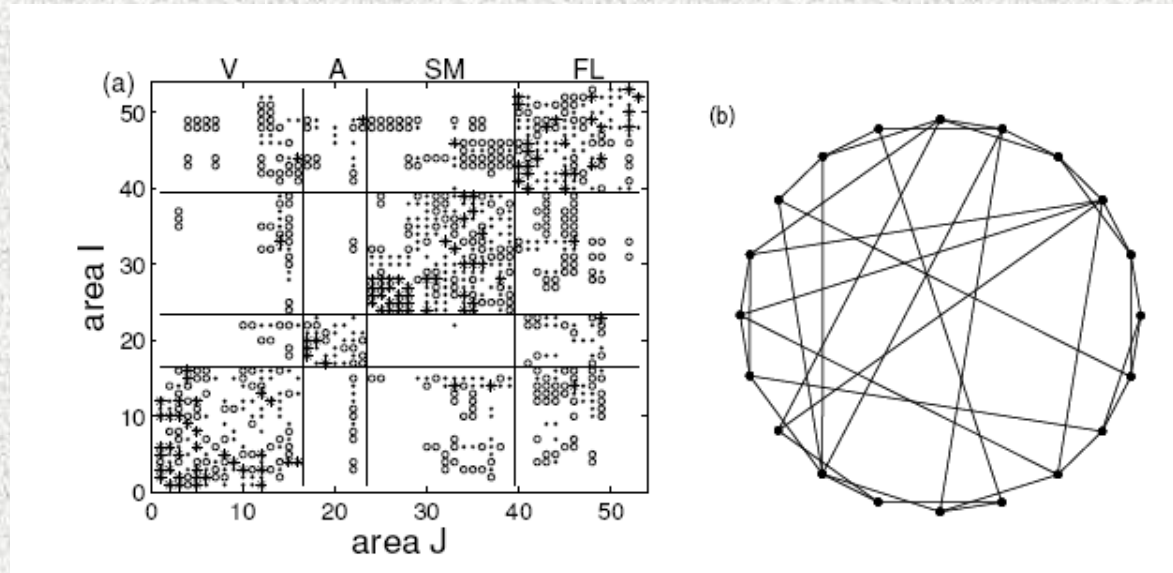
Scannell et al.,
Cereb. Cort., 1999

Modelling → Functional Organization in Networks

- Intention:

Macroscopic → Mesoscopic Modelling

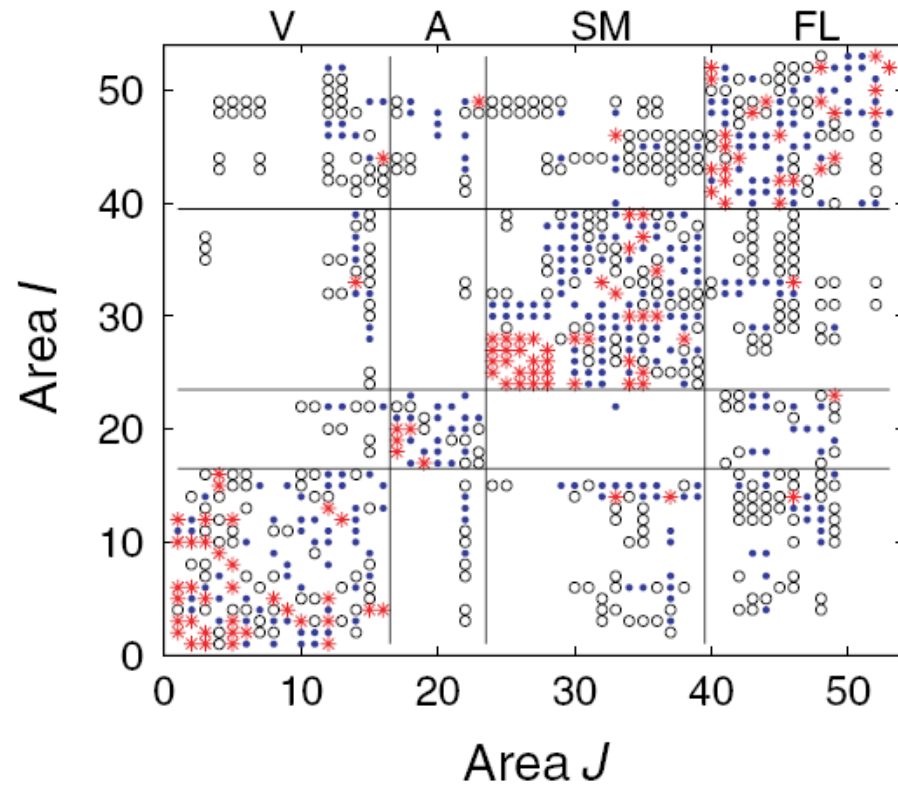
Hierarchical organization in complex brain networks



- a) Connection matrix of the cortical network of the cat brain (anatomical)
- b) Small world sub-network to model each node in the network (200 nodes each, FitzHugh Nagumo neuron models - excitable)

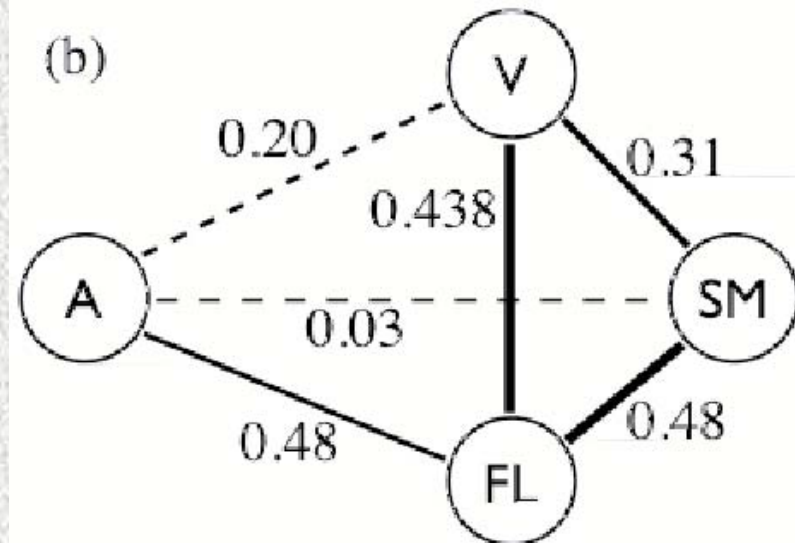
→ **Network of networks**

Phys Rev Lett 97 (2006), Physica D 224 (2006), New J. Phys. 9, 178 (2007)



Density of connections
between the four com-
munities

- Connections among the nodes: 2-3 ... 35
- 830 connections
- Mean degree: 15



Model for neuron i in area I

$$\begin{aligned}\epsilon \dot{x}_{I,i} &= f(x_{I,i}) + \frac{g_1}{k_a} \sum_j^{N_a} M_I^L(i, j)(x_{I,j} - x_{I,i}) \\ &\quad + \frac{g_2}{\langle w \rangle} \sum_J^N M^C(I, J) L_{I,J}(i)(\bar{x}_J - x_{I,i}), \\ \dot{y}_{I,i} &= x_{I,i} + a_{I,i} + D\xi_{I,i}(t),\end{aligned}$$

where

$$f(x_{I,i}) = x_{I,i} - \frac{x_{I,i}^3}{3} - y_{I,i}.$$

FHN model

Transition to synchronized firing

g – coupling strength – control parameter

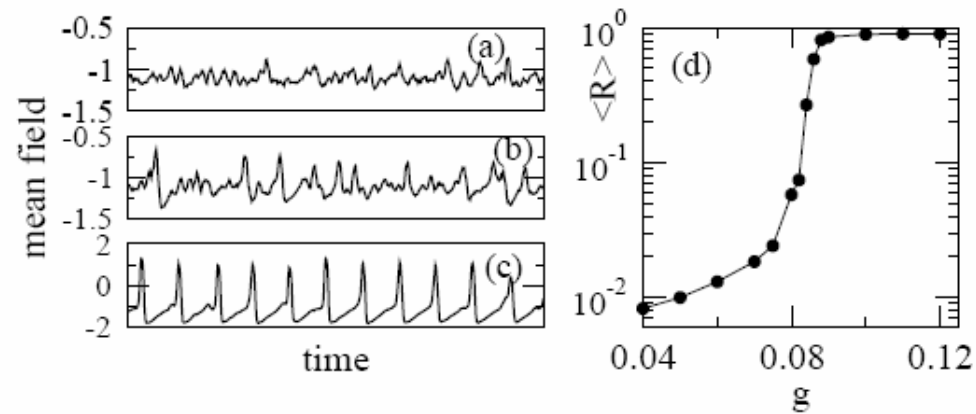


FIG. 2: Mean activity X of one area at various couplings (a) $g = 0.06$, (b) $g = 0.082$ (c) $g = 0.09$. The average correlation coefficient $\langle R \rangle = \frac{1}{N(N-1)} \sum_{I \neq J}^N R(I, J)$ ($N = 53$) vs. g .

Network topology vs. Functional organization in networks

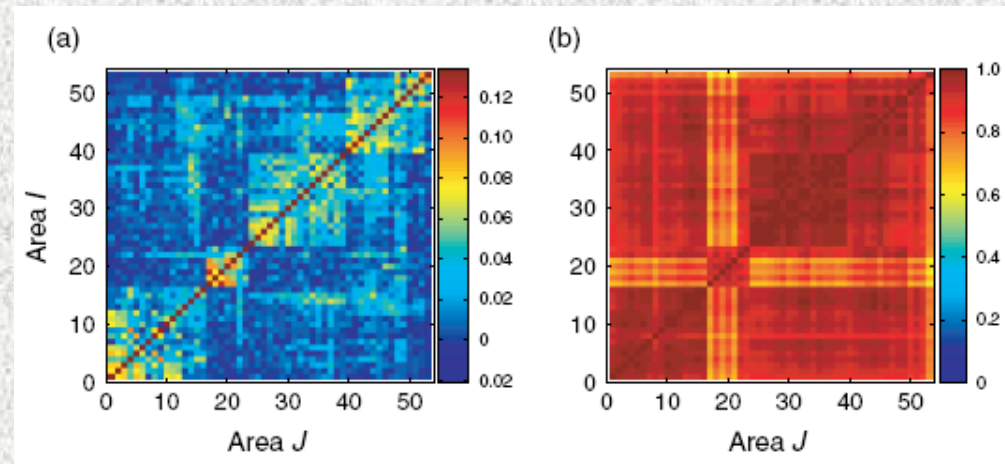


Figure 7. Correlation matrices R_{IJ} at weak coupling $g = 0.07$ (a) and strong coupling $g = 0.12$ (b).

Weak-coupling dynamics → non-trivial organization
→ relationship to underlying network topology

Dendrograms

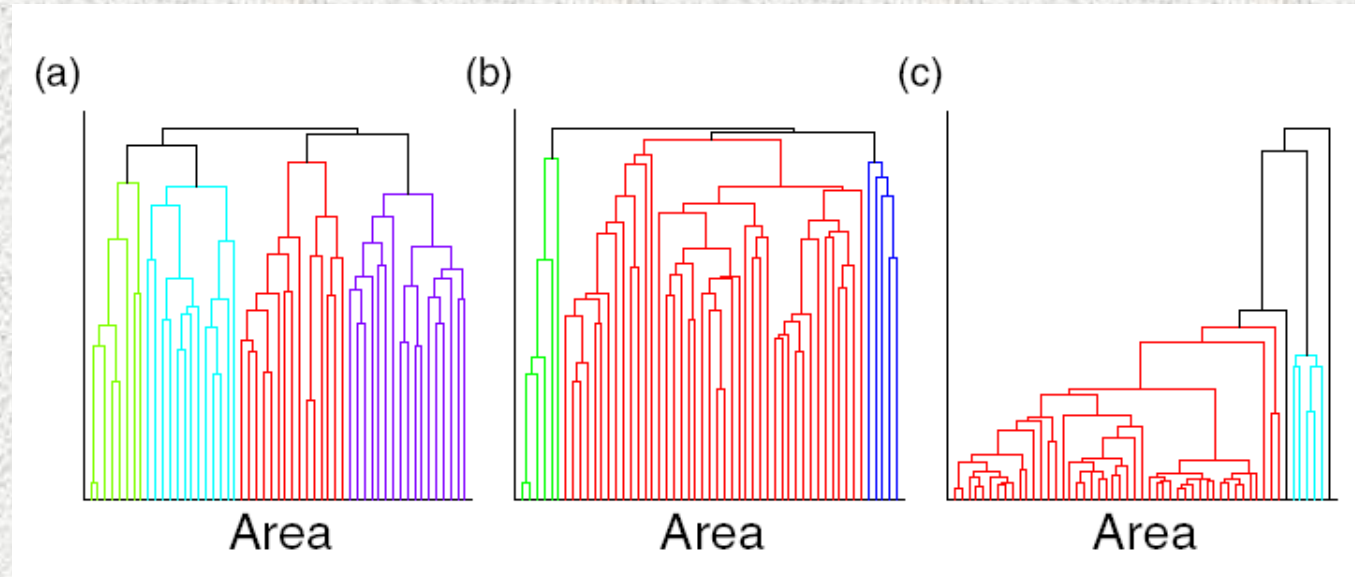


Figure 13. Typical hierarchical tree of the dynamical clusters in the weak coupling regime (a) $g = 0.07$, transient regime (b) $g = 0.082$ and strong coupling regime (c) $g = 0.12$.

Functional vs. Structural Coupling

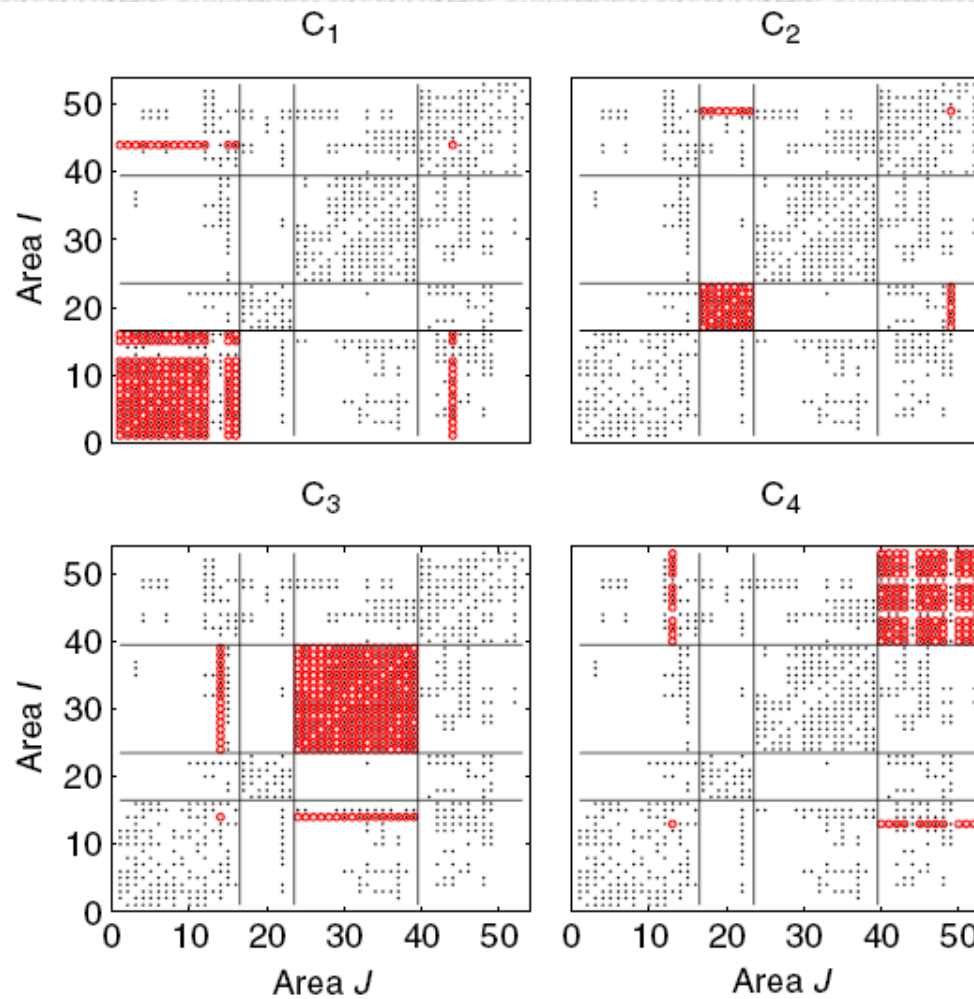
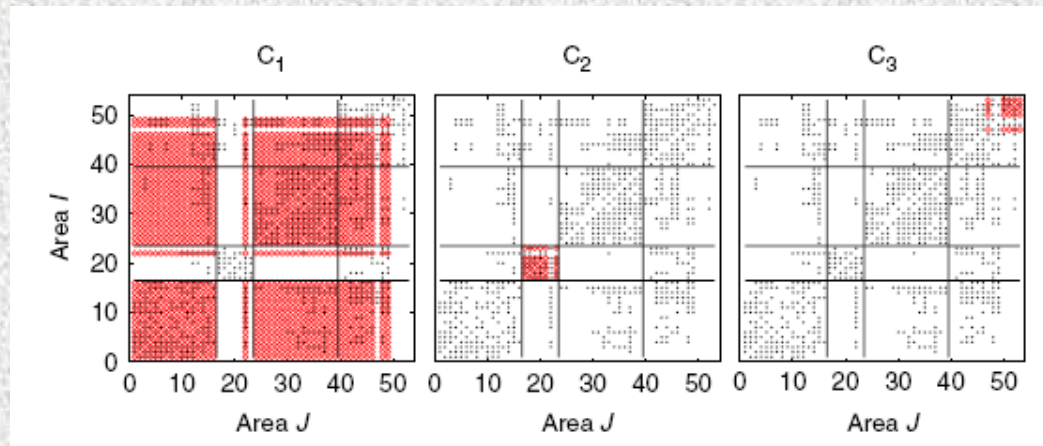


Figure 14. Four major dynamical clusters (\circ) with weak coupling strength $g = 0.07$, compared to the underlying anatomical connections (\cdot).

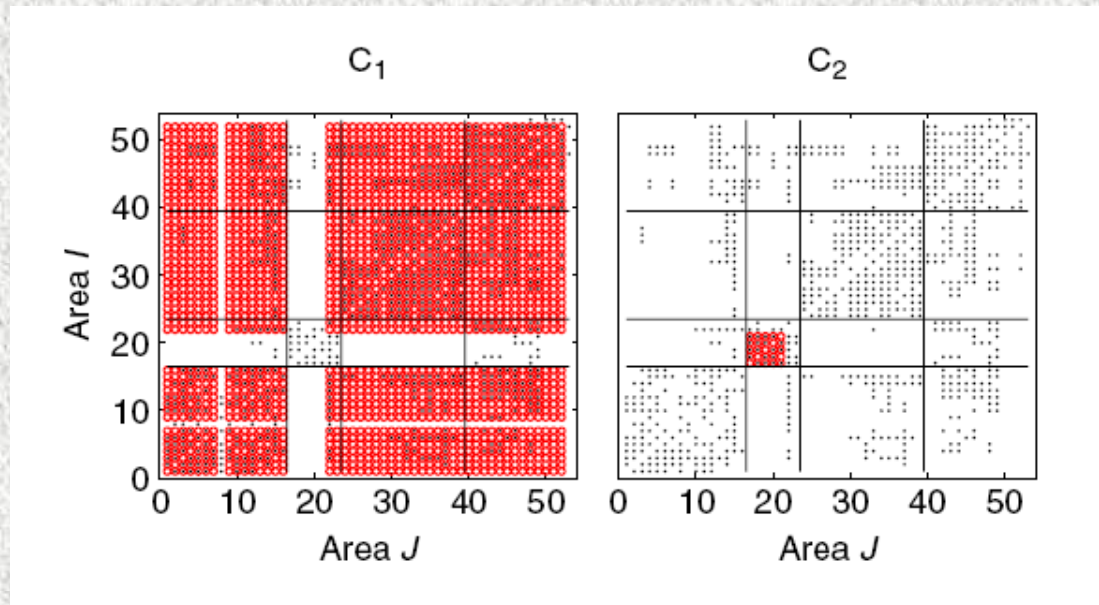
Intermediate Coupling



Intermediate Coupling:

3 main dynamical clusters

Strong Coupling



Summary - Take Home Messages

- There are rich synchronization phenomena in complex networks (**self-organized structure formation**) – hierarchical transitions
- This approach seems to be promising for modelling and understanding of some aspects of **brain dynamics** (e.g. cognition) but also **metabolic networks** etc.

Our papers on complex networks

Europhys. Lett. 69, 334 (2005)
Phys. Rev. E 71, 016116 (2005)
CHAOS 16, 015104 (2006)
Physica D 224, 202 (2006)
Phys: Rev. Lett. 96, 034101 (2006)
Phys. Rev. Lett. 96, 164102 (2006)
Phys. Rev. Lett. 96, 208103 (2006)
Phys. Rev. Lett. 97, 238103 (2006)
New J. Physics 9, 178 (2007)

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